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REDUCTION OF CERTAIN GENERALIZED KAMPÉ DE FÉRIET FUNCTIONS

PER W. KARLSSON

In addition to the reductions listed in a recent paper [4] we here give some expressions of hypergeometric functions of n variables as finite sums of simpler hypergeometric functions multiplied by elementary functions. We shall be concerned only with functions exhibiting certain symmetry properties, known as generalized Kampé de Fériet functions. One of these is defined by

$$(1)\qquad F^{1:2}_{1:1}\left[\begin{matrix}\alpha: & \beta_1,\beta_1'; \ldots ; \beta_n,\beta_n' ; \\ \gamma: & \delta_1; \ldots ; \delta_n ;\end{matrix}\; z_1; \ldots ; z_n\right]$$

$$= \sum\nolimits_{j_1=0}^{\infty} \cdots \sum\nolimits_{j_n=0}^{\infty} \frac{(\alpha)_{j_1+\ldots+j_n}}{(\gamma)_{j_1+\ldots+j_n}} \prod\nolimits_{k=1}^{n} \frac{(\beta_k)_{j_k}(\beta_k')_{j_k} z_k^{j_k}}{(\delta_k)_{j_k} j_k!}$$

for $|z_k| < 1$, $k \in \{1,\ldots,n\}$, and by analytical continuation elsewhere; none of the quantities $\gamma, \delta_1,\ldots,\delta_n$ may be zero or a negative integer. (The principle of notation due to Burchnall and Chaundy [3] has been chosen.)

Definitions of the functions $F^{p:r}_{q:s}$, $p+r \leqq q+s+1$, are readily obtained from (1) by deletion and/or addition of parameters and factors. In the sequel we shall, in addition to $F^{1:2}_{1:1}$, consider $F^{1:1}_{1:0}$, which is Lauricella's F_D, $F^{1:0}_{1:0}$, which is the Kummer function ${}_1F_1$ with the sum of the n variables as argument, and $F^{1:0}_{1:1}$; in the latter cases, the series are convergent for all values of the z-variables.

To express the results conveniently, we shall let the symbol $\sum^* f(\pm x_1,\ldots,\pm x_n)$ denote a sum over all sign combinations. ($\sum^*$ thus contains 2^n terms.)

Theorem 1. *When $a+\frac{1}{2}$ is not zero or a negative integer and* $|\arg(1-x_k^2)| < \pi$, $k \in \{1,\ldots,n\}$,

$$(2)\qquad F^{1:2}_{1:1}\left[\begin{matrix}\frac{1}{2} & : b_1, b_1+\frac{1}{2}; \ldots ; b_n, b_n+\frac{1}{2}; \\ a+\frac{1}{2} & : \quad \frac{1}{2} \quad ; \ldots ; \quad \frac{1}{2} \quad ;\end{matrix}\; x_1^2; \ldots ; x_n^2\right]$$

$$= 2^{-n} \sum\nolimits^* U(\pm x_1,\ldots,\pm x_n)$$

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where

$$(3)\qquad U(x_1,\dots,x_n)$$
$$\equiv F_D\left[\begin{matrix}a:2b_1;\dots;2b_n;\\2a:\end{matrix}\ 2x_1/(1+x_1);\dots;2x_n/(1+x_n)\right]\prod_{k=1}^n(1+x_k)^{-2b_k}.$$

THEOREM 2. *When* $a+\frac{1}{2}$ *is not zero or a negative integer,*

$$(4)\qquad F^{1:0}_{1:1}\left[\begin{matrix}\frac{1}{2}\ :\\ a+\frac{1}{2}:\frac{1}{2};\dots;\frac{1}{2};\end{matrix}\ \tfrac{1}{4}x_1^2;\dots;\tfrac{1}{4}x_n^2\right]$$
$$=2^{-n}\sum{}^{*}\,V(\pm x_1,\dots,\pm x_n)$$

where

$$(5)\qquad V(x_1,\dots,x_n)\equiv {}_1F_1\left[\begin{matrix}a;\\2a;\end{matrix}\ 2x_1+\dots+2x_n\right]\exp[-x_1-\dots-x_n].$$

PROOF. Theorem 2 is merely a confluent counterpart: to obtain (4) and (5), replace in (2) and (3) each variable x_k by $x_k/2b_k$ and let each b_k tend to infinity, cf. [1, § 48]. Theorem 1, which remains, will now be proved with the restrictions $\mathrm{Re}\,a>0$ and $|x_k|<1$, $k\in\{1,\dots,n\}$; the general validity follows by analytical continuation. It should be noted that, for negative integral values of a, the F_D in (3) and the ${}_1F_1$ in (5) do exist as limits.

