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BORDISM AND GEOMETRIC DIMENSION

HANS ANTON SALOMONSEN

Contents.

0. Introduction.

In sections 1-5 we define and establish some elementary properties of the bordism groups of a space with "coefficients" in a virtual bundle. These groups are, except for slight notational differences, identical with those defined by Wall in [4]. In sections 6, 7 we set up an exact sequence, which through a range of dimensions generalizes the Gysin sequence for a sphere bundle. Using this sequence we define in section 8 an obstruction $\gamma_k(\psi)([M^n])$, which for $n \leqq 2k-1$ is shown to vanish iff the virtual bundle ψ over the manifold M^n is of geometric dimension $\leqq k$. In section 9 this result is specialized to define an obstruction for existence of immersions in the metastable range. We show in section 10 that the order of $\gamma_k(\psi)([M^n])$ is a power of 2, and a bound for the order is given. This result together with a result of R. Brown [1] is used in section 11 to show that immersions

$$G_{n,m} \subseteq \mathsf{R}^{2nm-\alpha(nm)}$$

exist in a large number of cases, where $G_{n,m}$ is the real Grassmannian

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manifold of n-planes in R^{n+m} and $\alpha(nm)$ is the number of $1's$ in the dyadic expansion of nm.

The results in this paper extends some of the results in the authors thesis [3] written under the supervision of F. P. Peterson.

We shall in a later paper carry out a similar program for smooth embeddings of manifolds in Euclidean space.

1. Virtual bundles.

Let X be a topological space. We define a virtual bundle φ over X to be an ordered pair of vector bundles $\xi_+{}^k$ and $\xi_-{}^k$ over X, written

$$\varphi = \xi_+ - \xi_- .$$

The vector bundles shall always be of the same dimension.

We define

$$-\varphi = \xi_- - \xi_+ .$$

If $\psi = \zeta_+ - \zeta_-$ is a virtual bundle over X we define

$$\varphi + \psi = \xi_+ \oplus \zeta_+ - \xi_- \oplus \zeta_- .$$

If $\psi = \zeta_+ - \zeta_-$ is a virtual bundle over a space Y we define a virtual bundle

$$\varphi \times \psi = \xi_+ \times \zeta_+ - \xi_- \times \zeta_-$$

over $X \times Y$.

If $f: Y \to X$ is a continuous mapping we define the pull-back

$$f^*\varphi = f^*\xi_+ - f^*\xi_-$$

over Y. If f is an inclusion or a bundle projection we shall frequently denote $f^*\varphi$ by φ.

If ξ^k is a vector bundle over X we define a virtual bundle

$$\xi^0 = \xi^k - \varepsilon^k ,$$

where ε^k is the trivial bundle. The virtual bundle $\varepsilon^t - \varepsilon^t$ is denoted by 0^t.

We denote the total space of ξ by $E\xi$, and the complement of the zero-section in $E\xi$ is denoted by $E_0\xi$. We identify X with its image under the zero-section.

Finally if ξ and ζ are isomorphic vector bundles and the isomorphism is $g: \xi \to \zeta$, we shall write

$$\xi \cong_g \zeta .$$

2. φ-manifolds.

By a manifold, M we shall mean a smooth manifold with or without boundary. The tangent bundle is denoted by τ_M.

Let (X,A) be a pair of topological spaces, and let $\varphi = \xi_+{}^k - \xi_-{}^k$ be a virtual bundle over X. Then we define a $\varphi-$ manifold over (X,A) to be a triple

$$\mathscr{M} = (M^n, f, F) ,$$

where M^n is a compact n-dimensional manifold with boundary ∂M,

$$f:(M,\partial M) \to (X,A)$$

is a mapping of pairs, and

$$F: \tau_M \oplus f^*\xi_- \cong \varepsilon^n \oplus f^*\xi_+$$

is a stable bundle isomorphism, i.e. F is an equivalence class of bundle isomorphisms

$$F': \tau_M \oplus f^*\xi_- \oplus \varepsilon^N \cong \varepsilon^n \oplus f^*\xi_+ \oplus \varepsilon^N ,$$

where the equivalence relation is generated by the relations

1) $F' \sim F' \times 1_{\varepsilon 1}$
2) $F' \sim F''$ if F' is homotopic to F'' through bundle isomorphisms.

We define the dimension of $\mathscr{M}$ by

$$\dim \mathscr{M} = \dim M = n .$$

We note that for $k \geqq 2$ the set of stable bundle isomorphisms

$$\tau_M{}^n \oplus f^*\xi_-{}^k \cong \varepsilon^n \oplus f^*\xi_+{}^k$$

are in one-one correspondence with the set of homotopy classes of isomorphisms.

Let M^n be a manifold with boundary ∂M. Then the inward normal of ∂M in M defines a homotopy class of isomorphisms

$$\tau_M | \partial M \cong \tau_{\partial M} \oplus \varepsilon^1 .$$

Using this isomorphism we define the boundary of the $\varphi-$ manifold $\mathscr{M}$ over (X,A) by

$$\partial \mathscr{M} = (\partial M, f|\partial M, \partial F) ,$$

where

$$\partial F: \tau_{\partial M} \oplus f^*\xi_- \cong \varepsilon^{n-1} \oplus f^*\xi_+$$

is a stable isomorphism such that the composition

$$\varepsilon^1 \oplus \tau_{\partial M} \oplus f^*\xi_- = \tau_{\partial M} \oplus \varepsilon^1 \oplus f^*\xi_- \cong \tau_M | \partial M \oplus f^*\xi_- \cong_F \varepsilon^n \oplus f^*\xi_+ = \\ = \varepsilon^1 \oplus (\varepsilon^{n-1} \oplus f^*\xi_+)$$

is equivalent to $(-1)^n \times \partial F$. Then $\partial\mathscr{M}$ is a φ-manifold over $A = (A, \emptyset)$.

We define a φ-manifold

$$-\mathscr{M} = (M, f, -F) ,$$

where

$$-F = F \times (-1) : \tau_M \oplus f^*\xi_- \oplus \varepsilon^1 \cong \varepsilon^n \oplus f^*\xi_+ \oplus \varepsilon^1 .$$

Let $g:(Y,B) \to (X,A)$ be a mapping of pairs, and let $\mathscr{M} = (M,f,F)$ be a $g^*\varphi$-manifold over (Y,B). Then we define a φ-manifold over (X,A) by

$$g^*\mathscr{M} = (M, g \circ f, F) .$$

Let

$$\mathscr{M} = (M,f,F) \quad \text{and} \quad \mathscr{N} = (N,g,G)$$

be φ-manifolds of the same dimension over (X,A) Then we define a φ-manifold

$$\mathscr{M} + \mathscr{N} = ((M \cup N, f \cup g, F \cup G)$$

by disjoint union. We write

$$\mathscr{M} - \mathscr{N} = \mathscr{M} + (-\mathscr{N}) .$$

The empty manifold defines a φ-manifold

$$\mathscr{O} = (\emptyset, \emptyset, \emptyset) .$$

If $\partial\mathscr{M} = -\partial\mathscr{N}$ we can by identification of boundaries form

$$\mathscr{M} \cup \mathscr{N} = (M \cup N, f \cup g, F \cup G) ,$$

which is a φ-manifold with $\partial(\mathscr{M} \cup \mathscr{N}) = \mathscr{O}$.

Let $\psi = \zeta_+ - \zeta_-$ be a virtual bundle over a space Y and let $B \subset Y$ be a subspace. Furthermore let $\mathscr{M} = (M^m, f, F)$ be a φ-manifold over (X,A), and let $\mathscr{N} = (N^n, g, G)$ be a φ-manifold over (Y,B). Then we can form the cross product

$$\mathscr{M} \times \mathscr{N} = (M \times N, f \times g, F \times G)$$

to be a $\varphi \times \psi$-manifold over

$$(X,A) \times (Y,B) = (X \times Y, X \times B \cup A \times Y) .$$

Here $M \times N$ will have corners on the boundary. We can either remove them by some standard technique or just allow them to be there. It is unimportant which choice we make.

