MATH. SCAND. 31 (1972), 237–247

ASYMPTOTIC FORMULAE FOR THE COEFFICIENTS OF A CLASS OF MODULAR FORMS

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1.

Let $\Gamma(1)$ denote the full modular group, i.e., the group of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a,b \mid c,d) \qquad ad-bc=1 ,$$

where a, b, c, d are integers, and $\Gamma_0(n)$ denote the subgroup defined by $c \equiv 0 \pmod{n}$. A multiplier system $v=v(\Gamma,-k)$ of dimension $-k$, k integral, for a group $\Gamma \subset \Gamma(1)$ is a character of the group Γ. A modular form on Γ of dimension $-k$ and multiplier system $v=v(\Gamma,-k)$ is a function $F(\tau)$, meromorphic in the fundamental domain $\Delta(\Gamma)$ of Γ, which satisfies

$$F(M\tau) = v(M)(c_M\tau+d_M)^k F(\tau)$$

for all $M=(\cdot,\cdot \mid c_M,d_M) \in \Gamma$ (see Lehner [6, ch. 8]). The set $(\Gamma,-k,v)$ of all such modular forms is a vector space over the complex number field. We denote by $C^0(\Gamma,-k,v)$ the subspace consisting of all cuspforms, i.e., modular forms which are regular in $\Delta(\Gamma)$ and zero at the cusps of $\Delta(\Gamma)$. In particular, if $F(\tau) \in (\Gamma,0,1)$, then $F(\tau)$ is called a modular function on Γ (see Ford [2, ch. 7]).

We will first determine a class of modular forms

$$f(\tau) \in (\Gamma_0(p^{k_0}),-k,v), \qquad k>0 ,$$

whose Fourier coefficients, $a(n)$, at the cusp $\tau=i\infty$, can be expressed as

$$a(n) = \varrho(n)+R(n)$$

where $\varrho(n)$ is given in terms of $\sigma_k(n)$ or $v_{k,p}(n)$, $u_{k,p}(n)$ (see (2,1)–(2,3)) and $R(n)$ is a remainder term estimated by $R(n)=O(n^{\frac{1}{2}k})$.

This is a generalization of the results given in Dirdal [1], where the author studied a special class of modular forms on $\Gamma_0(3)$.

2.

Let

(2,1) $$\sigma_k(n) = \sum\nolimits_{d\mid n} d^k ,$$

Received April 30, 1972.

$$(2,2) \qquad v_{k,p}(n) = \sum_{d|n} \omega_p(d) d^{k-1} ,$$

$$(2,3) \qquad u_{k,p}(n) = \sum_{d|n} \omega_p(n/d) d^{k-1} ,$$

$$A_{k,p} = i^k s_p (k-1)!\, p^k (2\pi)^{-k} \sum_{n=1}^{\infty} \omega_p(n) n^{-k} ,$$

where

$$\begin{aligned} \omega_p(m) &= (m/p) && \text{if } p \text{ odd} , \\ &= \chi(m) && \text{if } p=2 \end{aligned}$$

$$\begin{aligned} s_p &= i^{-k^2} p^{-\frac{1}{2}} && \text{if } p \text{ odd} , \\ &= -i 2^{k-1} && \text{if } p=2 . \end{aligned}$$

Here and in the following χ denotes the real character (mod 4), p a prime and $(\cdot/p)$ the Legendre symbol. Further B_k denote the Bernoulli numbers

$$B_0 = 1,\ B_1 = -\tfrac{1}{2},\ B_2 = \tfrac{1}{6},\ B_3 = 0,\ B_4 = -\tfrac{1}{30}, \ldots .$$

$B_k = 0$ when k is odd and > 1.

By M we will always denote the matrix $M=(a,b \mid c,d)$, where a, b, c, d are inteters and $ad-bc=1$, and by T we mean the matrix

$$(2,4) \quad \begin{aligned} T &= T_{t_1,t_2,m} \\ &= \left(\alpha, -(\alpha(t_1+p^{k_0-m-1}t_2)+1)/p^{k_0-m} \mid p^{k_0-m}, -(t_1+p^{k_0-m-1}t_2)\right) , \end{aligned}$$

where $k_0 \geqq 2$, $1 \leqq m \leqq k_0-1$, $(t_1+p^{k_0-m-1}t_2, p)=1$ and α has a value such that $1 \leqq \alpha < p^{k_0-m}$ and $T \in \Gamma_0(p^{k_0-m})$. Here $(\cdot,\cdot)$ denotes the greatest common divisor.

