ASYMPTOTIC FORMULÆ FOR THE COEFFICIENTS
OF A CLASS OF MODULAR FORMS

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1.

Let $\Gamma(1)$ denote the full modular group, i.e., the group of matrices

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a, b \mid c, d), \quad ad - bc = 1,
\]

where $a, b, c, d$ are integers, and $\Gamma_0(n)$ denote the subgroup defined by $c \equiv 0 \pmod{n}$. A multiplier system $\nu = \nu(\Gamma, -k)$ of dimension $-k$, $k$ integral, for a group $\Gamma \subseteq \Gamma(1)$ is a character of the group $\Gamma$. A modular form on $\Gamma$ of dimension $-k$ and multiplier system $\nu = \nu(\Gamma, -k)$ is a function $F(\tau)$, meromorphic in the fundamental domain $\Delta(\Gamma)$ of $\Gamma$, which satisfies

\[
F(M\tau) = \nu(M)(c_M\tau + d_M)^kF(\tau)
\]

for all $M = (\cdot, \cdot \mid c_M, d_M) \in \Gamma$ (see Lehner [6, ch. 8]). The set $(\Gamma, -k, \nu)$ of all such modular forms is a vector space over the complex number field. We denote by $C^0(\Gamma, -k, \nu)$ the subspace consisting of all cuspforms, i.e., modular forms which are regular in $\Delta(\Gamma)$ and zero at the cusps of $\Delta(\Gamma)$. In particular, if $F(\tau) \in (\Gamma, 0, 1)$, then $F(\tau)$ is called a modular function on $\Gamma$ (see Ford [2, ch. 7]).

We will first determine a class of modular forms

\[
f(\tau) \in (\Gamma_0(p^k), -k, \nu), \quad k > 0,
\]

whose Fourier coefficients, $a(n)$, at the cusp $\tau = i\infty$, can be expressed as

\[
a(n) = \varrho(n) + R(n)
\]

where $\varrho(n)$ is given in terms of $\sigma_k(n)$ or $v_{k, p}(n)$, $u_{k, p}(n)$ (see (2,1)–(2,3)) and $R(n)$ is a remainder term estimated by $R(n) = O(n^{k+1})$.

This is a generalization of the results given in Dirdal [1], where the author studied a special class of modular forms on $\Gamma_0(3)$.

2.

Let

\[
(2,1) \quad \sigma_k(n) = \sum_{d|n} d^k,
\]

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\[(2,2) \quad v_{k,p}(n) = \sum_{d|n} \omega_p(d) d^{k-1}, \]
\[(2,3) \quad u_{k,p}(n) = \sum_{d|n} \omega_p(n/d) d^{k-1}, \]
\[A_{k,p} = i^k s_p (k-1)! p^k (2\pi)^{-k} \sum_{n=1}^{\infty} \omega_p(n) n^{-k}, \]
where
\[\omega_p(m) = \frac{m}{p} \quad \text{if } p \text{ odd}, \]
\[= \chi(m) \quad \text{if } p = 2, \]
\[s_p = i^{-k^2} p^{-\frac{1}{2}} \quad \text{if } p \text{ odd}, \]
\[= -i^{2k-1} \quad \text{if } p = 2. \]

Here and in the following \(\chi\) denotes the real character \((\mod 4)\), \(p\) a prime and \((\cdot/p)\) the Legendre symbol. Further \(B_k\) denote the Bernoulli numbers
\[B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \ldots. \]
\(B_k = 0\) when \(k\) is odd and \(> 1\).

By \(M\) we will always denote the matrix \(M = (a, b | c, d)\), where \(a, b, c, d\) are integers and \(ad - bc = 1\), and by \(T\) we mean the matrix
\[(2,4) \quad T = T_{t_1, t_2, m} = \left( \alpha, - (\alpha(t_1 + p^{k_0-m-1} t_2) + 1)/p^{k_0-m}, \quad p^{k_0-m}, \quad -(t_1 + p^{k_0-m-1} t_2) \right), \]
where \(k_0 \geq 2,\ 1 \leq m \leq k_0 - 1,\ (t_1 + p^{k_0-m-1} t_2, p) = 1\) and \(\alpha\) has a value such that \(1 \leq \alpha < p^{k_0-m}\) and \(T \in \Gamma_0(p^{k_0-m})\). Here \((\cdot, \cdot, \cdot)\) denotes the greatest common divisor.