3. Bordism groups.

Let (X,A) be a pair of topological spaces, and let φ be a virtual bundle over X. Then we shall introduce an equivalence relation among φ-manifolds over (X,A) as follows.

We say that

$$\mathscr{M} \sim \mathscr{O}$$

if there exists a φ-manifold $\mathscr{U}$ over (A,A) with $\partial\mathscr{U} = -\partial\mathscr{M}$ and a φ-manifold $\mathscr{W}$ over (X,X) such that

$$\partial\mathscr{W} = \mathscr{M} \cup \mathscr{U}$$

The pair $(\mathscr{W},\mathscr{U})$ is called a bordism. Now we define

$$\mathscr{M} \sim \mathscr{N} \quad \text{if} \quad \mathscr{M} - \mathscr{N} \sim \mathscr{O} .$$

It is easy to show that $\sim$ is an equivalence relation. The set of equivalence classes of φ-manifolds over (X,A) of dimension n is denoted by

$$\Omega_n(X,A\,;\varphi) .$$

The class defined by $\mathscr{M}$ is denoted by $[\mathscr{M}]$.

An Abelian group structure on $\Omega_n(X,A\,;\varphi)$ is defined by

$$[\mathscr{M}]+[\mathscr{N}] = [\mathscr{M}+\mathscr{N}] ,$$
$$-[\mathscr{M}] = [-\mathscr{M}] ,$$

and

$$0 = [\mathscr{O}] .$$

Let $g:(Y,B)\to(X,A)$ be a mapping of pairs. Then there is an induced homomorphism

$$g^* : \Omega_n(Y,B;g^*\varphi) \to \Omega_n(X,A\,;\varphi)$$

given by

$$g_*[\mathscr{M}] = [g_*\mathscr{M}] .$$

We define a boundary homomorphism

$$\partial : \Omega_n(X,A\,;\varphi) \to \Omega_n(A\,;\varphi)$$

by

$$\partial[\mathscr{M}] = [\partial\mathscr{M}] .$$

The bordism groups have a number of properties similar to the Eilenberg-Steenrod axioms.

Exactnes. Let $A \subset B \subset X$ be spaces and φ a virtual bundle over X. Then there is a long exact sequence

$$\ldots \xrightarrow{\partial} \Omega_n(B,A;\varphi) \to \Omega_n(X,A;\varphi) \to \Omega_n(X,B;\varphi) \xrightarrow{\partial} \Omega_{n-1}(B,A;\varphi) \to \ldots$$

where the unnamed maps are induced by inclusions and ∂ is the composition

$$\Omega_n(X,B;\varphi) \xrightarrow{\partial} \Omega_n(B;\varphi) \to \Omega_n(B,A;\varphi) .$$

Homotopy invariance.

I. Let φ be a virtual bundle over X, let $A \subset X$ be a subspace, let

$$f_i : (Y,B) \to (X,A), \quad i = 0,1$$

be maps, and let F be a homotopy from f_0 to f_1. Then there is a commutative diagram

$$\begin{tikzcd}
\Omega_n(Y, B; f_0{}^*\varphi) \arrow[d, "\theta_F", "\cong"'] \arrow[dr, "f_{0*}"] & \\
\Omega_n(Y, B; f_1{}^*\varphi) \arrow[r, "f_{1*}"'] & \Omega_n(X, A; \varphi)
\end{tikzcd}$$

where the isomorphism θ_F is defined by the isomorphism

$$f_0{}^*\varphi \cong f_1{}^*\varphi$$

determined by F.

II. If f: $(Y,B) \to (X,A)$ induces isomorphisms.

$$f_* : \pi_i(Y) \to \pi_i(X), \quad i \leqq n ,$$
$$f_* : \pi_i(B) \to \pi_i(A), \quad i \leqq n-1 ,$$

and epimorphisms

$$f_* : \pi_{n+1}(Y) \to \pi_{n+1}(X) ,$$
$$f_* : \pi_n(B) \to \pi_n(A)$$

with respect to all base points, then

$$f_* : \Omega_i(Y, B; f^*\varphi) \to \Omega_i(X, A; \varphi)$$

is an isomorphism for $i \leqq n$ and an epimorphism for $i = n+1$.

Excision. Let $U \subset A \subset X$ be spaces and let φ be a virtual bundle over X. Assume that there exists a continuous function

$$h : X \to [0,1]$$

such that

$$h|U = 0 \quad \text{and} \quad h|(X-A) = 1 .$$

Then the inclusion induces an isomorphism

$$\Omega_n(X-U,\, A-U;\, \varphi) \xrightarrow{\cong} \Omega_n(X,\, A;\, \varphi)\,.$$

Let ζ be a vector bundle over X. We put

$$0_\zeta = \zeta - \zeta\,.$$

Then for any subspace $A \subset X$ and virtual bundle φ over X we get isomorphisms

$$\Omega_n(X,\, A;\, \varphi) \cong \Omega_n(X,\, A;\, 0_\zeta + \varphi)$$
$$\Omega_n(X,\, A;\, \varphi) \cong \Omega_n(X,\, A;\, \varphi + 0_\zeta)$$

given by

$$[M^n, f, F] \mapsto [M^n, f, F \times 1_{f^*\zeta}]\,.$$

This shows that $\Omega_n(X,\, A;\, \varphi)$ only depends on the class of φ in $\widetilde{\mathrm{KO}}(X)$.

We define cross products

$$\times : \Omega_n(X,A;\, \varphi) \otimes \Omega_m(Y,B;\, \psi) \to \Omega_{n+m}((X,A) \times (Y,B);\, \varphi \times \psi)$$

by

$$[\mathscr{M}] \times [\mathscr{M}] = [\mathscr{M} \times \mathscr{N}]\,.$$

4. The fundamental class of a manifold.

Let M^m be a compact manifold with boundary ∂M. Then we define the fundamental class of M by

$$[M] = [M^m, 1_M, t_M] \in \Omega_m(M, \partial M;\, \tau_M{}^0)$$

where

$$t_M : \tau_M{}^m \oplus \varepsilon^m \cong \varepsilon^m \oplus \tau_M{}^m$$

is the isomorphism which interchange factors. It is then clear that

$$\partial[M] = [\partial M] \in \Omega_{m-1}(\partial M;\, \tau_{\partial M}{}^0)\,,$$

and if N^n is also a compact manifold then

$$[N \times M] = [N] \times [M] \in \Omega_{n+m}(N \times M, \partial(N \times M);\, \tau_{N\times M}{}^0)\,.$$

We note that the fundamental classes are universal in the following sense:

Let (X,A) be a pair of topological spaces and let $\varphi = \xi_+{}^k - \xi_-{}^k$ be a virtual bundle over X. We consider a φ-manifold (M^m, f, F) over (X,A). We can then choose an isomorphism

$$\theta : f^*\xi_-{}^k \oplus \zeta^p \cong \varepsilon^{k+p}$$

where ζ is a bundle over M of dimension $p > m$. Now the composition $(F \times 1_\zeta) \circ (1 \times \theta^{-1})$:

$$\tau_M \oplus \varepsilon^{k+p} \cong \tau_M \oplus f^*\xi_- \oplus \zeta \cong \varepsilon^m \oplus f^*\xi_+ \oplus \zeta$$

determines a homotopy class of isomorphisms

$$\omega : f^*\xi_+ \oplus \zeta \cong \tau_M \oplus \varepsilon^{k+p-m}$$

such that the composition $(1 \times \omega) \circ (F \times 1_\zeta) \circ (1 \times \theta^{-1})$:

$$\tau_M \oplus \varepsilon^{k+p} \cong \varepsilon^m \oplus f^*\xi_+ \oplus \zeta \cong \varepsilon^m \oplus \tau_M \oplus \varepsilon^{k+p-m}$$

equals $t_M \times 1_{\varepsilon^{k+p-m}}$. We may now consider the composition $f^* \circ (\omega^{-1}, \theta^{-1})_*$:

$$\begin{aligned} \Omega_m(M, \partial M; \tau_M{}^0) &= \Omega_m(M, \partial M; \tau_M{}^0 + 0^{k+p-m}) \cong \Omega_m(M, \partial M; f^*\varphi + 0_\zeta) \\ &= \Omega_m(M, \partial M; f^*\varphi) \to \Omega_m(X, A; \varphi) . \end{aligned}$$

It is clear that under this composition $[M]$ is mapped into $[M, f, F]$.

