THE GROUP PROPERTY OF THE INARIANT $S$ OF VON NEUMANN ALGEBRAS

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Abstract.

We prove that if $M$ is any countably decomposable factor, the invariant $S(M)$ defined in [1] is a closed subgroup of the group of positive real numbers. Moreover multiplication by any element of $S(M)$ leaves the spectrum of any state on $M$ invariant.

THEOREM 1. (a) Let $M$ be a countably decomposable factor, then the non zero elements of the intersection $S(M)$ of the spectra of the modular operators $\Delta_\varphi$ associated with $\varphi$, when $\varphi$ runs through all faithful normal states on $M$, is a closed subgroup of the multiplicative group of positive real numbers.

(b) For any faithful normal state $\varphi$ on $M$ the spectrum of $\Delta_\varphi$ is invariant under multiplication by $S(M)$.

To prove the theorem we need a few lemmas.

Let $\mathcal{A}$ be an achieved generalized left Hilbert algebra, $\Delta$ the modular operator of $\mathcal{A}$.

LEMMA 2. Let $V$ be any compact interval of $]0, \infty[$ and $\chi$ the characteristic function of $V$. If $\xi \in \mathcal{A}$ such that $\chi(\Delta)\xi = \xi$ then for all integers $n \in \mathbb{Z}$ we have $\xi \in \mathcal{B}(\Delta^n)$ and $\Delta^n \xi \in \mathcal{A}$.

PROOF. It is not hard to see that there exists a function $f \in L_1(\mathbb{R})$ such that

$$\lambda^n = \int_{-\infty}^{+\infty} \lambda^t f(t) dt \quad \text{for all } \lambda \in V.$$

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It then follows that
\[
\Delta^n \xi = \Delta^n \chi(\Delta) \xi = \int_{-\infty}^{+\infty} \Delta^u \chi(\Delta) \xi f(t) \, dt \\
= \int_{-\infty}^{+\infty} \Delta^u \xi f(t) \, dt.
\]
Clearly \(\Delta^n \xi \in \mathcal{D}(\mathbb{A})\) and \(\|\pi(\Delta^n \xi)\| \leq \|f\|_1 \|\pi(\xi)\|\) so that \(\Delta^n \xi \in \mathcal{A}\).

**Lemma 3.** Let \(V_1\) and \(V_2\) be two compact intervals of \(]0, \infty[\) and
\[
V = \{pq \mid p \in V_1, q \in V_2\}.
\]
Let \(\chi_1\), \(\chi_2\) and \(\chi\) be the characteristic functions of respectively \(V_1\), \(V_2\) and \(V\). Then for any \(\xi_1 \in \mathcal{A}\), \(\xi_2 \in \mathcal{A}\) such that \(\chi_1(\Delta) \xi_1 = \xi_1\) and \(\chi_2(\Delta) \xi_2 = \xi_2\) we have
\[
\chi(\Delta) \xi_1 \xi_2 = \xi_1 \xi_2.
\]

