MATH. SCAND. 32 (1973), 133–144

INVARIANT WEIGHTS ON SEMI-FINITE VON NEUMANN ALGEBRAS

NILS H. PETERSEN

1. Introduction.

In [10] Størmer proves, that if φ is a faithful normal state on a semi-finite von Neumann algebra invariant with respect to a group of *-automorphisms of the algebra acting ergodically on the center, then there exists an invariant faithful normal semi-finite trace, and φ is a Radon-Nikodym derived of this trace. Hence if the group acts ergodically on the algebra, φ itself becomes a trace (and the algebra is finite). The purpose of this paper is to examine the situation where φ no longer is assumed to be a state but a semi-finite weight. I refer to [1] and [7] for the general theory of weights (also contained in [12]) and to [2] and [12] for the theory of weights on von Neumann algebras and the connection between weights and Hilbert algebras. For the general theory of Hilbert algebras I refer to [11] and [12], as well as to [5] for general von Neumann algebra theory.

Basically the result is negative. The paper closes with an example of a II_∞ factor on a separable Hilbert space and an ergodically acting group of *-automorphisms leaving a faithful normal semi-finite weight invariant, but not the trace.

Before this it is proved that if a normal weight, invariant with respect to an ergodic group on a semi-finite factor satisfies a condition, called L^1-continuity, then it is the trace and is the unique invariant normal semi-finite weight. The question whether the uniqueness always holds (without the assumption of L^1-continuity) is left open.

I use the notation from [5] and [12]. For a Hilbert algebra Δ is always the modular operator, J the isometric (unitary) involution, $^\sharp$ the involution of the Hilbertalgebra etc. For a weight φ, $\mathscr{M}_\varphi$ denotes the linear span of the defining order ideal $\mathscr{M}_\varphi{}^+$. Also $\mathscr{N}_\varphi = \{x \mid \varphi(x^*x) < +\infty\}$, etc. I take normal weights in the sense of [12] (φ is normal if it is the pointwise supremum of the normal linear positive functionals it majorizes).

I want to thank Erling Størmer both for his hospitality at the University of Oslo, and for guiding my work. Apart from general, helpful suggestions

Received February 7, 1972, in revised form June 8, 1972.

he formulated and proved Theorem 3.3. Also I thank Alfons Van Daele for helpful corrections and both him and Alan Hopenwasser for stimulating discussions as well as Francois Combes for fruitful conversations during his visit to the University of Oslo in December 71.

2. Automorphisms and Hilbert algebras.

LEMMA 2.1. *Let $\mathscr{A}$ be a Hilbert algebra, the Hilbert space $\mathscr{H}$ its completion, and $M=\mathscr{L}(\mathscr{A})$ the left von Neumann algebra. Let u be a unitary operator on $\mathscr{H}$, so that $u\pi(\xi)u^{-1}=\pi(u\xi)$ for all $\xi \in \mathscr{A}$ (especially u maps $\mathscr{A}$ onto $\mathscr{A}$), then*

i) *u is a $^\sharp$-automorphism of $\mathscr{A}$,*
ii) *u is an isometry of the Hilbert space $\mathscr{D}^\sharp$,*
iii) *u maps $\mathscr{A}'$ onto $\mathscr{A}'$,*
iv) *u is a $^\flat$-anti-automorphism of $\mathscr{A}'$ and $\pi'(u\eta)=u\pi'(\eta)u^{-1}$ for all $\eta \in \mathscr{A}'$,*
v) *u is an isometry of the Hilbert space $\mathscr{D}^\flat$,*
vi) *$u\Delta u^{-1}=\Delta$ and $uJu^{-1}=J$,*
vii) *if $\xi \in \mathscr{H}$ is left-(right-) bounded, then so is $u\xi$ and $u\pi(\xi)u^{-1}=\pi(u\xi)$ (respectively $u\pi'(\xi)u^{-1}=\pi'(u\xi)$).*

PROOF. i) Clearly

$$\pi(u(\xi_1\cdot\xi_2)) = u\pi(\xi_1\cdot\xi_2)u^{-1} = u\pi(\xi_1)u^{-1}u\pi(\xi_2)u^{-1} = \pi(u\xi_1)\pi(u\xi_2) = \pi((u\xi_1)\cdot(u\xi_2))$$

for all $\xi_1, \xi_2 \in \mathscr{A}$, so $u(\xi_1\cdot\xi_2)=(u\xi_1)\cdot(u\xi_2)$.
Similarly $u\xi^\sharp=(u\xi)^\sharp$ for all $\xi \in \mathscr{A}$.

ii) For $\xi \in \mathscr{A}$

$$\|u\xi\|_\sharp^2 = \|u\xi\|^2+\|(u\xi)^\sharp\|^2 = \|\xi\|^2+\|u\xi^\sharp\|^2 = \|\xi\|^2+\|\xi^\sharp\|^2 = \|\xi\|_\sharp^2 .$$

Since $\mathscr{A}$ is dense in the Hilbert space $\mathscr{D}^\sharp$, $u|\mathscr{A}$ has a unique isometric extension to $\mathscr{D}^\sharp$, but as this will be isometric in the norm from $\mathscr{H}$, this extension must coincide with u itself.