Also we note that the isomorphism

$$\Omega_m(M, \partial M; \tau_M{}^0) \cong \Omega_m(M, \partial M; f^*\varphi)$$

depends on F, but not on θ.

5. The Thom isomorphism.

Let $A \subset X \subset Y$ be topological spaces. Let ν^p be a vector-bundle over X. We assume given a homeomorphism $h : E\nu^p \to Y$ with $h | X = 1_X$ onto an open neighbourhood of X in Y.

Now for $\varphi = \xi_+ - \xi_-$ a virtual bundle over Y we define a homomorphism

$$\Phi_h : \Omega_n(X, A; -\nu^0 + \varphi) \to \Omega_{n+p}(Y, (Y - X) \cup A; \varphi)$$

as follows. Let

$$[M^n, f, F] \in \Omega_n(X, A; -\nu^0 + \varphi) .$$

Let $\tilde{f} : Ef^*\nu \to E\nu$ be the mapping covering f, and let $Df^*\nu$ and $Sf^*\nu$ be the disc- and sphere bundles with respect to some Riemannian metric on $f^*\nu$.

Then we form the composition $g = h \circ \tilde{f} | Df^*\nu$,

$$g : Df^*\nu \subset Ef^*\nu \to E\nu \to Y .$$

We make the obvious identification

$$\tau_{Df^*\nu} = \pi^*(\tau_M \oplus f^*\nu) ,$$

where $\pi : Ef^*\nu \to M$ is the bundle projection.

We now observe that the stable isomorphism

$$F : \tau_M \oplus f^*\nu \oplus f^*\xi_- \cong \varepsilon^n \oplus \varepsilon^p \oplus f^*\xi_+$$

extends in a unique way to a stable isomorphism

$$G : \tau_{Df^*\nu} \oplus g^*\xi_- \cong \varepsilon^{n+p} \oplus g^*\xi_+ \, .$$

So we define

$$\Phi_h([M^n, f, F]) = [Df^*\nu, g, G] \in \Omega_{n+p}(Y, (Y-X) \cup A ; \varphi) \, .$$

We shall show

Theorem 5.1. *Φ_h is an isomorphism.*

Proof. We shall first show that Φ_h is onto. Therefore let

$$x = [W^{n+p}, g, G] \in \Omega_{n+p}(Y, (Y-X) \cup A ; \varphi)$$

be given. Let $U = g^{-1}$ (im h). We may assume that

$$h^{-1} \circ (g \mid U) : U \to E\nu$$

is transversal to X. Then

$$M^n = g^{-1}(X) \subset W^{n+p}$$

is a submanifold of codimension p, and there is a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{g} & Y \\ \uparrow{\scriptstyle h_1} & & \uparrow{\scriptstyle h} \\ Ef^*\nu & \xrightarrow{\tilde{f}} & E\nu , \end{array}$$

where $\tilde{f}$ is the bundle mapping covering $f = g \mid M : M \to X$, and where h_1 is a diffeomorphism onto a tubular neighbourhood of M. Now it is easy to see that

$$x = [h_1(Df^*\nu), g \mid Df^*\nu, G \mid Df^*\nu] \, ,$$

but then it is clear that

$$x = \Phi_h([M^n, f, G \mid M]) \, ,$$

where we identify $\tau_W \mid M$ and $\tau_M \oplus f^*\nu$ via dh_1.

A similar construction applied to a bordism shows that Φ_h is also injective.

When $Y = E\nu^p$ and h is the identity we write

$$\Phi(\nu) = \Phi_{1_{E\nu}} : \Omega_n(X,A;-\nu^0+\varphi) \xrightarrow{\cong} \Omega_{n+p}(E\nu, E_0\nu \cup A;\varphi) .$$

From the Thom isomorphism we can now produce Gysin sequences.

Let $A \subset X \subset Y$ and let $h: E\nu^p \to Y$ be a homeomorphism as above. Then for any virtual bundle φ over Y we get

THEOREM 5.2. *There is a long exact sequence*

$$\ldots \to \Omega_n((Y-X)\cup A, B;\varphi) \to \Omega_n(Y,B;\varphi) \to \Omega_{n-p}(X,A;-\nu^0+\varphi) \\ \to \Omega_{n-1}((Y-X)\cup A, B;\varphi) \to \ldots .$$

for any subspace $B \subset (Y-X)\cup A$.

PROOF. In the long exact sequence for the triple $(Y, (Y-X)\cup A, B)$ we substitute $\Omega_{n-p}(X,A;-\nu^0+\varphi)$ for $\Omega_n(Y,(Y-X)\cup A;\varphi)$ via Φ_h.

In particular we get for $Y = E\nu^p$ and $A = \emptyset$:

COROLLARY 5.3. *Let ν^p be a vectorbundle over a space X. Then for any virtual bundle φ over X there is an exact sequence*

$$\ldots \to \Omega_n(E_0\nu;\varphi) \to \Omega_n(X;\varphi) \xrightarrow{e(\nu)} \Omega_{n-p}(X;-\nu^0+\varphi) \\ \xrightarrow{w(\nu)} \Omega_{n-1}(E_0\nu;\varphi) \to \ldots$$

We shall call $e(\nu)$ the Euler mapping.

6. Geometric dimension.

Let $\psi = \zeta_+{}^p - \zeta_-{}^p$ be a virtual bundle over a space X. We define a k-reduction of ψ to be a pair (μ^k, G), where μ^k is a k-dimensional vectorbundle over X, and

$$G : \mu^k \oplus \zeta_- \cong \varepsilon^k \oplus \zeta_+$$

is a stable bundle isomorphism.

We note that the geometric dimension $\operatorname{gdim}(\psi) \leqq k$ iff a k-reduction exists.

Consider the bundle

$$\operatorname{Iso}(\varepsilon^k \oplus \zeta_-, \varepsilon^k \oplus \zeta_+) \to X ,$$

whose fibre at $x \in X$ consists of all linear isomorphisms

$$f\colon \mathsf{R}^k \oplus (\zeta_-)_x \cong \mathsf{R}^k \oplus (\zeta_+)_x .$$

The general linear group Gl_k acts freely on the right by

$$(f)T = f \circ (T \times 1_{(\zeta_-)_x}), \quad T \in \mathrm{Gl}_k .$$

So we define

$$V^k(\psi) = \operatorname{Iso}(\varepsilon^k \oplus \zeta_-, \varepsilon^k \oplus \zeta_+)/\mathrm{Gl}_k .$$

Then the induced mapping $V^k(\psi) \to X$ is a fibrebundle projection with the fibre homotopy equivalent to a Stiefel manifold. There is an inclusion

$$V^k(\psi) \to V^k(\psi + 0^1)$$

defined by

$$f\mathrm{Gl}_k \to (f \times 1_R)\mathrm{Gl}_k .$$

So we can define

$$\tilde{V}^k(\psi) = \bigcup_{t=0}^{\infty} V^k(\psi + 0^t) .$$

The induced projection

$$\pi : \tilde{V}^k(\psi) \to X$$

is again a fibre bundle projection with $(k-1)$-connected fibre.