By \(x\) we denote \(x = \exp(2\pi i \tau) = e(\tau)\) with \(\text{Im}\tau > 0\). We use \([a]\) to denote the integral part of \(a\) and further we let \(\epsilon = 0, \pm 1\). Let \(f(\tau)\) be a function such that
\[f(\tau) = \sum_{n=0}^{\infty} a(n) x^n, \]
\[f(-(p^{k_0})^{-1}) = i^{k^2} p^{(k_0-1)\frac{1}{2}} t^{\frac{k}{2}} \sum_{n=0}^{\infty} a*(n) x^n, \]
and
\[f(T\tau) = v(T)(c_T \tau + d_T)^k f(\tau), \quad c_T = p^{k_0-m}, \quad d_T = -(t_1 + p^{k_0-m-1} t_2), \]
where \(v(T)\) is a constant depending on the matrix \(T\). Then we put
\[\mu_{k,p,\epsilon} = \sum_{(r/p) = \epsilon} \{a*(0) + a(0)i^{k^2} \sum_{l=1}^{p-1} e(-rl/p)v(T_{l,0,k_0-1})^{-1}\}; \]
\[\mu^{(1)}_{m,s,p,\epsilon} = a(0) \sum_{(r/p) = \epsilon} \sum_{l=0}^{p-1} e(-rl/p)v(T_{s,l,m})^{-1}, \]
for \((s,p) = 1, \ s = \pm 1, \ldots, \ \pm \frac{1}{2}(p^{k_0-m} - 1 + [2/p])\);
\[\mu^{(2)}_{s,\beta} = a(0) \sum_{l=0}^{p-1} v(T_{l,0,k_0-1}^{-1}\{\sum_{(r/p) = \epsilon} \beta \sum_{(r/p) = \epsilon} e(rp^{-1}(s-l)) \}
- \sum_{(r/p) = \epsilon} \beta e(rs p^{-1}) \sum_{(r/p) = \epsilon} e(-rlp^{-1})}, \]
for $(s,p) = 1$, $s = \pm 1, \ldots, \pm \frac{1}{2}(p-1)$, $p$ odd;

\[
\delta = \sum_{l=0}^{1} e(-\frac{1}{2}l)\nu(T_{1,1,l,k_0-2})^{-1};
\]

\[
\delta^{(1)} = a(0) \sum_{l=0}^{1} e(-\frac{1}{2}l)\{\nu(T_{-1,1,l,k_0-2})^{-1} + \nu(T_{1,1,l,k_0-2})^{-1}\},
\]

where $\beta$ is an arithmetical function of $r$ and

\[
((r/p)) = (r/p) \quad \text{if } p \text{ is odd,}
\]

\[
= r \quad \text{if } p = 2.
\]

Further, if $f(\tau)$ is regular for $\text{Im} \tau > 0$ and $f(\tau) \in (\Gamma_0(p^{k_0}), -k, \nu)$, we shall prove the following theorems:

**Theorem 1.** Let $k$ be even $>2$ and $\nu(M) = 1$, when $M \in \Gamma_0(p^{k_0})$. If $k_0 = 1$ then

\[
(p^k - 1)(2k)^{-1}B_k\alpha(n) = (a^*(0) - a(0)p^k)\sigma_{k-1}(n/p) + (a(0) - a^*(0))\sigma_{k-1}(n) + O(n^{ik}).
\]

If $k_0 \geq 2$, $\mu_{k,p,\epsilon}$ real and $\mu^{(1)}_{m,s,p,\epsilon} = 0$, for $k_0 > 2$, $1 \leq m \leq k_0 - 2$, then

\[
(p^k - 1)(2k)^{-1}B_k\alpha(pn) = a(0)\{\sigma_{k-1}(np^{-(k_0-2)}) - \sigma_{k-1}(np^{-(k_0-1)})\} +
\]

\[
+ p^{k-1}\mu_{k,p,0}\{\sigma_{k-1}(np^{-1}) - \sigma_{k-1}(n)\} + O(n^{ik}).
\]

Further, if $\mu^{(2)}_{s,1} = 0$, then

\[
\frac{1}{2}(p-1 + [2/p])(p^k - 1)(2k)^{-1}B_k\alpha(pn + r) = -\mu_{k,p,\epsilon}\sigma_{k-1}(pn + r) + O(n^{ik})
\]

for $\epsilon = ((r/p))$ and $r = 1, \ldots, p-1$.