vii) Let $\eta \in \mathscr{H}$ be right bounded. Then for all $\xi \in \mathscr{A}$

$$\pi(\xi)u\eta = uu^{-1}\pi(\xi)u\eta = u\pi(u^{-1}\xi)\eta = u\pi'(\eta)u^{-1}\xi ,$$

so that $u\eta$ is right bounded and $\pi'(u\eta)=u\pi'(\eta)u^{-1}$-
Let $\xi \in \mathscr{H}$ be left bounded. Then for all $\eta \in \mathscr{A}'$

$$\pi'(\eta)u\xi = uu^{-1}\pi'(\eta)u\xi = u\pi'(u^{-1})\xi = u\pi(\xi)u^{-1}\eta ,$$

as η is right bounded, so that vii) is proved.

vi) Let $\xi \in \mathscr{A}$, then we have from i)

$$J\Delta^{\frac{1}{2}}\xi = \xi^{\#} = u^{-1}(u\xi)^{\#} = u^{-1}J\Delta^{\frac{1}{2}}u\xi = (u^{-1}Ju)(u^{-1}\Delta^{\frac{1}{2}}u)\xi .$$

As $\mathscr{A}$ is dense in the Hilbert space $\mathscr{D}^{\#}$,

$$J\Delta^{\frac{1}{2}} = \text{the closure of } (u^{-1}Ju)(u^{-1}\Delta^{\frac{1}{2}}u)\,|\,\mathscr{A} .$$

As u is isometric in $\mathscr{D}^{\#}$, the norm defined by $(u^{-1}Ju)(u^{-1}\Delta^{\frac{1}{2}}u)$ is the same as $\| \quad \|_{\#}$, so that

$$J\Delta^{\frac{1}{2}} = (u^{-1}Ju)(u^{-1}\Delta^{\frac{1}{2}}u) .$$

From the uniqueness of the Polar decomposition this gives $J=u^{-1}Ju$ and $\Delta^{\frac{1}{2}}=u^{-1}\Delta^{\frac{1}{2}}u$, so $\Delta=u^{-1}\Delta u$.

Especially it follows, that for all functions f measurable with respect to the spectral measures of Δ, $uf(\Delta)u^{-1}=f(\Delta)$, so that in particular $u\Delta^{-\frac{1}{2}}=\Delta^{-\frac{1}{2}}u$. Therefore u maps $\mathscr{D}^{\flat}$ into $\mathscr{D}^{\flat}$ (hence also onto).

From vii) and the fact, that

$$\eta \in \mathscr{A}' \Leftrightarrow \eta \text{ is right bounded and } \eta \in \mathscr{D}^{\flat}$$

iii) now follows.

From vii), iv) follows as in the proof of i), and similarly v) follows as ii).

REMARK. The lemma and the proof are basically the same as lemma 2 in [10].

Let now M be a von Neumann algebra, φ a faithful normal semi-finite weight on M^{+}. π_{φ} denotes the cyclic representation associated with φ. Since φ is faithful it is an isometry of M on $\pi_{\varphi}(M)$. From [2] and [12] I have the following:

$\mathscr{A}_{\varphi}=\mathscr{M}_{\varphi}$, with the pre-Hilbert structure of φ is a Hilbert algebra, so that

$$\mathscr{A}_{\varphi}'' = \mathscr{N}_{\varphi} \cap \mathscr{N}_{\varphi}^{*} \quad \text{and} \quad \mathscr{L}(\mathscr{A}_{\varphi}) = \pi_{\varphi}(M) ,$$

where $\mathscr{L}(\mathscr{A}_{\varphi})$ is the left von Neumann algebra of $\mathscr{A}_{\varphi}$. $\mathscr{H}_{\varphi}$ is the completion of $\mathscr{A}_{\varphi}$. Let ψ be the canonical weight on $\mathscr{L}(\mathscr{A}_{\varphi})$ (see [2], [12]). From [2] or [12] it is then easy to see, that $\psi\circ\pi_{\varphi}=\varphi$.

Assume G is a group of *-automorphisms of M, and that φ is invariant with respect to G. As in ([2] and [4]) we use the obvious generalization of the Gelfand-Naimark-Segal construction, namely representing G on $\mathscr{H}_{\varphi}$ in the following way. $\mathscr{G}$ is the group of *-automorphisms of $\mathscr{L}(\mathscr{A}_{\varphi})$,

$$\mathscr{G} = \{\pi_{\varphi}\circ g\circ\pi_{\varphi}^{-1}\,|\, g \in G\} .$$

Each $\alpha_{g} \in \mathscr{G}$ is implemented by the unitary operator

$$u_g\xi = g(\xi)\,, \quad \text{where } \xi \in \mathscr{A}_\varphi = \mathscr{M}_\varphi \quad \text{and} \quad \alpha_g = \pi_\varphi \circ g \circ \pi_\varphi^{-1}\,.$$

Since for $x \in \pi_\varphi(\mathscr{M}_\varphi) = \pi(\mathscr{M}_\varphi) = \pi(\mathscr{A}_\varphi)$, $x = \pi(\zeta)$, we have

$$u_g x u_g^{-1}\xi = u_g\pi(\zeta)g^{-1}(\xi) = g(\zeta)\cdot\xi = \pi(g(\zeta))\xi = \alpha_g(\pi_\varphi(\zeta))\xi = \alpha_g(x)\xi\,,$$

as $\pi_\varphi = \pi$ on $\mathscr{M}_\varphi = \mathscr{A}_\varphi$, so that

$$u_g x u_g^{-1} = \alpha_g(x)\,, \quad x \in \pi(\mathscr{A}_\varphi)\,,$$

and since $\pi(\mathscr{A}_\varphi)$ is strongly dense in $\mathscr{L}(\mathscr{A}_\varphi)$ it follows that u_g implements α_g.