By x we denote $x=\exp(2\pi i\tau)=e(\tau)$ with $\operatorname{Im}\tau > 0$. We use $[a]$ to denote the integral part of a and further we let $\varepsilon = 0, \pm 1$. Let $f(\tau)$ be a function such that

$$f(\tau) = \sum_{n=0}^{\infty} a(n) x^n ,$$

$$f(-(p^{k_0}\tau)^{-1}) = i^{k2} p^{(k_0-1)k} \tau^k \sum_{n=0}^{\infty} a^*(n) x^n ,$$

and

$$f(T\tau) = v(T)(c_T\tau + d_T)^k f(\tau), \quad c_T = p^{k_0-m}, d_T = -(t_1+p^{k_0-m-1}t_2) ,$$

where $v(T)$ is a constant depending on the matrix T. Then we put

$$\mu_{k,p,\varepsilon} = \sum_{((r/p))=\varepsilon p} \{a^*(0) + a(0) i^{k^2} \sum_{l=1}^{p-1} e(-rl/p) v(T_{l,0,k_0-1})^{-1}\};$$

$$\mu^{(1)}{}_{m,s,p,\varepsilon} = a(0) \sum_{((r/p))=\varepsilon p} \sum_{l=0}^{p-1} e(-rl/p) v(T_{s,l,m})^{-1} ,$$

for $(s,p)=1$, $s = \pm 1, \ldots, \pm \frac{1}{2}(p^{k_0-m} - 1 + [2/p])$;

$$\begin{aligned} \mu^{(2)}{}_{s,\beta} = a(0) \sum_{l=1}^{p-1} v(T_{l,0,k_0-1})^{-1} \{ & \sum_{(r/p)=\varepsilon} \beta \sum_{(r/p)=\varepsilon} e(rp^{-1}(s-l)) \\ & - \sum_{(r/p)=\varepsilon} \beta e(rsp^{-1}) \sum_{(r/p)=\varepsilon} e(-rlp^{-1}) \} , \end{aligned}$$

for $(s,p)=1,\ s=\pm 1,\dots,\ \pm\frac{1}{2}(p-1),\ p$ odd;

$$\delta = \sum_{l=0}^{1} e(-\tfrac{1}{2}rl)v(T_{1,l,k_0-2})^{-1};$$

$$\delta^{(1)} = a(0)\sum_{l=0}^{1} e(-\tfrac{1}{2}l)\{v(T_{-1,l,k_0-2})^{-1}+v(T_{1,l,k_0-2})^{-1}\},$$

where β is an arithmetical function of r and

$$\begin{aligned}((r/p)) &= (r/p) \quad \text{if } p \text{ is odd},\\ &= r \qquad\quad \text{if } p=2.\end{aligned}$$

Further, if $f(\tau)$ is regular for $\operatorname{Im}\tau>0$ and $f(\tau)\in(\Gamma_0(p^{k_0}),-k,\nu)$, we shall prove the following theorems:

THEOREM 1. *Let k be even >2 and $\nu(M)=1$, when $M\in\Gamma_0(p^{k_0})$. If $k_0=1$ then*

$$\begin{aligned}&(p^k-1)(2k)^{-1}B_k a(n)\\ &\quad = (a^*(0)-a(0)p^k)\sigma_{k-1}(n/p)+(a(0)-a^*(0))\sigma_{k-1}(n)+O(n^{\frac{1}{2}k}).\end{aligned}$$

If $k_0\geqq 2$, $\mu_{k,p,\varepsilon}$ real and ${\mu^{(1)}}_{m,s,p,\varepsilon}=0$, for $k_0>2$, $1\leqq m\leqq k_0-2$, then

$$\begin{aligned}(p^k-1)(2k)^{-1}B_k a(pn) &= a(0)\{\sigma_{k-1}(np^{-(k_0-2)})-p^k\sigma_{k-1}(np^{-(k_0-1)})\}+\\ &\quad +p^{k-1}\mu_{k,p,0}\{\sigma_{k-1}(np^{-1})-\sigma_{k-1}(n)\}+O(n^{\frac{1}{2}k}).\end{aligned}$$

Further, if ${\mu^{(2)}}_{s,1}=0$, then

$$\tfrac{1}{2}(p-1+[2/p])(p^k-1)(2k)^{-1}B_k a(pn+r) = -\mu_{k,p,\varepsilon}\sigma_{k-1}(pn+r)+O(n^{\frac{1}{2}k})$$

for $\varepsilon=((r/p))$ and $r=1,\dots,p-1$.