**Theorem 2.** Let $p$ be odd, $k \equiv \frac{1}{2}(p-1) \pmod{2}$, $k > 1$ and $\nu(M) = (d/p)$ when $M \in \Gamma_0(p^{k_0})$.

If $k_0 = 1$, then

\[
A_{k,p}\alpha(n) = a(0)v_{k,p}(n) + (-1)^k p^{-1}a^*(0)u_{k,p}(n) + O(n^{ik}).
\]

If $k_0 \geq 2$, $\mu_{k,p,\epsilon}$ real and $\mu^{(1)}_{m,s,p,\epsilon} = 0$ for $k_0 > 2$, $1 \leq m \leq k_0 - 2$, then

\[
A_{k,p}\alpha(pn) = a(0)v_{k,p}(np^{-(k_0-2)}) + (-1)^kp^{-3/2}\mu_{k,p,0}u_{k,p}(n) + O(n^{ik}).
\]

Further, if $\mu^{(2)}_{s, (r/p)} = 0$, then

\[
\frac{1}{2}(p-1)(r/p)A_{k,p}\alpha(pn + r) = (-1)^k p^{-1}\mu_{k,p,\epsilon}v_{k,p}(pn + r) + O(n^{ik}),
\]

for $\epsilon = (r/p)$ and $r = 1, \ldots, p-1$. 
THEOREM 3. Let \( k \) be odd \( > 1 \), \( k_0 \geq 3 \) and \( \nu(M) = \chi(d) \) when \( M \in \Gamma_0(2^{k_0}) \). If \( \delta \) and \( \mu_{k, 2, \varepsilon} \) are real, \( \mu_{m, s, 2, \varepsilon} = 0 \) for \( k_0 > 2 + \varepsilon^2 \), \( 1 \leq m \leq k_0 - (2 + \varepsilon^2) \), then
\[
A_{k, 2} a(2n) = a(0)\nu_{k, 2}(n2^{-(k_0-3)}) - 2^{k-1}\mu_{k, 2, \varepsilon} u_{k, 2}(2n) + O(n^{1k}).
\]
Further, if \( \delta^{(1)} = 0 \), then
\[
A_{k, 2} a(2n + 1) = -\frac{1}{2} a(0)\delta v_{k, 2}(2n + 1) - 2^{k-1}\mu_{k, 2, 1} u_{k, 2}(2n + 1) + O(n^{1k}).
\]
Similarly results can be obtained for \( a^*(n) \). (For an example see section 5).

Asymptotic formulae for \( a(n) \) can easily be derived from these theorems. For example, if we put
\[
\varrho(n) = (a^*(0) - a(0)p^k)\sigma_{k-1}(n/p) + (a(0) - a^*(0))\sigma_{k-1}(n),
\]
and \( a(0) = a^*(0) \), then \( \varrho(n) > C_1 n^{k-1} \) since \( \sigma_{k-1}(n) > C_2 n^{k-1} \), where \( C_j \), \( j = 1, 2 \) are constants. Hence putting \( k_0 = 1 \) in Theorem 1, we have
\[
(p^k - 1)(2k)^{-1}B_k a(n) = \varrho(n)\{1 + O(n^{-1k+1})\}.
\]
This formula is also valid when \( a(0) = a^*(0) \neq 0 \) if \( p \mid n \).

3.