We define a k-dimensional vector bundle μ^k over $V^k(\psi)$ by

$$E(\mu^k) = \operatorname{Iso}(\varepsilon^k \oplus \zeta_-, \varepsilon^k \oplus \zeta_+) \times_{\mathrm{Gl}_k} \mathsf{R}^k .$$

The bundles so defined over $V^k(\psi + 0^t)$ are compatible. Therefore they induce a bundle, also denoted by μ^k, over $\tilde{V}^k(\psi)$. Over $V^k(\psi)$ there is a canonical isomorphism

$$\mu^k \oplus \zeta_- \cong \varepsilon^k \oplus \zeta_+$$

given by

$$([f,v],\, u) \mapsto f(v,u) .$$

If $s : X \to V^k(\psi)$ is a cross section we get by "pull-back" over s an isomorphism

$$s^*\mu^k \oplus \zeta_- \cong \varepsilon^k \oplus \zeta_+ .$$

This procedure establishes a one–one correspondence between homotopy classes of cross sections in the bundle $V^k(\psi) \to X$ and equivalence classes of pairs (σ^k, g), where σ^k is a k-dimensional vector bundle over X and

$$g : \sigma^k \oplus \zeta_- \cong \varepsilon^k \oplus \zeta_+$$

is a homotopy class of isomorphisms. The equivalence is given by

$$(\sigma_1^k, g_1) \sim (\sigma_2^k, g_2)$$

iff there exists a bundle isomorphism $\theta : \sigma_1 \cong \sigma_2$ such that $g_1 = g_2 \circ (\theta \times 1_{\zeta_-})$.

Now in order to study whether k-reductions of ψ exist we consider

$$\pi_* : \Omega_n(\tilde{V}^k(\psi); \varphi) \to \Omega_n(X; \varphi)$$

where φ is a virtual bundle over X. We note that if ψ admits k-reductions over all compact subsets of X then π_* is onto. We have π_* embedded in the long exact sequence

$$\ldots \to \Omega_n(\tilde{V}^k(\psi);\varphi) \xrightarrow{\pi_*} \Omega_n(X;\varphi) \to \Omega_n(X,\tilde{V}^k((\psi);\varphi)$$
$$\to \Omega_{n-1}(\tilde{V}^k(\psi);\varphi) \to \ldots,$$

where we define $\Omega_n(X,\tilde{V}^k(\psi);\varphi)$ to be bordism classes of triples (M^n,f,F), where now f is a commutative diagram

$$f\colon\quad \begin{array}{ccc} \partial M & \xrightarrow{f_\partial} & \tilde{V}^k(\psi) \\ \cap\downarrow & & \downarrow\pi \\ M & \xrightarrow{f} & X \end{array}$$

and F is a stable bundle isomorphism as usual.

In section 7 we shall define a homomorphism

$$\Psi : \Omega_n(X,\tilde{V}^k(\psi);\varphi) \to \Omega_{n-k-1}(X\times P^\infty;\ -k\lambda^0-\psi\otimes\lambda+\varphi)\,,$$

where λ is the Hopf line bundle over the real projective space P^∞ and $\psi\otimes\lambda=\zeta_+\otimes\lambda-\zeta_-\otimes\lambda$. It will be shown then (theorem 7.4) that Ψ is an isomorphism for $n\leqq 2k+1$ and an epimorphism for $n=2k+2$. Therefore by substitution via Ψ in the sequence above we get:

Theorem 6.1. *There is an exact sequence*

$$\Omega_{2k+1}(\tilde{V}^k(\psi);\varphi) \xrightarrow{\pi_*} \Omega_{2k+1}(X;\varphi) \xrightarrow{\gamma_k(\psi)} \Omega_k(X\times P^\infty;\ -k\lambda^0-\psi\otimes\lambda+\varphi)$$
$$\xrightarrow{w_k(\psi)} \Omega_{2k}(\tilde{V}^k(\psi);\varphi) \to \ldots$$

The mappings $\gamma_k(\psi)$ and $w_k(\psi)$ are defined by the construction of the sequence. In fact $\gamma_k(\psi)$ is a composition, which is defined for all values of $k\geqq 0$. This will be usefull in section 10.

7. A generalization of the Thom isomorphism.

Let $\varphi=\xi_+-\xi_-$ and $\psi=\zeta_+-\zeta_-$ be virtual bundles over X. We have an inclusion $V^k(\psi)\to V^{k+1}(\psi)$ given by

$$f\mathrm{Gl}_k \mapsto (1_{\mathsf{R}}\times f)\,\mathrm{Gl}_{k+1}\,.$$

There is a canonical isomorphism

$$\mu^{k+1}\,|\,V^k(\psi) \cong \varepsilon^1\oplus\mu^k\,.$$

The section is defined by

$$f\mathrm{Gl}_k \mapsto [1_{\mathbb{R}} \times f, e_1]\,,$$

$e_1 \in \mathbb{R}^{k+1}$ being the first coordinate vector.

We shall for $p, i \geqq 0$ define homomorphisms

$$\Psi_k : \Omega_n\big(V^{k+p+i}(\psi),\, V^{k+p}(\psi)\,;\, \varphi\big) \to \Omega_{n-k-1}(X \times P^{p+i-1},\, X \times P^{p-1}; -k\lambda^0 - \psi\otimes\lambda + \varphi)$$

where λ is the Hopf line bundle, and in this case P^{p-1} is included in P^{p+i-1} by the last p coordinates. Ψ_k is defined by the following construction. Let (M^n, f, F) represent an element

$$x \in \Omega_n(V^{k+p+i}(\psi),\, V^{k+p}(\psi)\,;\, \varphi)\,.$$

We consider the vector bundle projection

$$E\big((\varepsilon^{k+p+i}\oplus f^*\zeta_+)\otimes\lambda\big) \to P(\varepsilon^{p+i}\oplus f^*\zeta_-)$$

where P denotes the projective bundle and λ the Hopf line bundle. Let

$$K : \mu^{k+p+i}\oplus\zeta_- \cong \varepsilon^{k+p+i}\oplus\zeta_+$$

be the canonical isomorphism over $V^{k+p+i}(\psi)$. We can then define a non-vanishing section s over $P(f^*\zeta_-)$ as follows. We choose a Riemannian metric on $f^*\zeta_-$. Let l be a line in $P(f^*\zeta_-)$, and let $v \in l$ be a vector of unit length. Then we define

$$s(l) = K(0,v)\otimes v\,.$$

Over $V^{k+p}(\psi)$, the bundle μ^{k+p+i} splits as a direct sum

$$\mu^{k+p+i} \,|\, V^{k+p}(\psi) = \varepsilon^i\oplus\mu^{k+p}\,.$$

We can therefore extend s over $P(\varepsilon^i\oplus f^*\zeta_-)\,|\,\partial M$ using the same formula as above.

We now extend the section s to all of $P(\varepsilon^{p+i}\oplus f^*\zeta_-)$ in such a way that it is transversal to the zero-section. Let

$$N \subset P(\varepsilon^{p+i}\oplus f^*\zeta_-)$$

be the preimage of the zero-section under s. Then N is a submanifold of dimension $n-k-1$, and by the construction of s

$$(N, \partial N) \subset (P(\varepsilon^{p+i}\oplus f^*\zeta_-) - P(f^*\zeta_-),\; P(\varepsilon^{p+i}\oplus f^*\zeta_-)\,|\,\partial M - P(\varepsilon^i\oplus f^*\zeta_-)\,|\,\partial M)$$

The fibre derivative of s defines an isomorphism

$$S : \tau_{P(\varepsilon^{p+i}\oplus f^*\zeta_-)}\,|\,N \cong \tau_N\oplus(\varepsilon^{k+p+i}\oplus f^*\zeta_+)\otimes\lambda\,.$$

Also there is a canonical isomorphism

$$Q : \varepsilon^1 \oplus \tau_{P(\varepsilon^{p+i} \oplus f^*\zeta_-)} \cong \tau_M \oplus (\varepsilon^{p+i} \oplus f^*\zeta_-) \otimes \lambda$$

coming from the isomorphism between $(\varepsilon^{p+i} \oplus f^*\zeta_-) \otimes \lambda$ and $\varepsilon^1 \oplus$ the fibre tangent bundle.