From the above calculation it also follows that $u_g\pi(\zeta)u_g^{-1} = \pi(g(\zeta)) = \pi(u_g\xi)$. So the following proposition is merely a summary of known facts.

PROPOSITION 2.2. *Let M be a von Neumann algebra, φ a faithful normal semi-finite weight on M^+, invariant with respect to a group G of *-automorphisms of M. Then G has a faithful unitary representation $g \mapsto u_g$ (for $g \in G$) on $\mathscr{H}_\varphi$, the completion of the Hilbert algebra $\mathscr{A}_\varphi$, so that*

$$u_g\pi(\xi)u_g^{-1} = \pi(u_g\xi) \quad \text{for all } g \in G \quad \text{and} \quad \xi \in \mathscr{A}_\varphi\,.$$

Furthermore $\pi_\varphi(M) = \mathscr{L}(\mathscr{A}_\varphi)$ and $\varphi = \psi \circ \pi_\varphi$, where π_φ is the representation of M on $\mathscr{H}_\varphi$ induced by φ, and ψ is the canonical weight on $\mathscr{L}(\mathscr{A}_\varphi)$.

3. Invariants weights and traces.

DEFINITION 3.1. Let M be a semi-finite von Neumann algebra, τ a faithful normal semi-finite trace. Let φ be a normal weight on M^+. We say that φ is *L^1-continuous* if for any sequence of elements A_n belonging to the unitball M_1^+, $\|A_n\|_1 \to 0$ implies $\varphi(A_n) \to 0$. (This definition is due to Størmer).

LEMMA 3.2. *In the above situation φ is semi-finite. In fact $\mathscr{M}_\varphi^+ \supset \mathscr{M}_\tau^+$.*

PROOF. Let $A \in \mathscr{M}_1^+$ be in $\mathscr{M}_\tau^+$. Then $A_n = n^{-1}A$ is a sequence with $\|n^{-1}A\|_1 \to 0$, so that $\varphi(A_n) \to 0$. Hence $\varphi(A) < +\infty$, that is $A \in \mathscr{M}_\varphi^+$.

REMARK. 1) In [3, remarques 4.11 (c)] Combes gives an example showing that there exist normal, semi-finite weights, not strictly semi-finite. The weights mentioned are all L^1-continuous, as they are derivatives of the trace on $\mathscr{B}(\mathscr{H})$. Hence L^1-continuity does not imply strictly semi-finiteness. The other implication is not true either, which the example in the next section will show.

2) As the trace on $\mathscr{B}(\mathscr{H})$ majorizes the norm, every state on $\mathscr{B}(\mathscr{H})$ is L^1-continuous. So L^1-continuity does not imply normality.

THEOREM 3.3. *Let M be a semi-finite von Neumann algebra with center $\mathscr{C}$, G a group of *-automorphisms of M leaving $\mathscr{C}$ elementwise fixed. Let τ be a faithful normal semi-finite trace on M^+, and φ a faithful L^1-continuous and G-invariant weight on M^+. Let Ψ be a center valued trace on M^+, faithful normal and semi-finite. Then Ψ is G-invariant.*

PROOF. As in ([5, Chapitre III § 4]) we identify $\mathscr{C}$ with $L_{\mathbf{C}}^{\infty}(Z,\nu)$ where Z is locally compact Hausdorff and ν a positive measure on Z. Let $\tilde{\mathscr{C}}^+$ be the positive measurable functions on Z (finite or not).

For all $g \in G$, $\Psi \circ g$ is again a faithful normal and semi-finite center valued trace on M^+, so that by [5, Chapitre III § 4, Théorème 2] there exists a unique $Q_g \in \tilde{\mathscr{C}}^+$, with $0 < Q_g(\zeta) < +\infty$ l.a.e. (locally almost everywhere) on Z, so that

$$\Psi(g(A)) = Q_g \cdot \Psi(A), \quad \text{for all } A \in M^+ .$$

By the uniqueness we get $Q_g \cdot Q_h = Q_{g \cdot h}$ l.a.e. for $g,h \in G$.

Assume that Ψ is not invariant, so that for some $g \in G$, $Q_g \neq 1$. Then there exists a $\delta > 0$, a measurable set Y (not of measure 0) and possibly another g so that $Q_g(\zeta) < 1-\delta$ for $\zeta \in Y$. Let F be the projection corresponding to $1_Y, F \in \mathscr{C}$.