We define a linear operator \( L_{p, r} \) acting on any power series
\[
g(x) = \sum_{n \geq N} b(n)x^n
\]
by
\[
L_{p, r} g(x) = \sum_{pn + r \geq N} b(pn + r)x^{pn + r}.
\]
Noting that
\[
\sum_{l=0}^{p-1} e(l(n-r)/p) = \begin{cases} p & \text{if } n \equiv r \pmod{p} \\ 0 & \text{if } n \not\equiv r \pmod{p} \end{cases}
\]
we obtain

**LEMMA 1.** If \( h(\tau) \) has a Laurent expansion in \( x \), then
\[
L_{p, r} h(\tau) = p^{-1} \sum_{l=0}^{p-1} e(-lr p^{-1}) h(\tau + lp^{-1}).
\]
In particular we observe \( L_{p, r_1} = L_{p, r_2} \) if \( r_1 \equiv r_2 \pmod{p} \).

Further we shall need the following lemma

**LEMMA 2.** If \( h(\tau) \) has a Laurent expansion in \( x \) and
\[
h(M \tau) = v(M)(c\tau + d)^{k} h(\tau), \quad M \in \Gamma_0(p^{k_0}), \quad k_0 \geq 2
\]
then
\[ \sum_{(\tau,p) = e_\tau} L_{\tau,p} h(M \tau) = v(M)(c\tau + d)^k \sum_{(\tau,p) = e_\tau} L_{\tau,p} h(\tau) \]
if \( v \) has the property \( v((a,b \mid c,d)) = v((a',b' \mid c',d')) \) when \( d \equiv d' \pmod{p^{k_0-1}}, \)
\( (a',b' \mid c',d') \in \Gamma_0(p^{k_0}) \).

**Proof.** We note that
\[ \frac{(ap\tau + pb)(c\tau + d)^{-1} + l}{p} = \frac{A(p\tau + m)p^{-1} + B}{pC(p\tau + m)p^{-1} + D}, \]
where \( A = a + lcp^{-1}, \quad B = (pb + ld - m(a + lcp^{-1}))/p, \quad C = c, \quad D = d - mcp^{-1}. \)
Here \( AD - BC = 1, \) and \( A, C \) and \( D \) are integers.
Since \( (p,a) = 1 \) we can solve the congruence
\[ am \equiv ld \pmod{p}, \quad 0 \leq l \leq p - 1, \]
that is, for each \( l \) we get a unique \( m, \) \( 0 \leq m \leq p - 1. \) With this value of \( m, B \) also becomes an integer thus
\[ (A,B \mid C,D) \in \Gamma_0(p^{k_0}). \]
Observing that \( D \equiv d \pmod{p^{k_0-1}} \) we have
\[ h(M \tau + lp^{-1}) = v(M)(c\tau + d)^{kh(\tau + mp^{-1})}. \]
From this, Lemma 1 and the fact that \( -l\tau \equiv -ma^2 \tau \pmod{p} \) we obtain
\[ L_{\tau,p}, h(M \tau) = v(M)(c\tau + d)^k L_{\tau,p}, a^2 h(\tau), \]
and the lemma follows immediately.

Put
\[ F_k(\tau) = -(2k)^{-1}B_k + \sum_{n=1}^{\infty} \sigma_{k-1}(n)x^n, \]
\[ V_{k,p}(\tau) = A_{k,p} + \sum_{n=1}^{\infty} v_{k,p}(n)x^n, \]
\[ U_{k,p}(\tau) = \sum_{n=1}^{\infty} u_{k,p}(n)x^n. \]
\[ F_k(\tau) \in (\Gamma(1), -k, 1), \quad k \text{ even} > 2. \]
When \( p \) is odd and \( k \equiv \frac{1}{2}(p - 1) \pmod{2}, \) \( k > 1, \) we have
\[ V_{k,p}(\tau) \in (\Gamma_0(p), -k, \nu_p), \]
\[ U_{k,p}(\tau) \in (\Gamma_0(p), -k, \nu_p), \]
\[ V_{k,p}((-p\tau)^{-1}) = i^{k(k+2)p^{-k-1}} k U_{k,p}(\tau), \]
\[ U_{k,p}((-p\tau)^{-1}) = i^{k(k+2)p^{-k-1}} k V_{k,p}(\tau), \]
where \( \nu_p(M) = (d/p). \)
When $k$ is odd $>1$;

(3.1) \[
V_{k,2}(\tau) \in \Gamma_0(4), -k, v_2), \\
U_{k,2}(\tau) \in \Gamma_0(4), -k, v_2), \\
V_{k,2}(-(4\tau)^{-1}) = -i2^{2k-1}2^k U_{k,2}(\tau), \\
U_{k,2}(-(4\tau)^{-1}) = -i2^k V_{k,2}(\tau),
\]

where $v_2(M) = \chi(d)$.

