AN OMITTING TYPES THEOREM
WITH AN APPLICATION TO THE CONSTRUCTION
OF GENERIC STRUCTURES

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Abstract.

We give a forcing-free construction of $f$-generic structures. The construction uses an omitting types theorem of independent interest.

Introduction.

Throughout we consider theories $T$ (i.e. deductively closed sets of sentences) formalized in some countable first order language $L$. (The countability of $L$ is an essential restriction.) There is associated with each such theory $T$ a certain class of structures $\mathcal{F}_T$, the class of $T-f$-generic structures, see [1]. This class is usually constructed using $f$-forcing; we will construct $\mathcal{F}_T$ by omitting types.

It is no surprise that $\mathcal{F}_T$ can be constructed in this way (by omitting types). The members of $\mathcal{F}_T$ are the completing models of $T'$ (= Th($\mathcal{F}_T$)) and hence are those structures which omit certain types $\Gamma_\varphi$ (see lemma 1); equivalently (as mentioned in [8, theorem 1.2]) the members of $\mathcal{F}_T$ are those structures which omit certain other types $p_\varphi$. The catch is that to define $\Gamma_\varphi$ or $p_\varphi$ we must refer to the forcing relation, whereas the method used here makes no use at all of forcing.

I am grateful to G. Cherlin for several comments on [10] which proved relevant to the problem discussed here; to R. Cusin for showing me a preprint containing theorem 1; and to A. Macintyre for several specific and general points.

Omitting types theorems.

Let $v_0, v_1, v_2, \ldots$ be the variables of the underlying language $L$.

By a type we will here mean a set of formulas $\Gamma$ such that the set $fv(\Gamma)$ of free variables occurring in $\Gamma$ is a subset of $\{v_0, \ldots, v_k\}$ for some
integer $k \geq 0$. We will sometimes indicate $fv(\Gamma)$ by writing $\Gamma(v_0, \ldots, v_k)$. We use the standard notions of realizing and omitting a type.

Let $T$ be some theory. A type $\Gamma$ is $T$-np (non-principal over $T$) if there is no formula $\psi$ consistent with $T$ such that $T \vdash \psi \rightarrow \gamma$, for each $\gamma \in \Gamma$. (Clearly it is sufficient to consider only those $\psi$ with $fv(\psi) \subseteq fv(\Gamma)$.)

The following theorem is well-known.

**THEOREM A.** Let $T$ be some fixed theory and $\Gamma$ some countable collection of $T$-np types. For each sentence $\sigma$ consistent with $T$ there is a countable structure $\mathfrak{A}$ such that

\begin{align*}
(Ai) & \quad \mathfrak{A} \models T, \\
(Aii) & \quad \mathfrak{A} \models \sigma, \\
(Aiii) & \quad \mathfrak{A} \text{ omits each type in } \Gamma.
\end{align*}

For each integer $n \geq 0$ let $\mathcal{V}_n(\exists_{n})$ be the set of formulas logically equivalent to formulas in prenex normal form whose prenex consists of $n$ blocks of quantifiers, the first block being universal (existential), the second block being existential (universal), the third block being universal (existential), etc. For each two structures $\mathfrak{A}, \mathfrak{B}$ let $\mathfrak{A} \ll_n \mathfrak{B}$ mean that $\mathfrak{A} \subseteq \mathfrak{B}$ and

$$\mathfrak{A} \models \varphi[x] \Rightarrow \mathfrak{B} \models \varphi[x]$$

holds for all formulas $\varphi \in \mathcal{V}_n$ and all $\mathfrak{A}$-assignments $x$. We note that for each theory $T$, $\mathfrak{A} \models T_n \cap \mathcal{V}_{n+1}$ if and only if $\mathfrak{A} \ll_n \mathfrak{B}$ for some $\mathfrak{B} \models T$.