We can choose a non-zero projection $E \in M$, where $E \leqq F$ so that $0 \neq \tau(E) < +\infty$. For all $\varepsilon > 0$ we can find $n \in \mathbf{N}$, so that

$$0 < Q_g{}^n(\zeta)F(\zeta) < \varepsilon, \quad \zeta \in Z ,$$

that is $Q_g{}^n \cdot F < \varepsilon \cdot F$. By ([5, Chapitre III § 4, Proposition 4]) there exists a normal trace ψ on $\tilde{\mathscr{C}}^+$, so that $\tau = \psi \circ \Psi$. Therefore

$$\begin{aligned} \tau(g^n(E)) &= \psi\big(\Psi(g^n(E))\big) = \psi(Q_g{}^n \Psi(E)) = \psi(Q_g{}^n F \Psi(E)) \\ &\leqq \psi(\varepsilon F \Psi(E)) = \psi(\varepsilon \Psi(E)) = \varepsilon \tau(E) . \end{aligned}$$

So $\tau(g^n(E)) \to 0$. This implies that $\varphi(g^n(E)) = \varphi(E) \to 0$. As φ is faithful this implies $E=0$, a contradiction. (This proof is due to Størmer.)

Note that the proof is very similar to the proof of lemma 2.1. in [9].

COROLLARY 3.4. *In the situation of Theorem 3.3 every normal, faithful, semi-finite trace on M^+ is G-invariant.*

REMARKS: 1) If φ is majorized by a trace it is L^1-continuous.

2) If φ is a normal state then φ is L^1-continuous. See [8, Lemma 2.1].

3) Francois Combes noted the following generalization:

Define φ to be L^1-preclosed if $\mathscr{M}_\tau \subset \mathscr{M}_\varphi$ and the injection from $\mathscr{N}_\tau$ into $\mathscr{N}_\varphi$ is preclosed in the τ- respectively φ-Hilbert structure, that is, if for $x_n \in \mathscr{N}_\tau$,

$$\tau(x_n{}^* x_n) \to 0 \quad \text{and} \quad \varphi((x_p - x_q)^*(x_p - x_q)) \to 0 ,$$

implies $\varphi(x_p{}^* x_p) \to 0$. It is an easy calculation to see that in the above proof $g^n(E)$ is a Cauchy-sequence in $\mathscr{H}_\varphi$, so that the conclusion still holds with the weaker assumtion that φ is L^1-preclosed.

Note in the following theorem that when G acts ergodically, then φ invariant implies that φ is faithful.

Theorem 3.5. *Let M be a semi-finite von Neumann algebra, G an ergodically acting group of *-automorphisms of M. Let τ be a normal, semi-finite, G-invariant trace on M^+. Let φ be a normal, semi-finite G-invariant weight on M^+. Then φ is a trace.*

Proof. Consider the standard representation on $\mathscr{H}_\varphi$. Let as in [12, § 13] $\mathscr{N}$ be the set of all left bounded elements ξ in $\mathscr{H}_\varphi$ such that $\pi(\xi) \in n_\tau$, the defining ideal of τ. From [12, § 13, 13.33] we have the Polar decomposition of the closure π of $\pi/\mathscr{N}$, where $\pi = \Lambda \circ K', K'$ positive selfadjoint on $\mathscr{H}_\varphi$, and Λ a unitary operator from $\mathscr{H}_\varphi$ onto $\mathscr{H}_\tau$, the Hilbert space corresponding to τ. As τ is invariant, the operator V_g defined on n_τ by $V_g(x) = u_g x u_g{}^{-1}$ (by proposition 2.2 we identify M and $\mathscr{L}(\mathscr{A}_\varphi)$) extends to a unitary operator on $\mathscr{H}_\tau$, for all $g \in G$. For all $g \in G$, $V_g \circ \Lambda \circ u_g{}^{-1}$ is then unitary from $\mathscr{H}_\varphi$ onto $\mathscr{H}_\tau$. Further $u_g K' u_g{}^{-1}$ is positive selfadjoint on $\mathscr{H}_\varphi$ and for $\xi \in \mathscr{N}, g \in G$

$$\begin{aligned}\pi(\xi) &= \pi(u_g u_g{}^{-1}\xi) = u_g \pi(u_g{}^{-1}\xi) u_g{}^{-1} = V_g(\pi(u_g{}^{-1}\xi)) \\ &= V_g \circ \Lambda(K' u_g{}^{-1}\xi) = (V_g \circ \Lambda \circ u_g{}^{-1}) \circ (u_g K' u_g{}^{-1})\xi ,\end{aligned}$$

where we have used that u_g maps $\mathscr{N}$ onto $\mathscr{N}$ (by lemma 2.1 and since τ is invariant).

Since u_g maps $\mathscr{N}$ onto $\mathscr{N}$ and is unitary, it is easy to see that $u_g K' u_g{}^{-1}$ and K' have the same domain (as K' is the closure of $K' | \mathscr{N}$) and that $u_g K' u_g{}^{-1}$ is the closure of $u_g K' u_g{}^{-1} | \mathscr{N}$. Thus we get

$$\pi = (V_g \circ \Lambda \circ u_g{}^{-1}) \circ (u_g K' u_g{}^{-1}) .$$

But from the uniqueness of the Polar decomposition it then follows that

$$u_g K' u_g{}^{-1} = K' .$$

As the u_g's act ergodically on $\mathscr{L}(\mathscr{A}_\varphi)'$ as well, K' is a scalar. From ([12, § 13, 13.35 and 13.40]) it follows that $\Delta = 1$, so that φ is a trace.