Finally we shall need the following result

**Lemma 3.** For $k$ odd $>1$ we have

\[
V_{k,2}(M\tau) = \chi(d)(c\tau+d)^k(V_{k,2}(\frac{1}{2}\tau) - V_{k,2}(\tau)), \\
U_{k,2}(M\tau) = \chi(d)(c\tau+d)^k(2^{1-k}U_{k,2}(\frac{1}{2}\tau) - U_{k,2}(\tau)),
\]

when $M \in \Gamma_0(2)$, $4 \nmid c$.

**Proof.** We shall prove only the first part of this lemma, the proof of the second part is quite similar. It is easily seen that

(3.2) \[
L_2 V_{k,2}(\tau) = V_{k,2}(2\tau), \\
L_2 U_{k,2}(\tau) = 2^{k-1}U_{k,2}(2\tau).
\]

Here $L_2 = L_{2,0}$. Following the proof of Lemma 2

\[
V_{k,2}(\frac{1}{2}(M\tau+l)) = V_{k,2}\left((A\frac{1}{2}(\tau+m)+B)/(C\frac{1}{2}(\tau+m)+D)\right),
\]

and noting that $(A,B \mid C,D) \in \Gamma_0(4)$, we have from (3.1)

(3.3) \[
V_{k,2}(\frac{1}{2}(M\tau+l)) = \chi(D)(c\tau+d)^k V_{k,2}(\frac{1}{2}(\tau+m)).
\]

Observing that

\[
\begin{align*}
A &= a+c1, \\
D &\equiv a+c1 \pmod{4}, \\
m &\equiv b+l \pmod{2},
\end{align*}
\]

we conclude from Lemma 1 and (3.3)

\[
V_{k,2}(M\tau) = \frac{1}{2}\chi(d)(c\tau+d)^k\{\chi(ad)V_{k,2}(\frac{1}{2}(\tau+b)) + \\
&\quad + \chi(ad+cd)V_{k,2}(\frac{1}{2}(\tau+b+1))\}.
\]

(3.2) yields

\[
V_{k,2}(\frac{1}{2}(\tau+1)) = 2V_{k,2}(\tau) - V_{k,2}(\frac{1}{2}\tau),
\]

hence

\[
V_{k,2}(M\tau)
\]

\[
= \frac{1}{2}\chi(d)(c\tau+d)^k \left\{ \begin{align*}
\{\chi(ad) - \chi(ad+cd)\}V_{k,2}(\frac{1}{2}\tau) + 2\chi(ad+cd)V_{k,2}(\tau) & \quad \text{if } 2 \nmid b, \\
\{\chi(ad+cd) - \chi(ad)\}V_{k,2}(\frac{1}{2}\tau) + 2\chi(ad)V_{k,2}(\tau) & \quad \text{if } 2 \nmid b,
\end{align*} \right.
\]

which completes the proof since
\[ \chi(ad) = (-1)^b, \quad \chi(ad + cd) = (-1)^{b+1}. \]

4.

The fundamental domain \( \Delta(\Gamma_0(p^{k_0})), \ k_0 \geq 2 \), has the following cusps, \( \tau = i\infty, \ \tau = 0 \) and \( \tau = sp^{m-k_0} \) where
\[
(s,p) = 1, \quad 1 \leq m \leq k_0 - 1, \quad s = \pm 1, \ldots, \pm \frac{1}{2}(p^{k_0-m} - 1 + [2/p]) .
\]
The local variable at the cusp \( P \in \Delta(\Gamma_0(p^{k_0})) \) is given as
\[
t_P = e(V_P \tau) ,
\]
where \( V_P \) is a non-singular matrix. If \( P \neq i\infty \) then
\[
V_P = (0, -1 | \gamma_0 p^{k_0}, -\gamma_0 p^{k_0}P), \quad \gamma_0 = p^{k_0}/(p^{k_0}, p^{2m}).
\]