A type $\Gamma$ is $T-(n)$-np if $\Gamma \subseteq \mathcal{V}_{n+1}$ and there is no formula $\psi \in \exists_{n+1}$ consistent with $T$ such that $T \vdash \psi \rightarrow \gamma$, for each $\gamma \in \Gamma$.

We will need theorem A as well as the following refinement.

**THEOREM B.** Let $T$ be some fixed theory, $n \geq 0$ some integer, and $\Gamma$ some countable collection of $T-(n)$-np types. For each $\sigma \in \exists_{n+1}$ consistent with $T$ there is a countable structure $\mathfrak{A}$ such that

\begin{align*}
(Bi) & \quad \mathfrak{A} \models T \cap \mathcal{V}_{n+1}, \\
(Bii) & \quad \mathfrak{A} \models \sigma, \\
(Biii) & \quad \mathfrak{A} \text{ omits each type in } \Gamma.
\end{align*}

Clearly theorems A, B are of the same family. Theorem B is also related to a theorem of Chang, [2].

Following [2] we say a type $\Gamma$ is $T-(n+2)$-existential if $\Gamma \subseteq \exists_{n+2}$ and there is no type $\Delta \subseteq \exists_{n+1}$ consistent with $T$ such that for each $\gamma \in \Gamma$ there is some $\delta \in \Delta$ with $T \vdash \delta \rightarrow \gamma$. 
Theorem C. Let $T$ be some fixed theory, $n \geq 0$ some integer. For each model $\mathfrak{M}$ of $T$ is some structure $\mathfrak{A}$, of the same cardinality as $\mathfrak{M}$, such that

(Ci) $\mathfrak{A} \models T \cap \forall_{n+1}$,
(Cii) $\mathfrak{M} <_{n} \mathfrak{A}$,
(Ciii) $\mathfrak{A}$ omits each $T - (n + 2)$-existential type.

This theorem occurs in [2, § 3], however, the following remarks should be noted.

(a) Chang's $n$ is our $n + 1$.

(b) Chang assumes that $T$ is $\forall_{n+2}$-axiomatizable (our $n$) and proves $\mathfrak{A} \models T$. This makes no essential difference. We see from lemma 0 (below) that in the presence of (Ciii) we can strengthen (Ci) to $\mathfrak{A} \models T \cap \forall_{n+2}$.

(c) (Cii) is stronger than Chang's (ii), but Chang does verify (Cii), see [2, (3) on p. 67].

(Ciii) can be given in an equivalent, more understandable form. To do this we use the type $\Gamma(\varphi, n + 1)$ that is

$$\{\varphi\} \cup \{-\psi : \psi \in \exists_{n+1}, fv(\psi) \subseteq fv(\varphi), T \models \psi \rightarrow \varphi\}$$

for formulas $\varphi \in \forall_{n+1}$. Such a type is easily seen to be $T - (n + 2)$-existential, (see [2, p. 65, E.g. B]), and so is also $T - (n)$-np.

Lemma 0. Suppose $\mathfrak{A} \models T \cap \forall_{n+1}$. The following are equivalent.

(i) $\mathfrak{A}$ omits each $T - (n + 2)$-existential type.

(ii) $\mathfrak{A}$ omits $\Gamma(\varphi, n + 1)$, for each $\varphi \in \forall_{n+1}$.

(iii) For each model $\mathfrak{B}$ of $T$, if $\mathfrak{A} <_{n} \mathfrak{B}$ then $\mathfrak{A} <_{n+1} \mathfrak{B}$.

We can now make a direct comparison between B and C. Clearly

(Bi) = (Ci)

and

(Bii) $\subseteq$ (Cii)

Also, provided we have $\Gamma(\varphi, n + 1) \in \Gamma$ for each $\varphi \in \forall_{n+1}$,

(Biii) $\Rightarrow$ (Ciii).

The stronger version of B obtained by replacing (Bii) by (Cii) is false.
Proof of B.

Anyone familiar with the proof of A will be able to provide a proof of B himself. We will not give all the details of the proof, but just outline the main points.