THEOREM 2. *Let p be odd, $k\equiv\frac{1}{2}(p-1)\ (\mathrm{mod}\,2)$, $k>1$* and *$\nu(M)=(d/p)$ when $M\in\Gamma_0(p^{k_0})$.*

If $k_0=1$, then

$$A_{k,p}a(n) = a(0)v_{k,p}(n)+(-1)^k p^{-\frac{1}{2}}a^*(0)u_{k,p}(n)+O(n^{\frac{1}{2}k}).$$

If $k_0\geqq 2$, $\mu_{k,p,\varepsilon}$ real and ${\mu^{(1)}}_{m,s,p,\varepsilon}=0$ for $k_0>2$, $1\leqq m\leqq k_0-2$, then

$$A_{k,p}a(pn) = a(0)v_{k,p}(np^{-(k_0-2)})+(-1)^k p^{k-3/2}\mu_{k,p,0}u_{k,p}(n)+O(n^{\frac{1}{2}k}).$$

Further, if ${\mu^{(2)}}_{s,(r/p)}=0$, then

$$\tfrac{1}{2}(p-1)(r/p)A_{k,p}a(pn+r) = (-1)^k p^{-\frac{1}{2}}\mu_{k,p,\varepsilon}v_{k,p}(pn+r)+O(n^{\frac{1}{2}k}),$$

for $\varepsilon=(r/p)$ and $r=1,\dots,p-1$.

THEOREM 3. *Let k be odd >1, $k_0 \geqq 3$ and $v(M)=\chi(d)$ when $M \in \Gamma_0(2^{k_0})$. If δ and $\mu_{k,2,\varepsilon}$ are real, $\mu^{(1)}{}_{m,s,2,\varepsilon}=0$ for $k_0>2+\varepsilon^2$, $1 \leqq m \leqq k_0-(2+\varepsilon^2)$, then*

$$A_{k,2}a(2n) = a(0)v_{k,2}(n2^{-(k_0-3)})-2^{k-1}\mu_{k,2,0}u_{k,2}(2n)+O(n^{\frac{1}{2}k}) .$$

Further, if $\delta^{(1)}=0$, then

$$A_{k,2}a(2n+1) = -\tfrac{1}{2}a(0)\delta v_{k,2}(2n+1)-2^{k-1}\mu_{k,2,1}u_{k,2}(2n+1)+O(n^{\frac{1}{2}k}) .$$

Similarly results can be optained for $a^*(n)$. (For an example see section 5).

Asymptotic formulae for $a(n)$ can easily be derived from these theorems. For example, if we put

$$\varrho(n) = \left(a^*(0)-a(0)p^k\right)\sigma_{k-1}(n/p)+\left(a(0)-a^*(0)\right)\sigma_{k-1}(n) ,$$

and $a(0) \neq a^*(0)$, then $\varrho(n)>C_1n^{k-1}$ since $\sigma_{k-1}(n)>C_2n^{k-1}$, where C_j, $j=1,2$ are constants. Hence putting $k_0=1$ in Theorem 1, we have

$$(p^k-1)(2k)^{-1}B_k a(n) = \varrho(n)\{1+O(n^{-\frac{1}{2}k+1})\} .$$

This formula is also valid when $a(0)=a^*(0) \neq 0$ if $p \mid n$.

3.

We define a linear operator $L_{p,r}$ acting on any power series

$$g(x) = \textstyle\sum_{n \geqq N} b(n)x^n$$

by

$$L_{p,r}g(x) = \textstyle\sum_{pn+r \geqq N} b(pn+r)x^{pn+r} .$$

Noting that

$$\begin{aligned} \textstyle\sum_{l=0}^{p-1} e\left(l(n-r)/p\right) &= p \quad \text{if } n \equiv r \pmod{p} \\ &= 0 \quad \text{if } n \not\equiv r \pmod{p} \end{aligned}$$

we obtain

LEMMA 1. *If $h(\tau)$ has a Laurent expansion in x, then*

$$L_{p,r}h(\tau) = p^{-1}\textstyle\sum_{l=0}^{p-1} e(-lrp^{-1})h(\tau+lp^{-1}) .$$

In particular we observe $L_{p,r_1}=L_{p,r_2}$ if $r_1 \equiv r_2 \pmod{p}$.

Further we shall need the following lemma

LEMMA 2. *If $h(\tau)$ has a Laurent expansion in x and*

$$h(M\tau) = v(M)(c\tau+d)^k h(\tau), \quad M \in \Gamma_0(p^{k_0}),\ k_0 \geqq 2$$

then

$$\textstyle\sum_{((r/p))=\varepsilon p} L_{p,r} h(M\tau) \;=\; v(M)(c\tau+d)^k \sum_{((r/p))=\varepsilon p} L_{p,r} h(\tau)$$

if v has the property $v((a,b \mid c,d))=v((a',b' \mid c',d'))$ when $d\equiv d'$ $(\mathrm{mod}\, p^{k_0-1})$, $(a',b' \mid c',d') \in \Gamma_0(p^{k_0})$.

Proof. We note that

$$\frac{(ap\tau+pb)(c\tau+d)^{-1}+l}{p} = \frac{A(p\tau+m)p^{-1}+B}{C(p\tau+m)p^{-1}+D},$$

where $A=a+lcp^{-1}$, $B=\{pb+ld-m(a+lcp^{-1})\}/p$, $C=c$, $D=d-mcp^{-1}$. Here $AD-BC=1$, and A, C and D are integers.