**Proof.** By lemma 2 we know that \(\Delta^n \xi_1 \in \mathcal{A}\) for all \(n \in \mathbb{Z}\). With the notations of [2] and using [2, lemma 8.3] this implies that \(\Delta^n \xi_1 \in \mathcal{A}'\) for all \(n \in \mathbb{Z}\) and therefore \(\Delta^n \xi_1 \in \mathcal{A}'\). This holds also for \(\Delta^n \xi_2\) and by induction we get that \(\xi_1 \xi_2 \in \mathcal{D}(\mathbb{A})\) and that
\[
\Delta^n (\xi_1 \xi_2) = (\Delta^n \xi_1)(\Delta^n \xi_2).
\]
Let \(\Delta_1 = \Delta \chi_1(\Delta) + \alpha(1 - \chi_1(\Delta))\) for some \(\alpha \in V_1\); then \(\text{Sp} \Delta_1 \subset V_1\) and \(\Delta^n \xi_1 = \Delta_1^n \xi_1\). For any simply closed smooth curve \(\Gamma\) enclosing \(V_1\) we have
\[
\Delta^n (\xi_1 \xi_2) = \pi'(\Delta^n \xi_2) \Delta_1^n \xi_1 \\
= (2\pi)^{-1} \oint_{\Gamma} \pi'(\lambda \Delta) \Delta_1^n \xi_1 (\Delta_1 - \lambda)^{-1} \xi_1 \, d\lambda.
\]
As in the proof of lemma 2 we can find a function \(f \in L_1(\mathbb{R})\) such that
\[
(\Delta_1 - \lambda)^{-1} \xi_1 = \int_{-\infty}^{+\infty} \Delta^u \xi_1 f(t) \, dt
\]
and by the same arguments \((\Delta_1 - \lambda)^{-1} \xi_1 \in \mathcal{A}\) whenever \(\lambda \notin V_1\). So for any polynomial \(p\) we have
\[
p(\Delta) \xi_1 \xi_2 = (2\pi)^{-1} \oint_{\Gamma} ((\Delta_1 - \lambda)^{-1} \xi_1) p(\lambda \Delta) \xi_2 \, d\lambda.
\]
Now let $V_0$ be any compact interval disjoint from $V$ and $E_0 = \chi_0(\Delta)$ where $\chi_0$ is the characteristic function of $V_0$. Then
\[
\|p(\Delta)E_0\xi_1\xi_2\| = \|E_0p(\Delta)\xi_1\xi_2\| \\
\leq (2\pi)^{-1} \sup_{\Gamma} \|\pi(\Delta_1 - \lambda)^{-1}\xi_1\| \|p(\lambda\Delta)\xi_2\| \|\Gamma\|.
\]
Choose $\varepsilon$ sufficiently small such that the two open sets
\[
W_0 = \{z \mid z \in \mathbb{C}, \text{ distance}(z, V_0) < \varepsilon\}, \\
W = \{z \mid z \in \mathbb{C}, \text{ distance}(z, V) < \varepsilon\}
\]
have disjoint closures.

Then it is possible to choose $\Gamma$ such that the set
\[
\{pq \mid p \in V_2, q \text{ is inside } \Gamma\}
\]
is contained in $W$. Let $f$ be the analytic function on $W_0 \cup W$ which is 1 on $W_0$ and 0 on $W$. By Runge's theorem it is possible to find a sequence of polynomials $p_k$ tending uniformly to $f$ on $W_0 \cup W$. Then
\[
p_k(\Delta)E_0\xi_1\xi_2 \text{ tends to } E_0\xi_1\xi_2
\]
and
\[
p_k(\lambda\Delta\xi_2(\Delta))\xi_2 \text{ tends to } 0
\]
uniformly in $\lambda \in \Gamma$. Moreover $\|\pi((\Delta_1 - \lambda)^{-1}\xi_1)\|$ is uniformly bounded on $\Gamma$. Therefore $E_0\xi_1\xi_2 = 0$ and since this holds for all compact closed intervals disjoint from $V$,
\[
\chi(\Delta)\xi_1\xi_2 = \xi_1\xi_2.
\]
This completes the proof.

Let $\varphi$ be a faithful normal state on the von Neumann algebra $M$. Let $(M, H, \xi_0)$ be the G.N.S.-construction of $\varphi$ on $M$. As in [2] let $S = J\Delta^t$ be the corresponding involution. Remind that $JMJ = M'$, and that
\[
\sigma_t(x) = \Delta^{it}x\Delta^{-it} \quad \text{for } x \in M
\]
defines a one parameter group of automorphisms of $M$. In [2, lemma 15.8] it is proved that the subalgebra
\[
\{x \in M \mid \sigma_t(x) = x \text{ for all } t \in \mathbb{R}\}
\]
equals the set
\[
\{x \in M \mid \varphi(xy) = \varphi(yx) \text{ for all } y \in M\}.
\]
As in [2] we call this subalgebra $M_\varphi$.  