Let \( T, n, \Gamma, \sigma \) be given. We form a new language \( M \) from \( L \) by adjoining a sequence \( (a_i : i < \omega) \) of new constant symbols. We refer to these as parameters. We construct an \( M \)-structure \( \langle \mathcal{U}, (a_i : i < \omega) \rangle \) such that \( \mathcal{U} \) is the required \( L \)-structure and each element of \( \mathcal{U} \) is some \( a_i \). To do this we construct a set of \( M \)-sentences \( X \subseteq \exists_{n+1} \) such that the following hold.

1. \( T \cup X \) is consistent.
2. \( \sigma \in X \).
3. For each \( M \)-sentence \( \tau \in \exists_{n+1} \), if \( T \cup X \cup \{ \tau \} \) is consistent then \( \tau \in X \).
4. For each \( \Gamma(v_0, \ldots, v_k) \in \Gamma \) and parameters \( a_{i_0}, \ldots, a_{i_k} \) there is some \( \gamma(v_0, \ldots, v_k) \in \Gamma \) such that \( -\gamma(a_{i_0}, \ldots, a_{i_k}) \in X \).
5. For each formulas \( \varphi(v_0, \ldots, v_k), \psi(v_0, \ldots, v_k) \) if

\[
(\exists v_0, \ldots, v_k) \varphi \in X, \quad (\forall v_0, \ldots, v_k) \psi \in X
\]

then there are parameters \( a_{i_0}, \ldots, a_{i_k} \) such that

\[
\varphi(a_{i_0}, \ldots, a_{i_k}) \in X, \quad -\psi(a_{i_0}, \ldots, a_{i_k}) \in X.
\]

We say \( \varphi(a_{i_0}, \ldots, a_{i_k}), -\psi(a_{i_0}, \ldots, a_{i_k}) \) are instances of \( (\exists v_0, \ldots, v_k) \varphi, -\gamma(v_0, \ldots, v_k) \psi \).

We construct \( X \) as the union of a chain

\[
X_0 \subseteq X_1 \subseteq \ldots \subseteq X_m \subseteq \ldots, \quad m < \omega
\]

of finite \( M \)-sets \( X_m \subseteq \exists_{m+1} \). We put \( X_0 = \{ \sigma \} \) and at each step \( X_m \uparrow X_{m+1} \) we consider some triple

\[
(\tau, \Gamma(v_0, \ldots, v_k), (a_{i_0}, \ldots, a_{i_k}))
\]

where \( \tau \) is an \( M \)-sentence in \( \exists_{n+1} \), \( \Gamma \in \Gamma \), and \( a_{i_0}, \ldots, a_{i_k} \) are parameters.

We arrange the construction in such a way that every such triple is considered at some stage.

Given \( X_m \) we construct \( X_{m+1} \) so that the following hold.

6. If \( T \cup X_m \cup \{ \tau \} \) is consistent then \( \tau \in X_{m+1} \).
7. There is some \( \gamma \in \Gamma \) such that \( -\gamma(a_{i_0}, \ldots, a_{i_k}) \in X_{m+1} \).
8. If we have put into \( X_{m+1} \) some sentence of the form \( (\exists v_0, \ldots, v_k) \varphi \) or \( -\gamma(v_0, \ldots, v_k) \psi \) then we have also put in instances.
We note that (7) is possible since $T$ is $T-(n)$-np, so we do not have

$$T \cup X_m \vdash \gamma(a_{i_0} \ldots a_{i_k})$$

for all $\gamma \in T$. Also (8) is possible since only finitely many parameters occur in $X_m$, so there are unused parameters available as witnesses.

The construction.

For each theory $T$ let $\mathcal{S}_T$ be the class of submodels of $T$, that is the class of structures $\mathcal{A}$ such that $\mathcal{A} \subseteq \mathcal{B}$ for some model $\mathcal{B}$ of $T$. Two theories $T,T'$ are co-theories (mutually model consistent) if $\mathcal{S}_T = \mathcal{S}_{T'}$, equivalently if $T \cap \forall_1 = T' \cap \forall_1$. A structure $\mathcal{A}$ is a completing model of $T$ if $\mathcal{A} \in \mathcal{S}_T$ and

$$\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A} < \mathcal{B}$$

holds for all models $\mathcal{B}$ of $T$.