Since $(p,a)=1$ we can solve the congruence

$$am \equiv ld\,(\mathrm{mod}\, p), \quad 0 \leqq l \leqq p-1\ ,$$

that is, for each l we get a unique m, $0\leqq m \leqq p-1$. With this value of m, B also becomes an integer thus

$$(A,B \mid C,D) \in \Gamma_0(p^{k_0})\ .$$

Observing that $D\equiv d$ $(\mathrm{mod}\, p^{k_0-1})$ we have

$$h(M\tau+lp^{-1}) \;=\; v(M)(c\tau+d)^k h(\tau+mp^{-1})\ .$$

From this, Lemma 1 and the fact that $-lr\equiv -ma^2r$ $(\mathrm{mod}\, p)$ we obtain

$$L_{p,r}h(M\tau) \;=\; v(M)(c\tau+d)^k L_{p,a^2r}h(\tau)\ ,$$

and the lemma follows immediately.

Put

$$\begin{aligned} F_k(\tau) &= -(2k)^{-1}B_k + \textstyle\sum_{n=1}^{\infty} \sigma_{k-1}(n)x^n\ ,\\ V_{k,p}(\tau) &= A_{k,p} + \textstyle\sum_{n=1}^{\infty} v_{k,p}(n)x^n\ ,\\ U_{k,p}(\tau) &= \textstyle\sum_{n=1}^{\infty} u_{k,p}(n)x^n\ . \end{aligned}$$

From Gunning [3] and Kolberg [5] we obtain

$$F_k(\tau) \in (\Gamma(1), -k, 1), \qquad k \text{ even } > 2\ .$$

When p is odd and $k\equiv \frac{1}{2}(p-1)$ $(\mathrm{mod}\, 2)$, $k>1$, we have

$$\begin{aligned} &V_{k,p}(\tau) \in (\Gamma_0(p), -k, \nu_p)\ ,\\ &U_{k,p}(\tau) \in (\Gamma_0(p), -k, \nu_p)\ ,\\ &V_{k,p}(-(p\tau)^{-1}) \;=\; i^{k(k+2)}p^{k-\frac{1}{2}}\tau^k U_{k,p}(\tau)\ ,\\ &U_{k,p}(-(p\tau)^{-1}) \;=\; i^{k(k+2)}p^{\frac{1}{2}}\tau^k V_{k,p}(\tau)\ , \end{aligned}$$

where $\nu_p(M)=(d/p)$.

When k is odd >1;

$$
\begin{aligned}
(3.1) \qquad & V_{k,2}(\tau) \in (\Gamma_0(4), -k, \nu_2)\,, \\
& U_{k,2}(\tau) \in (\Gamma_0(4), -k, \nu_2)\,, \\
& V_{k,2}(-(4\tau)^{-1}) = -i2^{2k-1}\tau^k U_{k,2}(\tau)\,, \\
& U_{k,2}(-(4\tau)^{-1}) = -i2\tau^k V_{k,2}(\tau)\,,
\end{aligned}
$$

where $\nu_2(M)=\chi(d)$.

Finally we shall need the following result

Lemma 3. *For k odd >1 we have*

$$
\begin{aligned}
V_{k,2}(M\tau) &= \chi(d)(c\tau+d)^k\big(V_{k,2}(\tfrac{1}{2}\tau)-V_{k,2}(\tau)\big)\,, \\
U_{k,2}(M\tau) &= \chi(d)(c\tau+d)^k\big(2^{1-k}U_{k,2}(\tfrac{1}{2}\tau)-U_{k,2}(\tau)\big)\,,
\end{aligned}
$$

when $M \in \Gamma_0(2)$, $4 \nmid c$.

Proof. We shall prove only the first part of this lemma, the proof of the second part is quite similar. It is easily seen that

$$
\begin{aligned}
(3.2) \qquad L_2 V_{k,2}(\tau) &= V_{k,2}(2\tau)\,, \\
L_2 U_{k,2}(\tau) &= 2^{k-1}U_{k,2}(2\tau)\,.
\end{aligned}
$$

Here $L_2=L_{2,0}$. Following the proof of Lemma 2

$$
V_{k,2}(\tfrac{1}{2}(M\tau+l)) = V_{k,2}\big((A\tfrac{1}{2}(\tau+m)+B)/(C\tfrac{1}{2}(\tau+m)+D)\big)\,,
$$

and noting that $(A,B \mid C,D) \in \Gamma_0(4)$, we have from (3.1)

$$
(3.3) \qquad V_{k,2}(\tfrac{1}{2}(M\tau+l)) = \chi(D)(c\tau+d)^k V_{k,2}(\tfrac{1}{2}(\tau+m))\,.
$$