We first rewrite (3) by utilizing the single Eulerian integral representation of F_D (cf. e.g., [1, § 38]) and the duplication formula for the Gamma function. The result is

$$\frac{2^{1-2a}\Gamma(\frac{1}{2})\Gamma(a)}{\Gamma(a+\frac{1}{2})}U(x_1,\dots,x_n)$$
$$=\int_0^1u^{a-1}(1-u)^{a-1}\prod_{k=1}^n\{(1+x_k)^{-2b_k}(1-(2x_ku)/(1+x_k))^{-2b_k}\}\,du\,.$$

Next, the interval of integration is bisected, and the substitutions $u=\frac{1}{2}(1-\sqrt{s})$, and $u=\frac{1}{2}(1+\sqrt{s})$, respectively, are introduced. The integral then becomes

$$2^{-2a}\int_0^1 s^{-\frac{1}{2}}(1-s)^{a-1}\left\{\prod_{k=1}^n(1+x_k\sqrt{s})^{-2b_k}+\prod_{k=1}^n(1-x_k\sqrt{s})^{-2b_k}\right\}ds\,,$$

or, by the binomial theorem,

$$2^{-2a}\int_0^1 s^{-\frac{1}{2}}(1-s)^{a-1}\sum_{i_1=0}^\infty\cdots\sum_{i_n=0}^\infty\left\{A\prod_{k=1}^n(2b_k)_{i_k}(x_k\sqrt{s})^{i_k}/i_k!\right\}ds$$

with

$$A=1+(-1)^{i_1+\dots+i_n}.$$

Different sign combinations in U imply different expressions for A but no other alterations. Hence, to determine the right-hand side of (2) we only need the sum $\sum^* A$, which is easily found to be given as follows,

$$\sum{}^* A = 2\prod_{k=1}^{n}\left(1+(-1)^{i_k}\right) = \begin{cases} 2^{n+1}, & i_1,\ldots,i_n \text{ all even}, \\ 0, & \text{otherwise}. \end{cases}$$

It follows that

$$\frac{\Gamma(\frac{1}{2})\Gamma(a)}{\Gamma(a+\frac{1}{2})}\sum{}^* U(\pm x_1,\ldots,\pm x_n)$$

$$= 2^n \int_0^1 s^{-\frac{1}{2}}(1-s)^{a-1}\sum_{j_1=0}^{\infty}\cdots\sum_{j_n=0}^{\infty}\prod_{k=1}^{n}\left\{\frac{(2b_k)_{2j_k}x_k^{2j_k}s^{j_k}}{(2j_k)!}\right\}ds$$

$$= 2^n \sum_{j_1=0}^{\infty}\cdots\sum_{j_n=0}^{\infty} B\prod_{k=1}^{n}\frac{(b_k)_{j_k}(b_k+\frac{1}{2})_{j_k}x_k^{2j_k}}{(\frac{1}{2})_{j_k}j_k!},$$

where

$$B = \int_0^1 s^{j_1+\ldots+j_n-\frac{1}{2}}(1-s)^{a-1}\,ds$$

$$= \frac{\Gamma(j_1+\ldots+j_n+\frac{1}{2})\Gamma(a)}{\Gamma(j_1+\ldots+j_n+\frac{1}{2}+a)}$$

$$= \frac{\Gamma(\frac{1}{2})\Gamma(a)}{\Gamma(a+\frac{1}{2})}\,\frac{(\frac{1}{2})_{j_1+\ldots+j_n}}{(a+\frac{1}{2})_{j_1+\ldots+j_n}}.$$

Equation (2) now readily follows by comparison with the definition (1). This completes the proof.

It can be proved from one of the linear transformations of F_D (cf. e.g., [1, § 38]) that

$$U(-x_1,\ldots,-x_n) = U(x_1,\ldots,x_n); \tag{6}$$

similarly, Kummer's first transformation implies

$$V(-x_1,\ldots,-x_n) = V(x_1,\ldots,x_n). \tag{7}$$

The right-hand sides of (2) and (4) might thus be rewritten as non-symmetric sums of only 2^{n-1} terms.

Finally, the special case $n=1$ is considered. F_D then becomes Gauss's hypergeometric function ${}_2F_1$, and so also does the $F_{1:1}^{1:2}$ in (2) since its parameters with the value $\frac{1}{2}$ cancel; upon application of (6), (2) now takes the form

$$
{}_2F_1\left[\begin{matrix}b, b+\frac{1}{2};\\ a+\frac{1}{2};\end{matrix}\ x^2\right] = (1+x)^{-2b}{}_2F_1\left[\begin{matrix}a, 2b;\\ 2a;\end{matrix}\ 2x/(1+x)\right]. \tag{8}
$$

This is one of the possible forms of the classical quadratic transformation of ${}_2F_1$. (A simpler generalization of (8), involving one F_A and one F_C, was given by Bailey [2].) Similarly, (4) becomes

$$
{}_0F_1\left[\begin{matrix}\\ a+\frac{1}{2};\end{matrix}\ \tfrac{1}{4}x^2\right] = e^{-x}{}_1F_1\left[\begin{matrix}a;\\ 2a;\end{matrix}\ 2x\right], \tag{9}
$$

which is Kummer's second transformation.

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THE TECHNICAL UNIVERSITY, LYNGBY, DENMARK