The following lemma is (well-known and) easily proved.

**Lemma 1.** For each theory $T$ and structure $\mathcal{A} \in \mathcal{S}_T$, the following are equivalent.

(i) $\mathcal{A}$ is a completing model of $T$.

(ii) For each formula $\varphi$, $\mathcal{A}$ omits $\Gamma_\varphi$.

Here $\Gamma_\varphi$ is the type $\Gamma(\varphi,0)$ that is

$$\{\varphi\} \cup \{-\theta : \theta \in \exists_1, fv(\theta) \subseteq fv(\varphi), T \vdash \theta \rightarrow \varphi\}.$$

The following lemma is due to Cusin [4, theorem 1']; it is proved using lemma 1 and theorem A.

**Theorem 2.** For each theory $T$ the following are equivalent.

(i) $T$ is the theory of its completing models.

(ii) For each formula $\varphi$ consistent with $T$ there is some formula $\theta \in \exists_1$ consistent with $T$ such that $T \vdash \theta \rightarrow \varphi$.

From now on let $T$ be some fixed (but arbitrary) theory.

In [9] we showed that there is exactly one class of structures $\mathcal{F}$ such that

(1) $T, T^* = \text{Th}(\mathcal{F})$ are co-theories,

(2) $\mathcal{F}$ is the class of completing models of $T^*$. 
Uniqueness followed by standard model theoretic arguments (compactness, method of diagrams, etc.), but existence used \( f \)-forcing. Here we construct \( \mathcal{F} \) using theorems A, B.

From [10] it also follows that for each integer \( n \geq 0 \) there is at most one class \( \mathcal{F}_n \) such that

\[(n) \ T, T_n = \text{Th}(\mathcal{F}_n) \text{ are co-theories},\]
\[(nii) \ \mathcal{F}_n \text{ is the class of submodels } \mathcal{A} \text{ of } T \text{ such that} \]
\[
\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A} \prec_n \mathcal{B}
\]
holds for all models \( \mathcal{B} \) of \( T_n \).

These classes (when they exist) form a chain

\[(h) \ \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \cdots \supseteq \mathcal{F}_n \supseteq \cdots \supseteq \mathcal{F}\]

with \( \mathcal{F} = \bigcap \{ \mathcal{F}_n : n < \omega \} \). We will construct (h) step by step.

First note that \( \mathcal{F}_0 \) must be \( \mathcal{F}_T \), and so there is no existence problem here. (In fact there is no existence problem for \( \mathcal{F}_1 \) since \( \mathcal{F}_1 = \mathcal{E}_T \), a class constructed by means of theorem C. See [9] for details.) We must provide a construction of \( \mathcal{F}_{n+1} \) from \( \mathcal{F}_n \). This we do using [10, theorem 4].

Suppose we have \( \mathcal{F}_n \) (for some integer \( n \geq 0 \)), and consider the class \( \mathcal{K} \) of submodels \( \mathcal{A} \) of \( T \) such that

\[
\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A} \prec_{n+1} \mathcal{B}
\]
holds for all models \( \mathcal{B} \) of \( T_n \). Our problem is, of course, to show that \( \mathcal{K} \) is non-empty.

**Theorem 3.** For each sentence \( \sigma \in \exists_{n+1} \) consistent with \( T_n \), there is some \( \mathcal{A} \in \mathcal{K} \) with \( \mathcal{A} \models \sigma \).

**Proof.** For each formula \( \varphi \) let \( \Gamma_\varphi \) be the type

\[
\{ \varphi \} \cup \{ \neg \theta : \theta \in \exists_1, \text{fv}(\theta) \subseteq \text{fv}(\varphi), \ T_n \vdash \theta \rightarrow \varphi \}.
\]

We first show that for \( \varphi \in \forall_{n+1}, \Gamma_\varphi \) is a \( T_n - (n) \)-np type.