Combining Q and S we get an isomorphism

$$L : \tau_N \oplus \varepsilon^1 \oplus (\varepsilon^{k+p+i} \oplus f^*\zeta_+) \oplus \lambda \cong \tau_M \oplus (\varepsilon^{p+i} \oplus f^*\zeta_-) \otimes \lambda$$

over N. Finally define the composition $G = (F \times 1) \circ (L \times 1_{f^*\xi_-})$:

$$\begin{aligned} \tau_N \oplus \varepsilon^1 \oplus (\varepsilon^{k+p+i} \oplus f^*\zeta_+) \otimes \lambda \oplus f^*\xi_- &\cong \tau_M \oplus (\varepsilon^{p+i} \oplus f^*\zeta_-) \otimes \lambda \oplus f^*\xi_- \\ &\cong \varepsilon^n \oplus (\varepsilon^{p+i} \oplus f^*\zeta_-) \otimes \lambda \oplus f^*\xi_+ . \end{aligned}$$

Let

$$\tilde{f} : P(\varepsilon^{p+i} \oplus f^*\zeta_-) \to P(\varepsilon^{p+i} \oplus \zeta_-)$$

be the bundle mapping covering the composition $\pi \circ f$:

$$M \to V^{k+p+i}(\psi) \to X .$$

Then $(N^{n-k-1}, \tilde{f} \mid N, G)$ defines an element in the group

$$\begin{aligned} A = \Omega_{n-k-1}\ (P(\varepsilon^{p+i} \oplus \zeta_-) - P(\zeta_-),\ P(\varepsilon^{p+i} \oplus \zeta_-) - P(\varepsilon^i \oplus \zeta_-); \\ 0^1 - k\lambda^0 + 0_{(p+i)\lambda} - \psi \otimes \lambda + \varphi) . \end{aligned}$$

Now we observe that the inclusion

$$\begin{aligned} &(X \times P^{p+i-1}, X \times P^{p-1}) = \\ &\quad (P(\varepsilon^{p+i}), P(\varepsilon^p)) \xrightarrow{\subset} (P(\varepsilon^{p+i} \oplus \zeta_-) - P(\zeta_-),\ P(\varepsilon^{p+i} \oplus \zeta_-) - P(\varepsilon^i \oplus \zeta_-)) \end{aligned}$$

is a homotopy equivalence. Therefore we have a canonical isomorphism

$$\begin{aligned} A &\cong \Omega_{n-k-1}(X \times P^{p+i-1}, X \times P^{p-1}; 0^1 - k\lambda^0 + 0_{(p+i)\lambda} - \psi \otimes \lambda + \varphi) \\ &= \Omega_{n-k-1}(X \times P^{p+i-1}, X \times P^{p-1}; -k\lambda^0 - \psi \otimes \lambda + \varphi) . \end{aligned}$$

We then define $\Psi_k(x)$ to be the element corresponding to $[N, \tilde{f} \mid N, G]$ under this isomorphism. It is easy to see that Ψ_k is well-defined. Also the following lemma follows easily from the construction of Ψ_k.

Lemma 7.1 *Let $0 \leqq p \leqq q \leqq r$. Then the following diagram of triple exact sequences is commutative up to sign*

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \Omega_n(V^{k+q}(\psi), V^{k+p}(\psi); \varphi) & \xrightarrow{\Psi_k} & \Omega_{n-k-1}(X \times P^{q-1}, X \times P^{p-1};\ -k\lambda^0 - \psi \otimes \lambda + \varphi) \\ \downarrow & & \downarrow \\ \Omega_n(V^{k+r}(\psi), V^{k+p}(\psi); \varphi) & \xrightarrow{\Psi_k} & \Omega_{n-k-1}(X \times P^{r-1}, X \times P^{p-1};\ -k\lambda^0 - \psi \otimes \lambda + \varphi) \\ \downarrow & & \downarrow \end{array}$$

$$\begin{array}{ccc}
\Omega_n(V^{k+r}(\psi),\ V^{k+q}(\psi);\varphi) & \xrightarrow{\Psi_k} & \Omega_{n-k-1}(X\times P^{r-1},\ X\times P^{q-1};\ -k\lambda^0-\psi\otimes\lambda+\varphi) \\
\downarrow & & \downarrow \\
\Omega_{n-1}(V^{k+q}(\psi),\ V^{k+p}(\psi);\varphi) & \xrightarrow{\Psi_k} & \Omega_{n-k-2}(X\times P^{q-1},\ X\times P^{p-1};\ -k\lambda^0-\psi\otimes\lambda+\varphi) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

Now we shall show

LEMMA 7.2. *Assume that* $t=\dim\zeta_-=\dim\zeta_+\geqq k+2$. *Then*

$$\Psi_k : \Omega_n(V^{k+1}(\psi),\ V_k(\psi);\varphi) \to \Omega_{n-k-1}(X;\ -\psi+\varphi)$$

is an isomorphism for $n\leqq 2k+1$ *and an epimorphism for* $n=2k+2$.

PROOF. We can lift the inclusion $V^k(\psi)\to V^{k+1}(\psi)$ to a mapping

$$V^k(\psi) \to E_0\mu^{k+1}$$

defined by

$$f\,\mathrm{Gl}_k \mapsto [1_{\mathsf{R}}\times f,\ e_1]$$

as above. We can extend this mapping over $V^k(\psi+0^1)$ as follows. Let

$$g : \mathsf{R}^k\oplus(\zeta_-)_x\oplus\mathsf{R} \cong \mathsf{R}^k\oplus(\zeta_+)_x\oplus\mathsf{R}$$

represent an element $[g]\in V^k(\psi+0^1)$. Let g' be the composition

$$g' : \mathsf{R}\oplus\mathsf{R}^k\oplus(\zeta_-)_x = \mathsf{R}^k\oplus(\zeta_-)_x\oplus\mathsf{R}\cong_g\mathsf{R}^k\oplus(\zeta_+)_x\oplus\mathsf{R} = \mathsf{R}\oplus\mathsf{R}^k\oplus(\zeta_+)_x\,.$$

Then we can define the extension $V^k(\psi+0^1)\to E_0\mu^{k+1}$ by

$$[g]\mapsto[g',\ e_1]\,.$$

Clearly this mapping is a homeomorphism.

It is easy to see that the inclusion $V^k(\psi)\to V^k(\psi+0^1)$ is an $(k+t-1)$-equivalence. Therefore the induced mapping

$$\Omega_n(V^{k+1}(\psi),\ V^k(\psi);\varphi) \to \Omega_n(V^{k+1}(\psi),\ E_0\mu^{k+1};\varphi)$$

is an isomorphism for $n\leqq k+t-1$ and an epimorphism for $n=k+t$.

Now we observe that the composition

$$\begin{aligned}
\Omega_n(V^{k+1}(\psi),\ V^k(\psi);\varphi) \to\ & \Omega_n(V^{k+1}(\psi),\ E_0\mu^{k+1};\varphi) \\
& \cong_{\Phi^{-1}(\mu)} \Omega_{n-k-1}(V^{k+1}(\psi);\ -\mu^0+\varphi) \\
& = \Omega_{n-k-1}(V^{k+1}(\psi);\ -\mu^0+\psi-\psi+\varphi) \\
& = \Omega_{n-k-1}(V^{k+1}(\psi);\ 0^{k+t+1}-\psi+\varphi) \\
& = \Omega_{n-k-1}(V^{k+1}(\psi);\ -\psi+\varphi) \\
& \xrightarrow{\pi_*} \Omega_{n-k-1}(X;\ -\psi+\varphi)
\end{aligned}$$

agrees with Ψ_k up to sign.

Since the fibre of the bundle $\pi: V^{k+1}(\psi) \to X$ is k-connected π_* is an isomorphism for $n-k-1 \leqq k$ or $n \leqq 2k+1$ and an epimorphism for $n = 2k+2$. Therefore Ψ_k is an isomorphism for

$$n \leqq \min(2k+1,\, k+t-1)$$

and an epimorphism for

$$n = \min(2k+2,\, k+t)\,.$$

We can now show

Theorem 7.3. *Assume that* $\dim \zeta_- = \dim \zeta_+ \geqq k+2$. *Then*

$$\Psi_k : \Omega_n(V^{k+i}(\psi), V^k(\psi); \varphi) \to \Omega_{n-k-1}(X \times P^{i-1};\ -k\lambda^0 - \psi \otimes \lambda + \varphi)$$

is an isomorphism for $n \leqq 2k+1$ *and an epimorphism for* $n = 2k+2$.

