A POSSIBLE CHARACTERIZATION OF GENERIC STRUCTURES

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Recently A. Robinson has introduced into model theory two kinds of forcing, which he calls finite forcing and infinite forcing. Details of these forcing notions can be found in [3], [2] for finite forcing and [4], [5] for infinite forcing. Several people have noticed that infinite forcing can be "explained" using standard model theoretic techniques (see for instance [1]). In this note I make several remarks which may eventually help to similarly explain finite forcing.

1. The main result.

Let $L$ be any first order language and let $T$ be any $L$-theory. Let $M$ be the class of $L$-structures which are substructures of models of $T$ (so that $M$ is elementary being the class of models of the universal part of $T$). Finite forcing is used to construct a subclass $F \subseteq M$ called the class of $T$-generic structures. We will consider how $F$ can be described without using forcing.

Remember that two theories $T_1, T_2$ are mutually model consistent if each model of the one is embeddable in a model of the other, equivalently if $T_1, T_2$ have the same universal part. Remember also that a model $A$ of a theory $T'$ is a completing model if for each model $B$ of $T'$,

$$A \subseteq B \Rightarrow A < B.$$

We can now state our main theorem.

**Theorem 1.** For any $L$-theory $T$ there is at most one class $F$ of $L$-structures such that

1. $T$ and $\text{Th}(F)$ are mutually model consistent,
2. $F$ is the class of completing models of $\text{Th}(F)$.

If such a class $F$ exists then $F$ is the class of $T$-generic structures and $\text{Th}(F) = T'$.

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To prove theorem 1 we need some notation.

For each integer \( n \geq 0 \) let \( \mathcal{V}_n \) be the set of formulas in prenex normal form whose prenex consists of \( n \) blocks of quantifiers, the first being universal, the second being existential, the third being universal, etc. For any two structures \( \mathcal{A}, \mathcal{B} \) let \( \mathcal{A} \prec_n \mathcal{B} \) mean that \( \mathcal{A} \subseteq \mathcal{B} \) and for each formula \( \varphi \in \mathcal{V}_n \) and \( \mathcal{A} \)-assignment \( a \),

\[
\mathcal{A} \models \varphi[a] \Rightarrow \mathcal{B} \models \varphi[a].
\]

From now on we suppose that \( \mathcal{F} \) satisfies (F1, 2) and we put \( T^* = Th(\mathcal{F}) \).

For each integer \( n \geq 0 \) let \( \mathcal{F}_n \) be the subclass of \( \mathcal{M} \) given by

\[
\mathcal{A} \in \mathcal{F}_n \iff (\forall \mathcal{B} \models T^*) [\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A} \prec_n \mathcal{B}]
\]

and let \( T_n = Th(\mathcal{F}_n) \). We see that \( \mathcal{F}_0 = \mathcal{M} \) and we have a descending chain

\[
\mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \ldots \supseteq \mathcal{F}.
\]

(The inclusion \( \mathcal{F} \subseteq \mathcal{F}_n \) follows from (F2)).

First we prove some simple facts about this chain.

**Lemma 2.** For each \( n \geq 0 \), \( T^* \cap \mathcal{V}_{n+1} \subseteq T_n \).

**Proof.** Consider any sentence \( \sigma \in T^* \cap \mathcal{V}_{n+1} \), and any structure \( \mathcal{A} \in \mathcal{F}_n \).

We show that \( \mathcal{A} \models \sigma \).

Now \( \mathcal{F}_n \subseteq \mathcal{M} \), and so (F1) gives us \( \mathcal{A} \subseteq \mathcal{B} \) for some model \( \mathcal{B} \) of \( T^* \).

In particular \( \mathcal{B} \models \sigma \). But, from the definition of \( \mathcal{F}_n \), \( \mathcal{A} \prec_n \mathcal{B} \), and so \( \mathcal{A} \models \sigma \), as required.

**Corollary 3.** For each model \( \mathcal{B} \) of \( T_n \) there is some model \( \mathcal{C} \) of \( T^* \) such that \( \mathcal{B} \prec_n \mathcal{C} \).

**Theorem 4.** For each \( n \geq 0 \) the following are equivalent.

(i) \( \mathcal{A} \in \mathcal{F}_{n+1} \).

(ii) There is some model \( \mathcal{B} \) of \( T_n \) such that \( \mathcal{A} \subseteq \mathcal{B} \), and for each model \( \mathcal{B} \) of \( T_n \),

\[
\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A} \prec_{n+1} \mathcal{B}.
\]

**Proof.** (i) \( \Rightarrow \) (ii). Suppose \( \mathcal{A} \in \mathcal{F}_{n+1} \). The existence of \( \mathcal{B} \) such that \( \mathcal{A} \subseteq \mathcal{B} \models T_n \) follows from (F1) (or corollary 3). Also, for any such \( \mathcal{B} \), corollary 3 shows that \( \mathcal{B} \prec_n \mathcal{C} \) for some model \( \mathcal{C} \) of \( T^* \). But \( \mathcal{A} \in \mathcal{F}_{n+1} \) and so \( \mathcal{A} \prec_{n+1} \mathcal{C} \). This gives \( \mathcal{A} \prec_{n+1} \mathcal{B} \), as required.

(ii) \( \Rightarrow \) (i) follows immediately from the definition of \( \mathcal{F}_{n+1} \) since \( T_n \subseteq T^* \).
Theorem 5. $\mathcal{F} = \bigcap_{n<\omega} \mathcal{F}_n$.

Proof. We have already noted that
\[ \mathcal{F} \subseteq \bigcap_{n<\omega} \mathcal{F}_n \]
so it is sufficient to show the reverse inclusion.