Combining theorem 3.3. and 3.5. we get:

THEOREM 3.6. *Let M be a semi-finite factor. Let G be an ergodically acting group of *-automorphisms of M. Suppose φ is a normal L^1-continuous G-invariant weight on M^+. Then φ is the trace and furthermore φ is the unique normal semi-finite G-invariant weight on M^+.*

Note that by the note before theorem 3.5. it is also here enough to assume φ L^1-preclosed instead of L^1-continuous.

4. An example.

THEOREM 4.1. *There exists a II_∞-factor $\mathscr{B}$ on a separable Hilbert space, a faithful, normal, strictly semi-finite weight ψ on $\mathscr{B}^+$, and ergodic acting group of *-automorphisms of $\mathscr{B}$ leaving ψ invariant, but which does not leave the trace on $\mathscr{B}$ invariant.*

PROOF: Throughout the proof we will use the notation from [5, Chapitre I, § 9]. The factor $\mathscr{B}$ is chosen to be the factor of type II_∞ constructed in Théorème 1 of [5, Chapitre I, § 9]. As the group G used in the construction, we specify $G = \mathsf{Q}$, the rational numbers.

The trace φ on $\mathscr{B}^+$ is defined by: For $A \in \mathscr{B}^+$, A has a matrix of the form

$$R_{s,t} = T_{s-t}\mathscr{U}_{s-t}, \qquad \text{with } T_{s-t} \in \mathscr{A}$$

(here $\mathscr{A} = L_{\mathsf{C}}^\infty(\mathsf{R},\nu)$ with ν the Lebesgue measure), $s,t \in \mathsf{Q}$.

T_0 corresponds to a L^∞-function f_0 on R, and

$$\varphi(A) = \int_{\mathsf{R}} f_0(\zeta)d\nu(\zeta)\,.$$

(This is well-defined, since $f_0 \geqq 0$.)

Let now a be a positive, non zero rational number $\neq 1$. Define Ω_a on $L_{\mathsf{C}}^2(\mathsf{R},\nu)$ by

$$\Omega_a f(\zeta) = a^{\frac{1}{2}} f(a\zeta), \quad \text{for } \zeta \in \mathsf{R}, f \in L_{\mathsf{C}}^2(\mathsf{R},\nu)\,.$$

Then the following is immediate:

Ω_a is unitary, and for $s \in \mathsf{Q}$

$$\Omega_a^{-1}\mathscr{U}_s\Omega_a = \mathscr{U}_{s/a}, \quad \text{and} \quad \Omega_a^{-1} = \Omega_{a^{-1}}\,.$$

Further for $g \in L_{\mathsf{C}}^\infty$ and T_g being the corresponding operator in $\mathscr{A}$,

$$\Omega_a^{-1}T_g\Omega_a = T_{g_a}, \quad \text{where } g_a(\zeta) = g(a^{-1}\zeta)\,.$$

Now we define $\tilde{\Omega}_a$ on $\mathscr{H}$ by the matrix:

$$R_{s,t} = \begin{cases} \Omega_a & \text{if } s = at \\ 0 & \text{otherwise.} \end{cases}$$

It is clear, that $\tilde{\Omega}_a$ is unitary and maps $\mathscr{H}_t$ on $\mathscr{H}_{at}$ for $t \in \mathrm{Q}$. (Note that $\tilde{\Omega}_a$ obviously is not in $\mathscr{B}$).

Claim 1. *$\tilde{\Omega}_a$ implements a *-automorphism of $\mathscr{B}$.*

Proof. Since $\mathscr{B}$ is the weak closure of $\mathscr{B}_0$ it is enough to show, that for $S \in \mathscr{B}_0$, $\tilde{\Omega}_a^{-1} S \tilde{\Omega}_a$ is again in $\mathscr{B}_0$. Let

$$S = \Phi(T_g)\tilde{\mathscr{U}}_y, \quad g \in L_C^\infty, y \in \mathrm{Q}\,.$$

The matrix of S is then:

$$R_{s,t} = \begin{cases} T_g \mathscr{U}_y & \text{if } s-t = y \\ 0 & \text{otherwise.} \end{cases}$$

The matrix of $\tilde{\Omega}_a^{-1}\Phi(T_g)\tilde{\mathscr{U}}_y\tilde{\Omega}_a$ is defined by: The matrix element with indices s,t equals

$$\begin{aligned} J_s^* \tilde{\Omega}_a^{-1}\Phi(T_g)\tilde{\mathscr{U}}_y\tilde{\Omega}_a J_t &= J_s^*\tilde{\Omega}_a^{-1} J_{as} J_{as}^* \Phi(T_g)\tilde{\mathscr{U}}_y J_{at} J_{at}^* \tilde{\Omega}_a J_t \\ &= \Omega_a^{-1} J_{as}^* \Phi(T_g)\tilde{\mathscr{U}}_y J_{at}\Omega_a \\ &= \begin{cases} \Omega_a^{-1} T_g \mathscr{U}_y \Omega_a & \text{if } a(s-t) = y \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \Omega_a^{-1} T_g \Omega_a \Omega_a^{-1} \mathscr{U}_y \Omega_a & \text{if } a(s-t) = y \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} T_{g_a} \mathscr{U}_{y/a} & \text{if } a(s-t) = y \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} T_{g_a} \mathscr{U}_z & \text{if } s-t = z \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(where $z = y/a$).