Proof. The proof will be by induction on i. For $i=1$ the theorem is identical with lemma 7.2. Assume then the theorem for $i = q \geqq 1$. We shall consider the diagram from lemma 7.1 with $p=0$ and $r = q+1$.

Assume that

$$\Psi_k : \Omega_n(V^{k+q+1}(\psi), V^{k+q}(\psi); \varphi) \to \Omega_{n-k-1}(X \times P^q,\, X \times P^{q-1};\ -k\lambda^0 - \psi \otimes \lambda + \varphi)$$

is an isomorphism for $n \leqq 2k+1$ and an epimorphism for $n = 2k+2$. Then the induction step follows by "four lemma" arguments. Therefore it is now sufficient to prove the assumption.

We note that $P^{q-1} \subset P^q$ is a deformation retract of $P^q - [e_1]$. Therefore by the Thom isomorphism we have

$$\begin{aligned} &\Omega_t(X \times P^q,\, X \times P^{q-1};\ -k\lambda^0 - \psi \otimes \lambda + \varphi) \\ &\quad \cong \Omega_t(X \times P^q,\, X \times (P^q - [e_1]);\ -k\lambda^0 - \psi \otimes \lambda + \varphi) \\ &\quad \cong \Omega_{t-q}(X;\ -\psi + \varphi)\,. \end{aligned}$$

Finally we observe that the diagram

$$\begin{tikzcd} \Omega_n(V^{k+q+1}(\psi), V^{k+q}(\psi); \varphi) \arrow[r, "\Psi_k"] \arrow[dr, "\Psi_{k+q}"'] & \Omega_{n-k-1}(X \times P^q,\, X \times P^{q-1};\ -k\lambda^0 - \psi \otimes \lambda + \varphi) \arrow[d, "\cong"] \\ & \Omega_{n-k-q-1}(X;\ -\psi + \varphi) \end{tikzcd}$$

commutes up to sign. Then the assumption follows from lemma 7.2.

In view of lemma 7.2, which gives an approximation of the Thom isomorphism, we consider theorem 7.3 as a generalization of the Thom isomorphism theorem. We can therefore view the exact sequences

$$\Omega_{2k+1}(V^k(\psi);\varphi) \to \Omega_{2k+1}(V^{k+i}(\psi);\varphi) \to \Omega_k(X \times P^{i-1};\ -k\lambda^0 - \psi\otimes\lambda + \varphi) \\ \to \Omega_{2k}(V^k(\psi);\varphi) \to \dots,$$

obtained from the long exact sequence for the pair $(V^{k+i}(\psi),\ V^k(\psi))$ by substitution via Ψ_k, as generalized Gysin sequences where sphere bundles are replaced by Stiefel bundles.

If we take i large we can replace $V^{k+i}(\psi)$ by X in theorem 7.3. Furthermore, replacing ψ by $\psi + 0^t$ for t large, we get a homomorphism

$$\Psi : \Omega_n(X, \tilde{V}^k(\psi);\varphi) \to \Omega_{n-k-1}(X \times P^\infty;\ -k\lambda^0 - \psi\otimes\lambda + \varphi),$$

and we get from theorem 7.3:

THEOREM 7.4. *The homomorphism*

$$\Psi : \Omega_n(X, \tilde{V}^k(\psi);\varphi) \to \Omega_{n-k-1}(X \times P^\infty;\ -k\lambda^0 - \psi\otimes\lambda + \varphi)$$

is an isomorphism for $n \leqq 2k+1$ *and an epimorphism for* $n = 2k+2$.

8. Geometric dimension and manifolds.

Let X be a closed connected manifold of dimension n and let ψ be a virtual bundle over X. We shall then show

THEOREM 8.1. *Assume* $n \leqq 2k-1$ *and* $n \geqq 5$. *Then* $\operatorname{gdim}(\psi) \leqq k$

if and only if

$$\gamma_k(\psi)([X]) = 0 \in \Omega_{n-k-1}(X \times P^\infty;\ -k\lambda^0 - \psi\otimes\lambda + \tau_X{}^0).$$

PROOF. We have already noted that if $\operatorname{gdim}(\psi) \leqq k$ then

$$\pi_* : \Omega_n(\tilde{V}^k(\psi);\ \tau_X{}^0) \to \Omega_n(X;\ \tau_X{}^0)$$

in Theorem 6.1 is onto. Therefore $\gamma_k(\psi)([X]) = 0$ by exactness.

Assume now that $\gamma_k(\psi)([X]) = 0$. Then by Theorem 6.1

$$[X] = \pi_*(\varkappa)$$

for some element

$$\varkappa \in \Omega_n(\tilde{V}^k(\psi);\ \tau_X{}^0).$$

We now *claim*: $\varkappa$ contains a representative of the form (X^n, s, F) where

$$s : X \to \tilde{V}^k(\psi)$$

is a section

Since the existence of s implies $\operatorname{gdim}(\psi) \leqq k$ we only have to prove the claim. Let $(M_0{}^n, f_0, F_0)$ be a representative for $\varkappa$. Then since

$$\pi_* \varkappa = [M_0, \pi \circ f_0, F_0] = [X]$$

there is a bordism (W^{n+1}, g, G) from $(M_0, \pi \circ f_0, F_0)$ to $(X, 1_X, t_X)$. We can choose a continuous function

$$h : W \to [0, 1]$$

with

$$h^{-1}(0) = M_0, \; h^{-1}(1) = X \,.$$

Then

$$g \times h : (W; M_0, X) \to (X \times I; X \times 0, X \times 1)$$

defines a mapping af degree 1. We can now by Wall [4, Theorem 3.3] do surgery on $g \times h$ outside $X \subset W$ and thereby obtain a simple homotopy equivalence. By the s-cobordism theorem the resulting triple is diffeomorphic to

$$(X \times I; X \times 0, X \times 1) \,.$$

Therefore we see that the effect of the surgery on M_0 is to change it into X. Furthermore we note that only surgery in or below the middle dimension is used. Therefore there exists a sequence of triples

$$(M_i, \bar{f}_i, F_i), \qquad i = 0, 1, 2, \ldots, q$$

with

$$\bar{f}_0 = \pi \circ f_0 \qquad \text{and} \qquad M_q = X$$

such that $(M_{i+1}, \bar{f}_{i+1}, F_{i+1})$ is obtained from $(M_i, \bar{f}_i, F_i)$ by surgery on a class

$$\alpha_i \in \pi_t(\bar{f}_i), \qquad 2t \leqq n+2 \,.$$

We shall show by induction that $\bar{f}_i : M_i \to X$ lifts to $\tilde{V}^k(\psi)$.

A lifting f_0 of $\bar{f}_0$ is already given. Assume then that we have a commutative diagram

$$\begin{tikzcd} & \tilde{V}^k(\psi) \arrow[d, "\pi"] \\ M_i \arrow[ur, "f_i"] \arrow[r, "\bar{f}_i"'] & X \end{tikzcd}$$

Since the fibre of π is $(k-1)$-connected the induced map

$$\pi_t(f_i) \to \pi_t(\bar{f}_i)$$

is onto for $t \leqq k$. Therefore assuming

$$n+2 \leqq 2k+1 \quad \text{or} \quad n \leqq 2k-1$$

there is an element $\alpha_i' \in \pi_t(f_i)$ mapping to α_i. Since we can do surgery on α_i we can also do surgery on α_i' and as the result get a triple

$$(M_{i+1}, f_{i+1}, F_{i+1}) \quad \text{with} \quad \pi \circ f_{i+1} = \bar{f}_{i+1} .$$

From the induction we now have a lifting

$$f_q : X = M_q \to V^k(\psi)$$

of $\bar{f}_q$. On the other hand $\bar{f}_q$ is homotopic to the identity. Therefore f_q is homotopic to a section s and the claim follows.