Let $e$ be a non zero projection of $M_\varphi$, we shall first determine the modular operator of the state $\varphi_e$ defined on the reduced von Neumann algebra $M_e$ by

$$\varphi_e(x) = \varphi(x)/\varphi(e).$$

The closed subspace

$$H_e = \text{Image} e \cap \text{Image} JeJ = eJeJH$$

is invariant by any element of the algebra $M_e$. So we can consider the algebra $M_1$ induced by $M_e$ in $H_e$ and the canonical homomorphism $\pi$ of $M_e$ onto $M_1$. The element $e\xi_0$ of $H$ is in $H_e$ because

$$JeJ\xi_0 = Je\xi_0 = Je\Delta^1\xi_0 = J\Delta^1e\xi_0 = e\xi_0$$

hence $e\xi_0 \in eJeJH$. Let $\xi_1 = e\xi_0/\|e\xi_0\|$; then it is easy to check that $(\pi, H_e, \xi_1)$ is the G.N.S.-construction of the state $\varphi_e$ on $M_e$. To check that $\xi_1$ is cyclic for $M_1$ in $H_e$ it is enough to prove that $x \in M$ implies $eJeJx\xi_0 \in M_1\xi_1$ which follows from the equality

$$eJeJx\xi_0 = exJeJ\xi_0 = exe\xi_0.$$

Now $e\Delta^t = \Delta^t e$ for all $t \in R$ and similarly $JeJ$ commutes with $\Delta^t$ for all $t$, so $\Delta$ leaves $H_e$ invariant and its restriction to $H_e$ is a closed positive operator.

Let $x \in M_1$, then there exists an $X$ in $M_e$ such that $\pi(X) = x$, in particular

$$\|e\xi_0\|x\xi_1 = xe\xi_0 = X\xi_0$$

and

$$\|e\xi_0\|x^*\xi_1 = x^*e\xi_0 = X^*\xi_0,$$

hence $Sx\xi_1 = x^*\xi_1$ and the involution $S_e$ corresponding to $(M_1, H_e, \xi_1)$ coincides with $S$ on $M_1\xi_1$. Similarly we get the coincidence of $F_e$ with $F$ on $M_1\xi_1$. It follows that $S_e = J_R^*\Delta_R^1$ where $J_R$ is the restriction of $J$ to $H_e$ and $\Delta_R$ the restriction of $\Delta$ to $H_e$. By the uniqueness of the polar decomposition of closed operators we get the equality $\Delta_e = \Delta_R$. Hence the modular operator of the state $\varphi_e$ on $M_e$ is the restriction of the modular operator of $\varphi$ on $M$ to the invariant subspace $eJeJH$.

**Definition 4.** For a faithful normal state $\varphi$ on $M$ put

$$\mathcal{E}_\varphi = \bigcap \{\text{spectrum of the modular operator of } \varphi_e \text{ on } M_e\}$$

where $e$ runs through all non zero projections of the center of $M_\varphi$.

**Lemma 5.** Let $\lambda_1 > 0$, $\lambda_1 \in \mathcal{E}_\varphi$ and $\lambda_2 > 0$, $\lambda_2 \in \text{Sp } \Delta$ then $\lambda_1\lambda_2 \in \text{Sp } \Delta$. 
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**Proof.** (a) We first show that if a bounded open interval $V$ of $]0, \infty[$ intersects $\text{Sp} \Delta$ there exists a non zero $x \in M$ with

$$\chi(\Delta)x\xi_0 = x\xi_0,$$

$\chi$ being the characteristic function of $V$. By hypothesis $\chi(\Delta) \neq 0$, so there is a $y \in M$ with $\chi(\Delta)y\xi_0 \neq 0$. Let $\chi_n$ be a sequence of $C^\infty$ functions on $]0, \infty[$ with $0 \leq \chi_n \leq \chi$ and $\chi_n(\Delta) \to \chi(\Delta)$ strongly when $n \to \infty$. Then there exists an $n$ with $\chi_n(\Delta)y\xi_0 \neq 0$. Further by [2] one has

$$\chi_n(\Delta)y\xi_0 \in M\xi_0,$$

and obviously

$$\chi(\Delta)\chi_n(\Delta)y\xi_0 = \chi_n(\Delta)y\xi_0.$$

(b) Let $V_1$ be a compact interval of $]0, \infty[$ with $\lambda_1$ in its interior, then let $e$ be a non zero projection of the center of $M_e^\varphi$. Since the interior of $V_1$ intersects $\text{Sp} \Delta_e$ there exists by (a) an element $x \neq 0$ of the reduced induced algebra $M_1$ of $M$ in $e\text{JeJH}$ such that