Suppose \( \psi \in \exists_{n+1} \) is such that \( T_n \vdash \psi \rightarrow \gamma \), for each \( \gamma \in \Gamma_\varphi \). (We will show that \( T_n \cup \{ \psi \} \) is inconsistent.)

If \( \psi \) is consistent with \( T_n \) then (nii) gives \( \mathcal{A} \models \psi[x] \) for some \( \mathcal{A} \in \mathcal{F}_n \) and \( \mathcal{A} \)-assignment \( x \). But then (nii) gives \( \mathcal{A} \models \theta[x] \) for some \( \theta \in \exists_1 \) where \( T_n \vdash \theta \rightarrow \psi \). Now we have \( T_n \vdash \psi \rightarrow \varphi \), and we can suppose that \( \text{fv}(\theta) \subseteq \text{fv}(\psi) \subseteq \text{fv}(\varphi) \), so that \( \neg \theta \in \Gamma_\varphi \). Thus we also have \( T_n \vdash \psi \rightarrow \neg \theta \), so that \( T_n \vdash \neg \theta \). This contradicts \( \mathcal{A} \models \theta[x] \), and so \( \Gamma_\varphi \) is \( T_n - (n) \)-np.
Now let $\Gamma = \{ \Gamma_\varphi : \varphi \in \mathcal{V}_{n+1} \}$, $\sigma \in \exists_{n+1}$ be consistent with $T_n$. Theorem B gives us some structure $\mathcal{A}$ such that

(i) $\mathcal{A} \models T_n \cap \forall_{n+1}$,
(ii) $\mathcal{A} \models \sigma$,
(iii) $\mathcal{A}$ omits each $\Gamma_\varphi$, for $\varphi \in \forall_{n+1}$.

We show that $\mathcal{A} \in \mathcal{K}$.

First (i) gives $\mathcal{A} \triangleleft_n \mathcal{B}$ for some $\mathcal{B} \models T_n$. In particular (using (ni)) $\mathcal{A}$ is a submodel of $T$.

Second, suppose that $\mathcal{A} \subseteq \mathcal{B}$ for some $\mathcal{B} \models T_n$, $\mathcal{A} \models \varphi[x]$ for some $\varphi \in \forall_{n+1}$ and $\mathcal{A}$-assignment $x$. From (iii) we get $\mathcal{A} \models \theta[x]$ for some $\theta \in \exists_1$ where $T_n \vdash \theta \rightarrow \varphi$. Thus $\mathcal{B} \models \theta[x]$ (since $\theta \in \exists_1$) and so $\mathcal{B} \models \varphi[x]$ (since $\mathcal{B} \models T_n$). Hence we get $\mathcal{A} \triangleleft_{n+1} \mathcal{B}$.

**Corollary 4.** $\emptyset \models \mathcal{K} \subseteq \mathcal{F}_n$, $T_n \subseteq \text{Th}(\mathcal{K})$, $\text{Th}(\mathcal{K}) \cap \forall_{n+1} \subseteq T_n$, in particular $T_n$, $\text{Th}(\mathcal{K})$ are co-theories.

To show that $\mathcal{K} = \mathcal{F}_{n+1}$ we must verify that

(K) $\mathcal{K}$ is the class of submodels $\mathcal{A}$ of $T$ such that

$$\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A} \triangleleft_{n+1} \mathcal{B}$$

holds for all models $\mathcal{B}$ of $\text{Th}(\mathcal{K})$.

First consider $\mathcal{A} \in \mathcal{K}$ (so that $\mathcal{A}$ is a submodel of $T$) and suppose $\mathcal{A} \subseteq \mathcal{B}$ for some $\mathcal{B} \models \text{Th}(\mathcal{K})$. In particular $\mathcal{B} \models T_n$ so that (by definition of $\mathcal{K}$) $\mathcal{A} \triangleleft_{n+1} \mathcal{B}$.