9. Immersions in the metastable range.

Let M^n be a closed manifold, $n \geqq 5$, and let Q^{n+k} be any smooth manifold, $n \leqq 2k-1$. Furthermore let

$$f : M^n \to Q^{n+k}$$

be a continuous mapping. We know by Hirsch [2, Theorem 5.7] that f is homotopic to an immersion iff f can be covered by a bundle injection $\tau_M \to \tau_Q$. Clearly this is the case iff

$$\operatorname{g\,dim}(f^* \tau_Q - \varepsilon^k \oplus \tau_M) \leqq k .$$

Therefore we get

THEOREM 9.1. *With the notation above let*

$$\psi = f^* \tau_Q - \varepsilon^k \oplus \tau_M .$$

Then

$$f : M^n \to Q^{n+k}, \quad 5 \leqq n \leqq 2k-1 ,$$

is homotopic to an immersion iff

$$\gamma_k(\psi)([M^n]) = 0 \in \Omega_{n-k-1}(M \times P^\infty, -k\lambda^0 - \psi \otimes \lambda + \tau_M{}^0) .$$

10. The spectral sequence for a double covering.

Let $\pi : \tilde{X} \to X$ be a double covering, and let λ be the associated linebundle over X, that is $\tilde{X} = S\lambda$. Let φ be a virtual bundle over X, and let $t : \tilde{X} \to \tilde{X}$ be the involution. Then the commutative diagram

$$\begin{tikzcd} \tilde{X} \arrow[dd, "t"'] \arrow[dr, "\pi"] & \\ & X \\ \tilde{X} \arrow[ur, "\pi"'] & \end{tikzcd}$$

defines by "pull-backs" of φ an equivalence

$$\pi^*\varphi = t^*\pi^*\varphi \, .$$

Therefore we can define an involution T on $\Omega_n(\tilde{X}; \varphi)$ $\left(=\Omega_n(\tilde{X}; \pi^*\varphi)\right)$ by the composition

$$T : \Omega_n(\tilde{X}; \varphi) = \Omega_n(\tilde{X}; t^*\pi^*\varphi) \xrightarrow{t_*} \Omega_n(\tilde{X}; \varphi) \, .$$

The Gysin sequences for $k \geqq 0$

$$\ldots \to \Omega_n(\tilde{X}; \varphi) \xrightarrow{\pi_*} \Omega_n(X; -k\lambda^0 + \varphi) \xrightarrow{e(\lambda)}$$
$$\Omega_{n-1}(X; -(k+1)\lambda^0 + \varphi) \xrightarrow{w(\lambda)} \Omega_{n-1}(\tilde{X}; \varphi) \xrightarrow{\pi_*} \ldots .$$

fit together into an exact triangle described by the following diagram.

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \\
 & & \| & & \| & & \\
\ldots \to 0 & \to & \Omega_n^0 & \to & 0 & \to & \Omega_{n-1}^0 & \to & \ldots \\
 & & \| & & & & \| & & \\
\ldots \to \tilde{\Omega}_n & \to & \Omega_n^0 & \to & 0 & \to & \Omega_{n-1}^0 & \to & \ldots \\
 & & \downarrow & & & & \| & & \\
\ldots \to \tilde{\Omega}_{n-1} & \to & \Omega_{n-1}^1 & \to & \tilde{\Omega}_{n-1} & \to & \Omega_{n-1}^0 & \to & \ldots \\
 & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & & & \vdots & & \\
 & & \downarrow & & & & \downarrow & & \\
\ldots \to \tilde{\Omega}_{n-k} & \to & \Omega_{n-k}^k & \to & \tilde{\Omega}_{n-k} & \to & \Omega_{n-k}^{k-1} & \to & \ldots \\
 & & \downarrow & & & & \downarrow & & \\
\ldots \to \tilde{\Omega}_{n-k-1} & \to & \Omega_{n-k-1}^{k+1} & \to & \tilde{\Omega}_{n-k-1} & \to & \Omega_{n-k-1}^k & \to & \ldots \\
 & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & & & \vdots & &
\end{array}$$

Here we write $\tilde{\Omega}_n$ for $\Omega_n(\tilde{X}; \varphi)$ and Ω_n^k for $\Omega_n(X; -k\lambda^0 + \varphi)$.

With a suitable indexing we then get a spectral sequence $(E^r{}_{n,k}, d^r)$ with

$$d^r : E^r{}_{n,k} \to E^r{}_{n-1,k-r} \, .$$

We shall show

Theorem 10.1. *Let $\tilde{X} \to X$ be a double covering, and let φ be a virtual bundle. Then there is a filtration*

$$\Omega_n(X, \varphi) = F_n{}^n \supset F_n{}^{n-1} \supset \ldots \supset F_n{}^0 \supset F_n{}^{-1} = 0$$

and a spectral sequence $(E^r{}_{n,k}, d^r)$ with

$$d^r : E^r{}_{n,k} \to E^r{}_{n-1,k-r}$$

coverging to $E^\infty{}_{*,*}$.

Furthermore

$$E^\infty{}_{n,k} = F_n{}^k / F_n{}^{k-1}$$

and

$$E^2{}_{n,k} = H_k(\mathsf{Z}_2; \Omega_{n-k}(\tilde{X}; \varphi)) .$$

Here $\Omega_{n-k}(\tilde{X}; \varphi)$ has the Z_2-module structure defined by T.

Proof. We put

$$F_n{}^k = \ker\left(e(\lambda)^{k+1} : \Omega_n(X; \varphi) \to \Omega_{n-k-1}(X; -(k+1)\lambda^0 + \varphi)\right) .$$

Then the only thing left is to calculate E^2. To this end we must identify d^1, which is given by the composition

$$\begin{array}{ccccc}
E^1{}_{n,k} & & \xrightarrow{\quad\quad d^1 \quad\quad} & & E^1{}_{n-1,k-1} \\
\| & & & & \| \\
\Omega_{n-k}(\tilde{X}; \varphi) & \xrightarrow{\pi_*} & \Omega_{n-k}(X; -k\lambda^0 + \varphi) & \xrightarrow{w(\lambda)} & \Omega_{n-k}(\tilde{X}; \varphi)
\end{array}$$

By a straightforward check of definitions it follows that

$$d^1 = 1 + (-1)^k T ,$$

and then the statement follows.

We can now estimate the order of $\gamma_k(\psi)$.

Theorem 10.2. *Let φ and ψ be virtual bundles over a space X. Let*

$$x \in \Omega_n(X; \varphi)$$

be an element with

$$\gamma_{k+1}(\psi)(x) = 0 \in \Omega_{n-k-2}(X \times P^\infty; -(k+1)\lambda^0 - \psi \otimes \lambda + \varphi) .$$

Then for $i<k$

$$2^{i+1}\gamma_{k-i}(\psi)(x) = 0 \in \Omega_{n-k+i-1}(X\times P^\infty; -(k-i)\lambda^0-\psi\otimes\lambda+\varphi) .$$

PROOF. We first observe that

$$\gamma_{i+1}(\psi) = e(\lambda)\circ\gamma_i(\psi) .$$

Now let $\sigma_i=\gamma_i(\psi)(x)$. Then

$$\sigma_i = e(\lambda)^i(\sigma_0), \quad \sigma_0 \in \Omega_{n-1}(X\times P^\infty; -\psi\otimes\lambda+\varphi) .$$

We consider the spectral sequence above for the double covering

$$X\times S^\infty \to X\times P^\infty$$

and the virtual bundle $-\psi\otimes\lambda+\varphi$. From the definition of the filtration we see that $e(\lambda)^p$ induces an injection

$$e(\lambda)^p : F^p_{n-1}/F^{p-1}_{n-1} \to \Omega_{n-p-1}(X\times P^\infty; -p\lambda_0-\psi\otimes\lambda+\varphi) .$$

From the assumption $\sigma_{k+1}=0$ we get that $\sigma_0\in F^k_{n-1}$.