Suppose $\mathcal{A} \subseteq \mathcal{F}_n$ for all $n \geq 0$, we must show that $\mathcal{A}$ is a completing model of $T^*$. Consider any $\mathcal{A} \subseteq \mathcal{B} \models T^*$. Since $\mathcal{A} \subseteq \mathcal{F}_n$ we have $\mathcal{A} <_n \mathcal{B}$, and this holds for all $n \geq 0$, hence $\mathcal{A} < \mathcal{B}$, as required.

Proof of Theorem 1. Suppose such a class $\mathcal{F}$ exists, and consider the hierarchy $(h)$.

We have $\mathcal{F}_0 = \mathcal{M}$ and so $\mathcal{F}_0$ is uniquely determined. Moreover theorem 4 shows that each $\mathcal{F}_{n+1}$ is uniquely determined in terms of $\mathcal{F}_n$, and so each $\mathcal{F}_n$ is uniquely determined. Finally theorem 5 shows that $\mathcal{F}$ is uniquely determined.

We must now show that $\mathcal{F}$ is the class of $T$-generic structures and $T^* = T'$.

First from $(F_2)$ and [2, theorem 4.9] we see that $T^*$ is forcing complete, i.e.
\[ T^* = T^*'. \]

Also from [2, theorem 2.19] we have
\[ T^* = (T^* \cap \mathcal{V}_1)', \quad T' = (T \cap \mathcal{V}_1)'. \]

However $(F_1)$ shows that
\[ T^* \cap \mathcal{V}_1 = T \cap \mathcal{V}_1 \]
so that
\[ T^* = T'. \]

Finally $(F_2)$ and [2, theorem 3.4] show that $\mathcal{F}$ is the class of $T$-generic structures.

This completes the proof of theorem 1.

2. Further remarks.

Some properties of $T$-generic structures can be derived from theorem 1 and the hierarchy $(h)$. For instance we will prove the following theorem, (c.f. [2, theorem 3.7]).

Theorem 6. For any two structures $\mathcal{A}, \mathcal{B}$,
\[ \mathcal{A} <_1 \mathcal{B} \in \mathcal{F} \Rightarrow \mathcal{A} \in \mathcal{F}. \]
This theorem follows from the following two lemmas.

**Lemma 7.** For each integer $n \geq 0$, and any two structures $\mathcal{A}, \mathcal{B}$,

$$\mathcal{A} \prec_{n+1} \mathcal{B} \in \mathcal{F} \Rightarrow \mathcal{A} \prec_{n+2} \mathcal{B}. $$

**Proof.** Suppose that $\mathcal{A} \prec_{n+1} \mathcal{B} \in \mathcal{F}$, so that

(*)

$$\mathcal{A} \prec \mathcal{C}, \quad \mathcal{B} \prec_{n} \mathcal{C}$$

for some suitable $\mathcal{C}$. In particular we have

$$\mathcal{C} \equiv \mathcal{A} \vdash T^* \cap \forall_{n+1}$$

so that $\mathcal{C} \prec_{n} \mathcal{D}$ for some model $\mathcal{D}$ of $T^*$. But $\mathcal{B} \in \mathcal{F}$ and so

$$\mathcal{B} \prec \mathcal{D}, \quad \mathcal{C} \prec_{n} \mathcal{D}$$

which gives $\mathcal{B} \prec_{n+1} \mathcal{C}$. Thus, from (*), we get $\mathcal{A} \prec_{n+2} \mathcal{B}$, as required.

**Lemma 8.** For any two structures $\mathcal{A}, \mathcal{B}$,

$$\mathcal{A} \prec \mathcal{B} \in \mathcal{F} \Rightarrow \mathcal{A} \in \mathcal{F}.$$  

**Proof.** Suppose that $\mathcal{A} \prec \mathcal{B} \in \mathcal{F}$ and $\mathcal{A} \equiv \mathcal{C} \vdash T^*$. Thus we have a commuting diagram

$$\begin{array}{ccc}
\mathcal{A} & < & \mathcal{B} \\
\| & \| & \| \\
\mathcal{C} & \overset{f}{\rightarrow} & \mathcal{D}
\end{array}$$

where $f$ is an elementary embedding. In particular $\mathcal{D} \vdash T^*$ and so (since $\mathcal{B} \in \mathcal{F}$), $\mathcal{B} \prec \mathcal{D}$. This gives $\mathcal{A} \prec \mathcal{C}$, as required.

3. Open problems.

Theorem 1 says nothing about the existence of class $\mathcal{F}$. However it is known that for countable $L$ the class of $T$-generic structures exists and satisfies (F1, 2), see [2, theorems 3.3, 3.9, 3.4, and 4.1]. Thus for countable $L$ we have both existence and uniqueness. It has been noticed by Shelah, [6], and independently by Macintyre that for uncountable $L$ there are theories $T$ for which no $T$-generic structures exist. For these theories no class $\mathcal{F}$ exists.

Thus we have the following problem.

(A) Under what conditions does the class $\mathcal{F}$ exist?
Theorem 1 says the class $\mathcal{F}$ (if it exists) is the class of $T$-generic structures, however the converse of this is not known. Thus we can ask the following.

(B) Under what conditions does the class of $T$-generic structures give us a class $\mathcal{F}$.

There are many open problems concerning the behaviour of the heirachy $(h)$, (even for countable $L$). For instance we have the following.

(C) Under what conditions is $(h)$ finite?

(D) What are the possible patterns of equality between member of $(h)$?

REFERENCES


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