Observing that

$$
\begin{aligned}
A &= a+cl\,, \\
D &\equiv a+cl \pmod 4\,, \\
m &\equiv b+l \pmod 2\,,
\end{aligned}
$$

we conclude from Lemma 1 and (3.3)

$$
\begin{aligned}
V_{k,2}(M\tau) = \tfrac{1}{2}\chi(d)(c\tau+d)^k\{&\chi(ad)V_{k,2}(\tfrac{1}{2}(\tau+b))+ \\
&+\chi(ad+cd)V_{k,2}(\tfrac{1}{2}(\tau+b+1))\}\,.
\end{aligned}
$$

(3.2) yields

$$
V_{k,2}(\tfrac{1}{2}(\tau+1)) = 2V_{k,2}(\tau)-V_{k,2}(\tfrac{1}{2}\tau)\,,
$$

hence

$$
\begin{aligned}
&V_{k,2}(M\tau) \\
&= \tfrac{1}{2}\chi(d)(c\tau+d)^k \begin{cases} \{\chi(ad)-\chi(ad+cd)\}V_{k,2}(\tfrac{1}{2}\tau)+2\chi(ad+cd)V_{k,2}(\tau) & \text{if } 2 \mid b \\ \{\chi(ad+cd)-\chi(ad)\}V_{k,2}(\tfrac{1}{2}\tau)+2\chi(ad)V_{k,2}(\tau) & \text{if } 2 \nmid b\,, \end{cases}
\end{aligned}
$$

which completes the proof since

$$\chi(ad) = (-1)^b, \quad \chi(ad+cd) = (-1)^{b+1}.$$

4.

The fundamental domain $\Delta(\Gamma_0(p^{k_0}))$, $k_0 \geqq 2$, has the following cusps, $\tau = i\infty$, $\tau = 0$ and $\tau = sp^{m-k_0}$ where

$$(s,p) = 1, \quad 1 \leqq m \leqq k_0 - 1, \quad s = \pm 1, \ldots, \pm \tfrac{1}{2}(p^{k_0-m} - 1 + [2/p]).$$

The local variable at the cusp $P \in \Delta(\Gamma_0(p^{k_0}))$ is given as

$$t_P = e(V_P \tau),$$

where V_P is a non-singular matrix. If $P \neq i\infty$ then

$$V_P = (0, -1 \mid \gamma_0 p^{k_0}, -\gamma_0 p^{k_0} P), \quad \gamma_0 = p^{k_0}/(p^{k_0}, p^{2m}).$$

In order to determine the value of a function $g(\tau)$, $g(\tau) \in (\Gamma_0(p^{k_0}), -k, \nu)$ at a cusp $P \neq i\infty$, we have to express $g(\tau)$ as (see Lehner [6])

$$(\tau - P)^k e(-\varkappa_P V_P \tau) g(\tau) = \psi(t_P),$$

or as we will be doing

$$(V_P{}^*\tau - P)^k e(-\varkappa_P \tau) g(V_P{}^*\tau) = \psi(x), \quad V_P{}^* = (\gamma_0 p^{k_0} P, -1 \mid \gamma_0 p^{k_0}, 0),$$

where

$$\begin{aligned} e(-\varkappa_P) &= \nu(M_P), \\ M_P &= ((1 - \gamma_0 p^{k_0} P), \gamma_0 p^{k_0} P^2 \mid -\gamma_0 p^{k_0}, (1 + \gamma_0 p^{k_0} P)). \end{aligned}$$

Now, we turn to the proof of Theorem 1 in the case that $k_0 \geqq 2$ and p odd. Since $\nu(M_P) = 1$, $\varkappa_P = 0$ for all cusps $P \in \Delta(\Gamma_0(p^{k_0}))$. Put

$$g(\tau) = a(0)\{p^k F_k(p^{k_0}\tau) - F_k(p^{k_0-1}\tau)\} - p^{k-1}\mu_{k,p,0}\{F_k(p^2\tau) - F_k(p\tau)\},$$

$$\begin{aligned} (4.1) \quad H_\varepsilon(\tau) = (p^k - 1)(2k)^{-1} B_k \left(\textstyle\sum_{(r/p)=\varepsilon} 1\right) \textstyle\sum_{(r/p)=\varepsilon} L_{p,r} f(\tau) + \\ + \varepsilon^2 \mu_{k,p,\varepsilon} \textstyle\sum_{(r/p)=\varepsilon} L_{p,r} F_k(\tau) + (1 - \varepsilon^2) g(\tau). \end{aligned}$$