But this is the matrix of $\Phi(T_{g_a})\tilde{\mathscr{U}}_z$, so

$$\tilde{\Omega}_a^{-1}\Phi(T_g)\tilde{\mathscr{U}}_y\tilde{\Omega}_a = \Phi(T_{g_a})\tilde{\mathscr{U}}_{y/a}\,.$$

As $\mathscr{B}_0$ consists of sums of operators of this type, $\mathscr{B}_0$ is left invariant, so that the claim follows.

Claim 2. *φ is not invariant under this automorphism of $\mathscr{B}$.*

Proof. Let S be as in proof of claim 1, with $y=0$. $R_{0,0}=T_g$, for some $g \in L_{\mathrm{C}}^\infty$. Then $\tilde{\Omega}_a^{-1}S\tilde{\Omega}_a=\Phi(T_{g_a})$, that is the $(0,0)$ matrix element is T_{g_a}, so

$$\varphi(S) = \int_{\mathrm{R}} g(\zeta)d\nu(\zeta)$$

and

$$\varphi(\tilde{\Omega}_a^{-1}S\tilde{\Omega}_a) = \int_{\mathrm{R}} g(a^{-1}\zeta)d\nu(\zeta) = a\int_{\mathrm{R}} g(\zeta)d\nu(\zeta) .$$

As $a \neq 1$, claim 2 follows by choosing a $g \in L_{\mathrm{C}}^\infty$, integrable with respect to the Lebesgue measure and not a zero-function.

Now we define the weight ψ on $\mathscr{B}^+$. As above for $A \in \mathscr{B}^+$ define

$$\psi(A) = \int_{\mathrm{R}} f_0(\zeta)\cdot|\zeta|^{-1}d\nu(\zeta) ,$$

where $f_0 \in L_{\mathrm{C}}^\infty$ is derived as in the definition of φ. Consider the intervals $[n,n+1[$, where $n \in \mathrm{Z}$ and $n \geqq 1$ or $\leqq -2$, and $[1/(n+1), 1/n[$, $[-1/n, -1/(n+1)[$, $n \in \mathrm{N}$. They form a partition of $]-\infty,0[\cup]0,\infty[$. Calling them I_n (giving them some ordering) consider the positive linear normal functional on $\mathscr{B}$ defined by

$$\psi_n(A) = \int_{I_n} f(\zeta)|\zeta|^{-1}d\nu(\zeta) .$$

Consider the projection $\Phi(T_{\chi_n}) \in \mathscr{B}$, where χ_n is the characteristic function for I_n. Since

$$1-\Phi(T_{\chi_n}) = \Phi(1-T_{\chi_n}) = \Phi(T_{1-\chi_n}) ,$$

and $\psi_n(\Phi(T_{1-\chi_n})) = 0$, we have $\mathrm{Supp}\,\psi_n \subseteq \Phi(T_{\chi_n})$. Therefore the ψ_n's have orthogonal support, and $\sum_n \psi_n = \psi$, so by [3, Proposition 4.2 and 4.5] ψ is strictly semi-finite. Also it follows, that ψ is normal.

ψ is faithful as we shall show next. If for some $S \in \mathscr{B}^+$, $\psi(S)=0$, then $\psi_n(S)=0$. Therefore as $\psi_n(A)=\varphi(A\cdot\Phi(T_g))$ where $g(\zeta)=|\zeta|^{-1}\cdot\chi_n(\zeta)$, it follows that $f\cdot g=0$, so that f is zero a.e. on I_n, and therefore on R. The proof of [5, Proposition 1, Chapitre I, § 9] showing that this implies $S=0$ for a positive S carries over.

ψ is invariant with respect to the constructed *-automorphism of $\mathscr{B}$. To prove this, let $S \in \mathscr{B}^+$. S has the matrix

$$R_{s,t} = T_{s-t}\mathscr{U}_{s-t}, \quad T_{s-t} \in \mathscr{A} .$$

Let T_0 correspond to the L_{C}^∞-function f. Then

$$\psi(S) = \int_{\mathrm{R}} f(\zeta)|\zeta|^{-1}d\nu(\zeta) .$$

Now $\tilde{\Omega}_a^{-1}S\tilde{\Omega}_a$ has as its $(0,0)$-matrix element

$$J_0^*\tilde{\Omega}_a^{-1}J_0J_0^*SJ_0J_0^*\tilde{\Omega}_aJ_0 = \Omega_a^{-1}T_0\Omega_a ,$$

which, as in the beginning of the proof, corresponds to the L_{C}^{∞}-function f_a. Therefore

$$\psi(\tilde{\Omega}_a^{-1}S\tilde{\Omega}_a) = \int_{\mathsf{R}} f_a(\zeta)|\zeta|^{-1}d\nu(\zeta) = \int_{\mathsf{R}} f(a^{-1}\zeta)|\zeta|^{-1}d\nu(\zeta)$$
$$= \int_{\mathsf{R}} f(\zeta)(a|\zeta|)^{-1}a\,d\nu(\zeta) = \int_{\mathsf{R}} f(\zeta)|\zeta|^{-1}d\nu(\zeta) = \psi(S)\,.$$