Secondly, consider any $\mathcal{A}$ satisfying the property of (K), in particular $\mathcal{A}$ is a submodel of $T$. Now suppose that $\mathcal{A} \subseteq \mathcal{B}$ for some $\mathcal{B} \models T_n$. Then (since $\text{Th}(\mathcal{K}) \cap \forall_{n+1} \subseteq T_n$) we have $\mathcal{B} \triangleleft_n \mathcal{C}$ for some $\mathcal{C} \models \text{Th}(\mathcal{K})$. Thus (K) gives $\mathcal{A} \triangleleft_n \mathcal{C}$, and so $\mathcal{A} \triangleleft_{n+1} \mathcal{B}$. Hence $\mathcal{A} \in \mathcal{K}$, as required.

We have now constructed the chain (h) except for $\mathcal{F}$. We also have a chain

$$T_0 \subseteq T_1 \subseteq \ldots \subseteq T_n \subseteq \ldots \quad n < \omega$$

of co-theories of $T$. Let $T^* = \bigcup \{ T_n : n < \omega \}$ so that $T_n, T^*$ are co-theories. To show that $\mathcal{F}$ exists it is sufficient to show that $T^*$ is the theory of its completing models. We do this using theorem 2.

Consider any formula $\varphi$ consistent with $T^*$. We have $\varphi \in \forall_{n+1}$ for some $n$, and $\varphi$ is consistent with $T_n$. Conditions (ni,ii) now give $T_n \vdash \theta \rightarrow \varphi$ for some $\theta \in \exists_1$ consistent with $T_n$. But $T_n, T^*$ are co-theories, so $\theta$ is consistent with $T^*$. Also $T^* \vdash \theta \rightarrow \varphi$, so we may apply theorem 2.
Other remarks.

(1) Theorems A, B depend heavily on the countability of \( L \) and \( \mathcal{F} \), however, we can replace "\( \mathcal{F} \) countable" by "\( \mathcal{F} \) meager in the appropriate stone space". Does this lead to interesting results about \( f \)-generic structures?

(2) The construction of \( \mathcal{F}_T \) given here is analogous to the construction of \( \mathcal{B}_T \) (the \( T-F \)-generic structures) using a certain chain

\[
\mathcal{G}_0 \supseteq \mathcal{G}_1 \supseteq \ldots \supseteq \mathcal{G}_n \supseteq \ldots \supseteq \mathcal{G}.
\]

This chain is built up step by step using a quantifier count as a measure of complexity (as we have done here). In particular the constructions at each step are similar but different. Cherlin noted that there was a certain construction \( \mathcal{K} \Rightarrow \mathcal{K}' \) such that when iterated gave a chain

\[
\mathcal{G}_0 \supseteq \mathcal{G}_0' \supseteq \mathcal{G}_0'' \supseteq \ldots \supseteq \mathcal{G}_0^{(n)} \supseteq \ldots \supseteq \mathcal{G}
\]

with \( \mathcal{G} = \cap \{ \mathcal{G}_0^{(m)} : n < \omega \} \). Details can be found in [3].

Is there a corresponding construction which gives \( \mathcal{F}_T \)? The following may be relevant.

(3) Theorems A, B are clearly related. Indeed if we put "\( n = \omega \)" in B we get A. Presumably there is a common generalization of A, B which is concerned with an unspecified set of formulas \( F \). This set \( F \) would have to satisfy certain restrictions. The general theorem would be such that \( F = \{ \text{all formulas} \} \) gives A and \( F = \forall_n \) gives B. (See [5] and [6].)

(4) Can B be deduced from A (or A from B)? Can A, B be proved using \( f \)-forcing?

(5) The whole of the method used here can be lifted to suitable countable fragments of \( L_{\omega_1 \omega} \) in the manner of [7].

REFERENCES