From the definition, $H_\varepsilon(\tau)$ is seen to be regular for $\operatorname{Im}\tau > 0$. Further, from Lemma 2 we conclude that

$$H_\varepsilon(\tau) \in (\Gamma_0(p^{k_0}), -k, 1).$$

It is immediately seen that $H_\varepsilon(\tau)$ has a zero at $\tau = i\infty$. Let $P = sp^{m-k_0}$. If $k_0 > 2$ we examine the class of cusps where $1 \leqq m \leqq k_0 - 2$. From (2,4) we have

$$T_{s,l,m}((V_P{}^*\tau) + lp^{-1}) = \alpha p^{-(k_0-m)} + \tau p^{2m}/(p^{k_0}, p^{2m}).$$

When using the transformation formula for $f(\tau)$ we get

$$\begin{aligned} f(\alpha p^{-(k_0-m)}+\tau p^{2m}/(p^{k_0},p^{2m})) &= f\big(T_{s,l,m}((V_P{}^*\tau)+lp^{-1})\big) \\ &= v(T_{s,l,m})p^{(k_0-m)k}(\gamma_0 p^{k_0}\tau)^{-k}f\big((V_P{}^*\tau)+lp^{-1}\big)\,. \end{aligned}$$

Hence, from this and Lemma 1 we obtain

$$\begin{aligned} L_{p,r}f(V_P{}^*\tau) = p^{(m-k_0)k-1}(\gamma_0 p^{k_0}\tau)^k \textstyle\sum_{l=0}^{p-1} e(-rlp^{-1})v(T_{s,l,m})^{-1}\cdot \\ \cdot f\big(\alpha p^{-(k_0-m)}+\tau p^{2m}/(p^{k_0},p^{2m})\big)\,. \end{aligned}$$

Similarly $L_{p,r}F_k(V_P{}^*\tau)$ is found. Consider the matrices

$$D_j = (\alpha_j, (-\alpha_j s+1)/p^{k_0-m-j} \mid p^{k_0-m-j}, -s), \quad j=1,2\,,$$

where $\alpha_j s\equiv 1 \pmod{p^{k_0-m-j}}$, and the corresponding transformations

$$D_j(p^j V_P{}^*\tau) = \alpha_j p^{-(k_0-m-j)}+\tau p^{2m+j}/(p^{k_0},p^{2m})\,.$$

From the transformation formula for $F_k(\tau)$ we obtain

$$\begin{aligned} F_k\big(\alpha_j p^{-(k_0-m-j)}+\tau p^{2m+j}/(p^{k_0},p^{2m})\big) &= F_k(D_j(p^j V_P{}^*\tau)) \\ &= p^{(k_0-m-j)k}(\gamma_0 p^{k_0-j}\tau)^{-k}F_k(V_P{}^*\tau)\,, \end{aligned}$$

or

$$\begin{aligned} &p^{k-1}F_k(p^j V_P{}^*\tau) \\ &\quad = (\gamma_0 p^{k_0}\tau)^k p^{(m+1-k_0)k-1}F_k\big(\alpha_j p^{-(k_0-j-m)}+\tau p^{2m+j}/(p^{k_0},p^{2m})\big)\,. \end{aligned}$$

Now, subjecting (4.1) to the transformation $\tau\to V_P{}^*\tau$ we obtain

$$\begin{aligned} (4.2)\quad &(\gamma_0 p^{k_0}\tau)^{-k}H_\varepsilon(V_P{}^*\tau) \\ &= p^{(m-k_0)k-1}(p^k-1)(2k)^{-1}B_k\big(\textstyle\sum_{(r/p)=\varepsilon} 1\big)\sum_{(r/p)=\varepsilon}\sum_{l=0}^{p-1} e(-rlp^{-1})\cdot \\ &\quad \cdot v(T_{s,l,m})^{-1}f\big(\alpha p^{-(k_0-m)}+\tau p^{2m}/(p^{k_0},p^{2m})\big)+ \\ &\quad +\varepsilon^2\mu_{k,p,\varepsilon}p^{(m-k_0)k-1}\textstyle\sum_{(r/p)=\varepsilon}\sum_{l=0}^{p-1} e(-rlp^{-1})\cdot \\ &\quad \cdot F_k\big(\alpha p^{-(k_0-m)}+\tau p^{2m}/(p^{k_0},p^{2m})\big)+ \\ &\quad +(1-\varepsilon^2)\{a(0)p^{(1-k_0)k}\big(F_k(\gamma_0\tau)-F_k(\gamma_0 p\tau)\big)- \\ &\quad -\mu_{k,p,0}p^{(m+1-k_0)k-1}\big(F_k\big(\alpha_2 p^{-(k_0-2-m)}+\tau p^{2m+2}/(p^{k_0},p^{2m})\big)- \\ &\quad -F_k\big(\alpha_1 p^{-(k_0-1-m)}+\tau p^{2m+1}/(p^{k_0},p^{2m})\big)\big)\}\,. \end{aligned}$$