Consider now a unitary operator $\tilde{\mathscr{V}} \in \Phi(\mathscr{A}) = \tilde{\mathscr{A}}$. Its matrix has the form

$$Q_{s,t} = \begin{cases} T_u & \text{if } s = t \\ 0 & \text{otherwise,} \end{cases}$$

where T_u is a unitary member of $\mathscr{A}$. Let S be in $\mathscr{B}$, with matrix $R_{s,t}$. The $(0,0)$-element in the matrix of $\tilde{\mathscr{V}}^{-1}S\tilde{\mathscr{V}}$ is then

$$J_e^*\tilde{\mathscr{V}}^{-1}S\tilde{\mathscr{V}}J_e = T_u^{-1}R_{0,0}T_u = R_{0,0}\,,$$

since $R_{0,0} \in \mathscr{A}$, which is abelian. From this it follows that the *-automorphism of M which $\tilde{\mathscr{V}}$ implements, leaves ψ invariant.

Consider now the group of *-automorphisms of M spanned by the *-automorphisms implemented by $\tilde{\Omega}_a$, $a \in \mathsf{Q}_+$ and all the unitary members of $\tilde{\mathscr{A}}$. It is clear that this group leaves ψ invariant, but not φ. Assume that $S \in \mathscr{B}$ is invariant under the group. Then S commutes with $\tilde{\mathscr{A}}$, so that by the proof of Théorème 1 and by Lemme 2 in [5, Chapitre I, § 9] S itself belongs to $\tilde{\mathscr{A}}$. Therefore S has the matrix $R_{s,t}$ of the following form

$$R_{s,t} = \begin{cases} T_f & \text{if } s = t \\ 0 & \text{otherwise,} \end{cases}$$

where $f \in L_{\mathsf{C}}^{\infty}(\mathsf{R},\nu)$. But $\tilde{\Omega}_a^{-1}S\tilde{\Omega}_a$ has as its $(0,0)$-element in its matrix T_{f_a}. Since therefore for all $a \in \mathsf{Q}_+, f = f_a$ almost everywhere, f is constant on R_+ and R_- a.e. respectively.

For $g \in L_{\mathsf{C}}^{\infty}(\mathsf{R},\nu)$ let

$$(Wg)(x) = g(-1/x), \qquad x \in \mathsf{R}\setminus\{0\}\,.$$

This defines a *-automorphism of $L_{\mathsf{C}}^{\infty}(\mathsf{R},\nu)$.

For $g \in L_{\mathsf{C}}^{\infty}(\mathsf{R},\nu)$ let also for $y \in \mathsf{R}$

$$(W_y'g)(x) = g((x/(1+yx)), \qquad x \in \mathsf{R}\setminus\{-y^{-1}\}\,.$$

It is easy to check, that W'_y defines a one-parameter group of *-automorphisms of $L_{\mathsf{C}}^{\infty}(\mathsf{R},\nu)$.

Let now W^1 and W''_y be a unitary operator and a one-parameter group of unitary operators on $L_{\mathsf{C}}^2(\mathsf{R},\nu)$ implementing W and W'_y on $L_{\mathsf{C}}^{\infty}(\mathsf{R},\nu)$. (We know they exist, and moreover they are easy to calculate).

Then define $\mathscr{W}$ on $\mathscr{H}$ by the matrix:

$$R_{s,t} = \begin{cases} \mathscr{U}_s W''_s W^1 & \text{if } s = t \\ 0 & \text{otherwise.} \end{cases}$$

As in the beginning of the proof we have to check that $\mathscr{W}$, which is obviously a unitary operator on $\mathscr{H}$ implements a *-automorphism on $\mathscr{B}$; because it is then enough to show that for $g \in L_{\mathsf{C}}^\infty$ and $y \in \mathsf{Q}$, $\mathscr{W}^{-1}\Phi(T_g)\tilde{\mathscr{U}}_y\mathscr{W}$ is of the same form. Its matrix is defined by the following element with indices s,t

$$J_s{}^*\mathscr{W}^{-1}\Phi(T_g)\tilde{\mathscr{U}}_y\mathscr{W}J_t = (W^1)^{-1}(W''_s)^{-1}\mathscr{U}_s{}^{-1}T_gJ_s{}^*\tilde{\mathscr{U}}_yJ_t\mathscr{U}_tW''_tW^1$$

$$= \begin{cases} (W^1)^{-1}W''_{-s}\mathscr{U}_{-s}T_g\mathscr{U}_y\mathscr{U}_tW''_tW^1, & \text{if } s-t = y \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} (W^1)^{-1}W''_{-s}\mathscr{U}_{-s}T_g\mathscr{U}_sW''_sW^1(W^1)^{-1}W''_{-s}\mathscr{U}_{-s}\mathscr{U}_y\mathscr{U}_tW''_tW^1 & \text{if } s-t=y \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} T_{g_1}(W^1)^{-1}W''_{-y}W^1 & \text{if } s-t = y \\ 0 & \text{otherwise,} \end{cases}$$