In order to determine the value of a function \( g(\tau), g(\tau) \in (\Gamma_0(p^{k_0}), -k,v) \) at a cusp \( P \neq i\infty \), we have to express \( g(\tau) \) as (see Lehner [6])
\[
(\tau - P)^k e(-\kappa_P V_P \tau) g(\tau) = \psi(t_P),
\]
or as we will be doing
\[
(V_P^{-1} - P)^k e(-\kappa_P \tau) g(V_P^{-1} \tau) = \psi(x), \quad V_P^{-1} = (\gamma_0 p^{k_0}P, -1 | \gamma_0 p^{k_0}, 0),
\]
where
\[
e(-\kappa_P) = v(M_P), \quad M_P = ((1 - \gamma_0 p^{k_0}P), \gamma_0 p^{k_0}P^2 | -\gamma_0 p^{k_0}, (1 + \gamma_0 p^{k_0}P)).
\]

Now, we turn to the proof of Theorem 1 in the case that \( k_0 \geq 2 \) and \( p \) odd. Since \( v(M_P) = 1, \ \kappa_P = 0 \) for all cusps \( P \in \Delta(\Gamma_0(p^{k_0})). \) Put
\[
g(\tau) = \alpha(0\{p^k F_k(p^{k_0} \tau) - F_k(p^{k_0-1} \tau)\} - p^{k-1} \mu_{k,p,0} (F_k(p^2 \tau) - F_k(p \tau)) ,
\]
(4.1) \[ H_\varepsilon(\tau) = (p^k - 1)(2k)^{-1} B_k \left( \sum_{|\tau|} + 1 \right) \sum_{|\tau|} \varepsilon L_{p,r} f(\tau) + \\
+ \varepsilon^2 \mu_{k,p,0} \sum_{|\tau|} \varepsilon L_{p,r} F_k(\tau) + (1 - \varepsilon^2) g(\tau) .
\]

From the definition, \( H_\varepsilon(\tau) \) is seen to be regular for \( \text{Im} \ \tau > 0 \). Further, from Lemma 2 we conclude that
\[
H_\varepsilon(\tau) \in (\Gamma_0(p^{k_0}), -k, 1).
\]

It is immediately seen that \( H_\varepsilon(\tau) \) has a zero at \( \tau = i\infty \). Let \( P = sp^{m-k_0} \).