Similarly, when $m=k_0-1$, we get

$$
\begin{aligned}
(4.3)\quad & (p^{k_0}\tau)^{-k}H_\varepsilon(V_P{}^*\tau) \\
&= p^{-k-1}(p^k-1)(2k)^{-1}B_k\big(\textstyle\sum_{(r/p)=\varepsilon} 1\big) \sum_{(r/p)=\varepsilon} e(rsp^{-1})\cdot \\
&\quad \cdot\{\textstyle\sum_{n=0}^{\infty} a^*(n)x^n+\sum_{l=1}^{p-1} e(-rlp^{-1})v(T_{l,0,k_0-1})^{-1}f(\alpha p^{-1}+p^{k_0-2}\tau)\}+ \\
&+\varepsilon^2\mu_{k,p,\varepsilon}p^{-k-1}\textstyle\sum_{(r/p)=\varepsilon} e(rsp^{-1})\cdot \\
&\quad \cdot\{p^kF_k(p^{k_0}\tau)+\textstyle\sum_{l=1}^{p-1} e(-rlp^{-1})F_k(\alpha p^{-1}+p^{k_0-2}\tau)\}+ \\
&+(1-\varepsilon^2)\{a(0)p^{(1-k_0)k}(F_k(\tau)-F_k(p\tau))-\mu_{k,p,0}p^{-k-1}(F_k(p^{k_0-2}\tau)- \\
&-p^kF_k(p^{k_0-1}\tau))\}\,.
\end{aligned}
$$

Hence we conclude from (4.2) and (4.3) that $H_\varepsilon(\tau)$ has zeros at all cusps $P=sp^{m-k_0}$, where

$$(s,p)\;=\;1,\quad s\;=\;\pm 1,\ldots,\pm\tfrac{1}{2}(p^{k_0-m}-1),\quad 1\leqq m\leqq k_0-1\,.$$

The value of $H_\varepsilon(\tau)$ at $\tau=0$ is obtained from (4.3) if we put $s=0$,

$$(p^{k_0}\tau)^{-k}H_\varepsilon\big(-(p^{k_0}\tau)^{-1}\big)\;=\;\textstyle\sum_{n=1}^{\infty}\gamma_{\varepsilon,0}(n)x^n\,,$$

thus $H_\varepsilon(\tau)$ has a zero at $\tau=0$.

Hence

$$H_\varepsilon(\tau)\in C^0(\Gamma_0(p^{k_0}),-k,1)\,.$$

The coefficients $\gamma_{\varepsilon,P}(n)$ in the expansion of $H_\varepsilon(\tau)$ at the cusp P are estimated by (cf. Hecke [4, pp. 644–707])

$$\gamma_{\varepsilon,P}(n)\;=\;O(n^{\frac{1}{2}k})\,.$$

Thus the proof is completed when equating coefficients of (4.1) at the cusp $\tau=i\infty$.

Similar results are obtained for $a^*(n)$ when equating coefficients of (4.1) at the cusp $\tau=0$.

The rest of Theorem 1, Theorem 2 and 3 are proved in quite a similar way.

5.

As an application of the results given in section 2, we will find asymptotic formulae for $r_k(n)$ and $t_k(n)$, k even. $r_k(n)$ is the number of representations of n as a sum of k squares (with the usual convention as to sign and order). Similarly, $t_k(n)$ is the number of representations of n as a sum of k triangular numbers.

In particular we put $r_k(0)=t_k(0)=1$. Then we have

$$\sum_{n=0}^{\infty} r_k(n)x^n = \prod_{n=1}^{\infty} (1-(-x)^n)^k(1+(-x)^n)^{-k},$$
$$\sum_{n=0}^{\infty} t_k(n)x^n = \prod_{n=1}^{\infty} (1-x^n)^{-k}(1-x^{2n})^{2k}.$$

Dedekind's eta function is defined by

$$\eta(\tau) = x^{1/24}\prod_{n=1}^{\infty}(1-x^n).$$

If we put

$$E_k(\tau) = \eta(\tau)^{2k}/\eta(2\tau)^k; \quad E_k{}^*(\tau) = \eta(2\tau)^{2k}/\eta(\tau)^k,$$

then

$$E_k(\tau) = \sum_{n=0}^{\infty}(-1)^n r_k(n)x^n; \quad E_k{}^*(\tau) = x^{\frac{1}{8}k}\sum_{n=0}^{\infty} t_k(n)x^n.$$

From Weber [8, § 38] and Kolberg [5] we have

$$E_k(\tau) \in (\Gamma_0(2), -\tfrac{1}{2}k, \nu), \quad \nu(M) = e(\tfrac{1}{8}k(\tfrac{1}{2}dc-d-c+1))$$

and

$$E_k(-\tfrac{1}{2}\tau^{-1}) = (-i)^{\frac{1}{2}k}\tau^{\frac{1}{2}k}2^kE_k{}^*(\tau).$$

Hence

$$E_k(\tau) \in \left(\Gamma_0(16/(4,k)), -\tfrac{1}{2}k, \nu_k\right),$$

where

$$\begin{aligned}\nu_k(M) &= 1 &&\text{if } 4\mid k,\\ &= \chi(d) &&\text{if } 4\nmid k.\end{aligned}$$

From the Theorems 1 and 3 we then obtain the following known asymptotic formulae (see for example Hilfssatz 7 in Suetuna [7] or Hecke [4, pp. 644–707]).