$$x\xi_1 = \chi_1(\Delta_e)x\xi_1$$

where $\chi_1$ is the characteristic function of $V_1$. Now $x\xi_1 \in H_e$, hence

$$\chi_1(\Delta_e)x\xi_1 = \chi_1(\Delta)x\xi_1.$$ 

Since $x \in M_1$ there exists an $X$ in $M_e$ with $x\xi_1 = X\xi_0$, so

$$\chi_1(\Delta)X\xi_0 = X\xi_0, \quad X \neq 0, \quad X \text{ in } M_e.$$

We claim that for such $V_1$ the supremum $V\text{Supp}x$, where $x$ runs over all elements in $M$ with

$$\chi_1(\Delta)x\xi_0 = x\xi_0,$$

is equal to one. In fact it is a certain projection $\kappa$ with for all $t \in \mathbb{R}$, $\Delta^{it}k\Delta^{-it} = k$ because

$$\chi_1(\Delta)\Delta^{it}\Delta^{-it}x\xi_0 = \Delta^{it}x\Delta^{-it}x\xi_0$$

if $\chi_1(\Delta)x\xi_0 = x\xi_0$. Also for all unitary $u \in M^\varphi$, $\hbar k = k$ because

$$\chi_1(\Delta)uxu^*\xi_0 = \chi_1(\Delta)uJux\xi_0 = uJux(\chi_1(\Delta)x\xi_0$$

since $u$ and $Jux$ commute with $\Delta$. So we know that $k$ belongs to the center of $M^\varphi$ hence $1 - k$ is a projection $e$ in the center of $M^\varphi$. If $e \neq 0$, there exists an $X \in M$ with

$$X\xi_0 = \chi_1(\Delta)X\xi_0, \quad X \neq 0, \quad eX = Xe = X,$$

so $\text{Supp}X \leq e$ which contradicts $\text{Supp}X \leq k$ if $X \neq 0$. 
(c) Now let $W$ be any neighbourhood of $\lambda_1\lambda_2$ in $]0,\infty[$, choose $V_1$ and $V_2$ compact intervals containing respectively $\lambda_1$ and $\lambda_2$ in their interior and such that $V_1 \cdot V_2 \subset W$. Let $\chi_1$, $\chi_2$ and $\chi$ be the respective characteristic functions of $V_1$, $V_2$ and $V$. By (a) there exists $x \in M$ with $x \neq 0$ and
$$x\xi_0 = \chi_2(\Delta)x\xi_0,$$
by (b) there exists $y \in M$ with
$$y\xi_0 = \chi_1(\Delta)y\xi_0$$
and $yx \neq 0$ because $1 = \text{VSupp} y$, when $y$ runs over all elements in $M$ satisfying
$$\chi_1(\Delta)y\xi_0 = y\xi_0.$$ 
If we apply lemma 3 to the left generalised Hilbert algebra $\mathcal{A} = M\xi_0$ we get
$$\chi(\Delta)yxx_0 = yxx_0$$
hence $V$ intersects the spectrum of $\Delta$. It then follows that $\lambda_1\lambda_2 \in \text{Sp} \Delta$ as far as $W$ was arbitrary.

**Proof of the Theorem.** Since the theorem is obvious in the semi-finite case we assume $M$ is type III. It is enough to prove b). Let $\varphi$ be a faithful normal state on $M$, let $\lambda_2 > 0$, $\lambda_2 \in \text{Sp} \Delta_\varphi$, let $\lambda_1 > 0$, $\lambda_1 \in S(M)$, then $\lambda_1\lambda_2 \in \text{Sp} \Delta_\varphi$ will follow from the inclusion $S(M) \subset \mathbb{S}_\varphi$. This inclusion is true because for each non zero projection $e$ in the center of $M_\varphi$, $M_\varphi$ is isomorphic to $M$ and hence $\text{Sp} \Delta_\varphi \supset S(M)$ because $\varphi_e$ is a faithful normal state on $M_\varphi$.

This result will be used later to improve the classification of type III factors.

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**References**


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