Since $E^2{}_{n-1,p}=H_p(\mathsf{Z}_2, \Omega_{n-p-1}(X; -\psi+\varphi))$ is a vector space over Z_2 for $p>0$ also

$$F^p_{n-1}/F^{p-1}_{n-1} = E^\infty{}_{n-1,p}$$

is a vector space over Z_2 for $p>0$. Therefore

$$2^i\sigma_0 \in F^{k-i}_{n-1} \quad \text{for} \quad i\leqq k .$$

Then we get for $i<k$

$$2^{i+1}\sigma_{k-i} = e(\lambda)^{k-i}(2^{i+1}\sigma_0) = 0 .$$

Together with Theorem 9.1 this theorem proves the following conjecture of Gitler: In the metastable range the odd obstructions for immersions all vanish.

11. Immersions of Grassmannian manifolds.

We denote the real Grassmannian manifold of n-planes in R^{n+m} by $G_{n,m}$. In this section let τ denote the tangentbundle of $G_{n,m}$. Over $G_{n,m}$ we have the canonical bundles ξ^n and η^m, whose fibres at $p\in G_{n,m}$ are p and its orthogonal complement respectively. We have canonical isomorphisms

$$\xi^n\oplus\eta^m \cong \varepsilon^{n+m}$$
$$\tau \cong \xi^n\otimes\eta^m .$$

Let $\mathfrak{N}_*$ denote the unoriented bordism ring. Let ξ^k denote the universal bundle over BO_k. The Thom construction defines an isomorphism

$$\Omega_n(BO_k; -(\xi^k)^0) \cong \pi^S{}_{n+k}(MO_k)$$

where $\pi^S{}_*$ denotes stable homotopy.

R. Brown showed in [1] that the canonical mapping for $N > n$

$$\pi^S{}_{2n-\alpha(n)}(MO_{n-\alpha(n)}) \to \pi^S{}_{n+N}(MO_N) = \mathfrak{N}_n$$

is an epimorphism, where $\alpha(n)$ is the number of 1's in the dyadic expansion of n. Therefore we conclude that the canonical mapping

$$BO_{n-\alpha(n)} \to BO_N, \qquad N > n$$

induces an epimorphism

$$\Omega_n(BO_{n-\alpha(n)}; -(\xi^{n-\alpha(n)})^0) \to \Omega_n(BO_N; -(\xi^N)^0)\,.$$

If we take N sufficiently large the mapping $\tilde{V}^k((\xi^N)^0) \to BO_k$ classifying μ^k can be made into an arbitrarily high equivalence. So for N large the mapping

$$\pi_* : \Omega_n\big(\tilde{V}^{n-\alpha(n)}((\xi^N)^0); -(\xi^N)^0\big) \to \Omega_n(BO_N; -(\xi^N)^0)$$

is an epimorphism. Therefore

$$\gamma_{n-\alpha(n)}((\xi^N)^0) : \Omega_n(BO_N; -(\xi^N)^0) \to \Omega_{\alpha(n)-1}(BO_N \times P^\infty; -(n-\alpha(n))\lambda^0 - (\xi^N)^0\otimes\lambda - (\xi^N)^0)$$

is 0. Let $\nu: G_{n,m} \to BO_N$ classify the stable normal bundle of $G_{n,m}$. Then $\nu^*(-(\xi^N)^0)$ is equivalent to τ^0. Therefore there is an induced mapping

$$\nu_* : \Omega_{nm}(G_{n,m}; \tau^0) \to \Omega_{nm}(BO_N; -(\xi^N)^0)$$

and a commutative diagram

$$\begin{array}{ccc}
\Omega_{nm}(G_{n,m}; \tau^0) & \xrightarrow{\gamma_{nm-\alpha(nm)}(-\tau^0)} & \Omega_{\alpha(nm)-1}\big(G_{n,m} \times P^\infty; -(nm-\alpha(nm))\lambda^0 + \tau^0\otimes\lambda + \tau^0\big) \\
\downarrow{\scriptstyle \nu_*} & & \downarrow{\scriptstyle (\nu\times 1)_*} \\
\Omega_{nm}(BO_N; -(\xi^N)^0) & \xrightarrow{0} & \Omega_{\alpha(nm)-1}\big(BO_N \times P^\infty; -(nm-\alpha(nm))\lambda^0 - (\xi^N)^0\otimes\lambda - (\xi^N)^0\big)
\end{array}$$

We shall use this diagram to show

THEOREM 1. *Let $n < m$, $\alpha(nm) \leqq n$, $n \equiv m+1$ (mod 2). Then there exists an immersion*

$$G_{n,m} \subseteq \mathbb{R}^{2nm-\alpha(nm)}\,.$$

PROOF. When $n < m$ and $n \equiv m+1$ (mod 2) a calculation of the normal Stiefel-Whitney classes of $G_{n,m}$

$$\overline{W} = W(\xi^n \otimes \eta^m)^{-1}$$

shows that

$$\nu_* : H_i(G_{n,m};\, Z_2) \to H_i(BO_N;\, Z_2)$$

is an isomorphism for $i \leqq n$. We now appeal to

LEMMA 2. *Let $f: X \to Y$ be a mapping of* CW*-complexes with only a finite number of cells in each dimension. Assume that*

$$f_* : H_i(X;\, Z_2) \to H_i(Y;\, Z_2)$$

is an isomorphism for $i < n$ and an epimorphism for $i = n$.

Then for any virtual bundle φ over Y

$$f_* : \Omega_i(X;\, f^*\varphi) \to \Omega_i(Y;\, \varphi)$$

is an isomorphism modulo odd torsion for $i < n$ and an epimorphism modulo odd torsion for $i = n$.

The lemma implies that the kernel of $(\nu \times 1)^*$ consists of odd torsion in dimensions $\leqq n-1$. Therefore when $\alpha(nm) \leqq n$ we get from the diagram that

$$\gamma_{nm-\alpha(nm)}(-\tau^0)[G_{n,m}]$$

has odd order.

On the other hand Theorem 10.2 tells us that the order is a power of 2. So

$$\gamma_{nm-\alpha(nm)}(-\tau^0)[G_{n,m}] = 0\ .$$

From Theorem 9.1 we then get that an immersion

$$G_{n,m} \subseteq R^{2nm-\alpha(nm)}$$

exist provided $5 \leqq nm \leqq 2(nm-\alpha(nm))-1$. This condition is satisfied except for $n=1$ and $m=2$ or 4. But in these cases the result is well-known from the Whitney immersion theorem.

PROOF OF LEMMA 2. We may assume that X and Y are finite cell-complexes since otherwise we could restrict f to the $(n+1)$-skeletons to get that situation. Now φ is equivalent to a virtual bundle of the form $-\xi^0$ for some vector bundle ξ^N over Y. By the Thom construction we then get isomorphisms

$$\Omega_i(X;\, -f^*\xi^0) \cong \pi^S{}_{N+i}(Tf^*\xi^N)$$

and

$$\Omega_i(Y;\, -\xi^0) \cong \pi^S{}_{N+i}(T\xi^N)\,,$$

with T denoting the Thom complex.

Since f_* is an n-equivalence on homology with Z_2-coefficients we get from the Thom isomorphism that

$$(Tf)_* : H_*(Tf^*\xi; Z_2) \to H_*(T\xi; Z_2)$$

is an $(N+n)$-equivalence. Then from the Whitehead theorem modulo odd torsion we get that

$$(Tf)_* : \pi^S{}_*(Tf^*\xi) \to \pi^S{}_*(T\xi)$$

is an $(N+n)$-equivalence modulo odd torsion. But this is the same as

$$f_* : \Omega_*(X; -f^*\xi^0) \to \Omega_*(Y; -\xi^0)$$

being an n-equivalence modulo odd torsion as claimed.

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MATEMATISK INSTITUT, AARHUS UNIVERSITET, DANMARK