If $4\mid k$, $k>4$ then

$$(5.1)\quad (2^{\frac{1}{2}k}-1)k^{-1}B_{\frac{1}{2}k}r_k(n) = \{\varrho_{k,n}\sigma_{\frac{1}{2}k-1}(n)-2^{\frac{1}{2}k}\sigma_{\frac{1}{2}k-1}(n/2\lambda_k)\}\{1+O(n^{-\frac{1}{2}k+1})\},$$

and if $4\nmid k$, $k>2$

$$(5.2)\quad A_{\frac{1}{2}k,2}r_k(n) = \{v_{\frac{1}{2}k,2}(n)+(-1)^{\frac{1}{4}(k-2)+n-1}2^{\frac{1}{2}k-1}u_{\frac{1}{2}k,2}(n)\}\{1+O(n^{-\frac{1}{4}k+1})\}$$

where

$$\begin{aligned}\varrho_{k,n} &= (-1)^n &&\text{if } 8\mid k &\qquad \lambda_k &= 1 &&\text{if } 8\mid k\\ &= 1 &&\text{if } 8\nmid k & &= 2 &&\text{if } 8\nmid k.\end{aligned}$$

The technique used in proving Theorems 1–3 is to construct a cuspform from a given modular form. Hence, to each result in these theorems there corresponds a certain cuspform whose explicit form is seen immediately hereof. Thus from Theorems 1 and 3 we obtain:

If $4 \mid k,\ k>4$ then

$$(5.3)\quad (2^{\frac{1}{2}k}-1)k^{-1}B_{\frac{1}{2}k}L_{2,1}E_k(\tau)+\mu_{\frac{1}{2}k,2,1}L_{2,1}F_k(\tau) \in C^0(\Gamma_0(4), -\tfrac{1}{2}k, 1)\ ,$$

and if $4 \nmid k,\ k>2$

$$(5.4)\quad A_{\frac{1}{2}k,2}L_{2,1}E_k(\tau)+\tfrac{1}{2}r_k(0)\delta L_{2,1}V_{\frac{1}{2}k,2}(\tau)+ \\ +2^{\frac{1}{2}k-1}\mu_{\frac{1}{2}k,2,1}L_{2,1}U_{\frac{1}{2}k,2}(\tau) \in C^0(\Gamma_0(8), -\tfrac{1}{2}k, \nu)\ ,$$

where $\nu(M)\ =\ \chi(d),\quad \mu_{\frac{1}{2}k,2,1} = \begin{cases} (-1)^{\frac{1}{4}k+1} & \text{if } 4 \mid k \\ (-1)^{\frac{1}{4}(k-2)} & \text{if } 4 \nmid k \end{cases}$ and $\delta = 2$.

Now, subjecting (5.3) and (5.4) to the transformations $\tau \to -1/4\tau$, $\tau \to -1/8\tau$, respectively, and using (5.1) and (5.2) we obtain after some calculation:

If $4 \mid k,\ k>4$ *then*

$$2^{k-1}(2^{\frac{1}{2}k}-1)k^{-1}B_{\frac{1}{2}k}t_k(n) \\ = \{(\lambda_k-1)\sigma_{\frac{1}{2}k-1}(2n+\tfrac{1}{4}k)+2^{\frac{1}{2}k-1}\big(\sigma_{\frac{1}{2}k-1}(\tfrac{1}{4}(2n+\tfrac{1}{4}k))- \\ -\sigma_{\frac{1}{2}k-1}((2n+\tfrac{1}{4}k)/2\lambda_k)\big)\}\{1+O(n^{-\frac{1}{2}k+1}\}\ ,$$

and if $4 \nmid k,\ k>2$

$$2^{k-1}A_{\frac{1}{2}k,2}t_k(n)\ =\ v_{\frac{1}{2}k,2}(4n+\tfrac{1}{2}k)\{1+O(n^{-\frac{1}{2}k+1}\}\ .$$

In this paper we have studied modular forms on $\Gamma_0(p^{k_0})$. Carrying on by the same method which led us to Theorems 1–3, we may obtain analogous results for modular forms on $\Gamma_0(q)$, q an integer $\geqq 1$.

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