where $g_1 \in L_{\mathsf{C}}^\infty$, since all the unitary operators on $L_{\mathsf{C}}^2(\mathsf{R},\nu)$ act on L_{C}^∞. Now $(W^1)^{-1}W''_{-y}W^1$ is a unitary operator on L_{C}^2 implementing the same automorphism on L^∞ as $\mathscr{U}_y$, as an easy calculation shows. Therefore

$$(W^1)^{-1}W''_{-y}W^1 = T_{g_2}\mathscr{U}_y \quad \text{for some } g_2 \in L_{\mathsf{C}}^\infty ,$$

so that the matrix is the matrix of $\Phi(T_{g_1\cdot g_2})\tilde{\mathscr{U}}_y$, and therefore $\mathscr{W}$ acts on $\mathscr{B}$. When for $S \in \mathscr{B}^+$, $\mathscr{W}^{-1}S\mathscr{W}$ has as its (0,0)-matrix element

$$J_0{}^*\mathscr{W}^{-1}J_0J_0{}^*SJ_0J_0{}^*\mathscr{W}J_0 = (W^1)^{-1}(J_0{}^*SJ_0)W^1$$

so if $J_0{}^*SJ_0$ corresponds to $f \in L_{\mathsf{C}}^\infty$, then the transformed corresponds to Wf so that

$$\psi(\mathscr{W}^{-1}S\mathscr{W}) = \int_{\mathsf{R}} f(-\zeta^{-1})\cdot|\zeta|^{-1}d\nu(\zeta)$$
$$= \int_{\mathsf{R}} f(\zeta)|\zeta|\cdot\zeta^{-2}d\nu(\zeta) = \int_{\mathsf{R}} f(\zeta)|\zeta|^{-1}d\nu(\zeta) = \psi(S) .$$

Therefore ψ is invariant. (By the way it is also obvious, that φ is not invariant).

Assume now that the group of *-automorphisms spanned by all the implementations of $\tilde{\Omega}_a$, $a \in \mathsf{Q}_+$, the unitary operators from $\tilde{\mathscr{A}}$ and $\mathscr{W}$ leave $S \in \mathscr{B}$ invariant. Then $S \in \tilde{\mathscr{A}}$, $S = \Phi(T_f)$, where f is constant on R_+ and R_- respectively a.e. Now $\mathscr{W}^{-1}S\mathscr{W}$ will be $\Phi(T_{Wf})$, where Wf has the same property with the values interchanged. Therefore f must be constant a.e. so that S is a constant, and the group acts ergodically.

Remark. Independently of this work, Gert Kjærgård Pedersen and Masamichi Takesaki have developed a Radon-Nikodym theory for weights.

Their work contains theorems stronger than the ones given here, and an example which is the same as the above with the exception, that the group in this paper is larger, so that it acts ergodically, (see [8]).

REFERENCES

1. F. Combes, *Poids sur une C*-algèbre*, J. Math. Pures Appl., 9e Série, 47 (1968), 57–100.
2. F. Combes, *Poids associé a une algèbre Hilbertienne a gauche*, Compositio Math. 23 (1971), 49–77.
3. F. Combes, *Poids et espérances conditionelles dans les algèbres de von Neumann*, Bull. Soc. Math. France, 99 (1971), 73–112.
4. A. Van Daele, *The upper envelope of invariant functionals majorized by an invariant weight*, University of Oslo Preprint Series No. 29, Dec. 1971.
5. J. Dixmier, *Les algèbres d'operateurs dans l'espace Hilbertien*, 2e edition, Gauthier-Villars Paris, 1969.
6. N. Dunford and J. Schwarz, *Linear Operators* II, Interscience Publishers, New York, 1963.
7. G. Kjærgård Pedersen, *Measure Theory for C*-algebras*. Math. Scand. 19 (1966), 131–145.
8. G. Kjærgård Pedersen and M. Takesaki, *The Radon-Nikodym theorem for von Neumann algebras*. Acta Math. 130 (1973), 53–88.
9. E. Størmer, *Types of von Neumann algebras associated with extremal invariant states*, Comm. Math. Phys. 6 (1967), 194–204.
10. E. Størmer, *States and invariant maps of operator algebras*, J. Functional analysis 5 (1970) 44–65.
11. E. Størmer, *Automorphisms and invariant states of operator-algebras*, Acta Math. 127 (1971), 1–9.
12. M. Takesaki, *Tomita's theory of modular Hilbert algebras and its applications*, (Lecture notes in mathematics 128) Springer-Verlag, Berlin · Heidelberg · New York, 1970.
13. M. Takesaki, *Operator algebras*, Lecture notes, University of California, Los Angeles.
14. M. Tomita, *Standard forms of von Neumann algebras*. The Vth functional analysis symposium of the Math. Soc. of Japan, Sendai, 1967.

UNIVERSITY OF COPENHAGEN, DENMARK
UNIVERSITY OF OSLO, NORWAY