If \( k_0 > 2 \) we examine the class of cusps where \( 1 \leq m \leq k_0 - 2 \). From (2,4) we have
\[
T_{s,t,m}(V_P^{-1} + l p^{-1}) = \alpha p^{-(k_0-m)} + \tau p^{2m}/(p^{k_0}, p^{2m}) .
\]
When using the transformation formula for $f(\tau)$ we get
\[
f(\alpha p^{-(k_0 - m)} + \tau p^{2m}/(p^{k_0}, p^{2m})) = f(T_{s,l,m}((V_P\tau) + lp^{-1}))
\]
\[
= v(T_{s,l,m})p^{(k_0 - m)k}(\gamma_0 p^{k_0}\tau)^{-k}f((V_P\tau) + lp^{-1}).
\]
Hence, from this and Lemma 1 we obtain
\[
L_p,rf(V_P\tau) = p^{(m-k_0)k-1}(\gamma_0 p^{k_0}\tau)^k \sum_{l=0}^{p-1} e(-rlp^{-1})v(T_{s,l,m})^{-1} \cdot f(\alpha p^{-(k_0 - m)} + \tau p^{2m}/(p^{k_0}, p^{2m})).
\]
Similarly $L_p,rf_k(V_P\tau)$ is found. Consider the matrices
\[
D_j = (\alpha_j, (-\alpha_j s + 1)/p^{k_0 - m - j} | p^{k_0 - m - j}, -s), \quad j = 1, 2,
\]
where $\alpha_j s \equiv 1 \pmod{p^{k_0 - m - j}}$, and the corresponding transformations
\[
D_j(p^j V_P\tau) = \alpha_j p^{-(k_0 - m - j)} + \tau p^{2m + j}/(p^{k_0}, p^{2m}).
\]
From the transformation formula for $F_k(\tau)$ we obtain
\[
F_k(\alpha_j p^{-(k_0 - m - j)} + \tau p^{2m + j}/(p^{k_0}, p^{2m})) = F_k(D_j(p^j V_P\tau))
\]
\[
= p^{(k_0 - m - j)k}(\gamma_0 p^{k_0 - j}\tau)^{-k}F_k(V_P\tau),
\]
or
\[
p^{k-1}F_k(p^j V_P\tau) = (\gamma_0 p^{k_0}\tau)^{k}p^{(m+1-k_0)k-1}F_k(\alpha_j p^{-(k_0 - j - m)} + \tau p^{2m + j}/(p^{k_0}, p^{2m})).
\]
Now, subjecting (4.1) to the transformation $\tau \to V_P\tau$ we obtain
\[
(4.2) \quad (\gamma_0 p^{k_0}\tau)^{-k}H_e(V_P\tau)
\]
\[
= p^{(m-k_0)k-1}(p^k - 1)(2k)^{-1}B_k(\sum_{\tau/p} e) \sum_{\tau/p} e \sum_{l=0}^{p-1} e(-rlp^{-1}.
\]
\[
\cdot v(T_{s,l,m})^{-1}f(\alpha p^{-(k_0 - m)} + \tau p^{2m}/(p^{k_0}, p^{2m}))+
\]
\[
+ e^{2}\mu_{k, p}, p^{(m-k_0)k-1} \sum_{\tau/p} e \sum_{l=0}^{p-1} e(-rlp^{-1}) \cdot F_k(\alpha p^{-(k_0 - m)} + \tau p^{2m}/(p^{k_0}, p^{2m}))+
\]
\[
+(1 - e^{2})\{a(0)p^{(1-k_0)k}(F_k(\gamma_0 p\tau) - F_k(\gamma_0 p\tau)) - \mu_{k, p, 0} p^{(m+1-k_0)k-1} \sum_{\tau/p} e \sum_{l=0}^{p-1} e(-rlp^{-1}) \cdot F_k(\alpha_2 p^{-(k_0 - 2 - m)} + \tau p^{2m + 2}/(p^{k_0}, p^{2m}))- \]
\[
- F_k(\alpha_1 p^{-(k_0 - 2 - m)} + \tau p^{2m + 1}/(p^{k_0}, p^{2m}))) \}.
Similarly, when \( m = k_0 - 1 \), we get

\[
(4.3) \quad (p^{k_0} \tau)^{-k} H_{\varepsilon}(V_P^* \tau)
\]

\[
= p^{-k-1}(p^k - 1)(2k)^{-1}B_k(\sum_{r|p}^1) \sum_{r|p}^1 e(rsp^{-1}) \cdot
\]

\[
\sum_{n=0}^\infty a^*(n)x^n + \sum_{l=1}^{p-1} e(-rlp^{-1})v(T_{l,0,k_0-1})^{-1}f(\alpha p^{-1} + p^{k_0-2}\tau) +
\]

\[
+ \varepsilon^2 \mu_{k,p,0} \varepsilon p^{-k-1} \sum_{r|\rho}^1 e(rsp^{-1}) \cdot
\]

\[
\sum_{l=1}^{p-1} e(-rlp^{-1})F_k(\alpha p^{-1} + p^{k_0-2}\tau) +
\]

\[
+ (1-\varepsilon^2)(a(0)p^{(1-k_0)k}(F_k(\tau) - F_k(p\tau)) - \mu_{k,p,0} p^{-k-1}(F_k(p^{k_0-2}\tau) -
\]

\[- p^{k_0}F_k(p^{k_0-1})) \}.
\]

Hence we conclude from (4.2) and (4.3) that \( H_{\varepsilon}(\tau) \) has zeros at all cusps \( P = sp^{m-k_0} \), where

\[
(s,p) = 1, \quad s = \pm 1, \ldots, \pm \frac{1}{2}(p^{k_0-m} - 1), \quad 1 \leq m \leq k_0 - 1.
\]

The value of \( H_{\varepsilon}(\tau) \) at \( \tau = 0 \) is obtained from (4.3) if we put \( s = 0 \),

\[
(p^{k_0} \tau)^{-k} H_{\varepsilon}(-(p^{k_0} \tau)^{-1}) = \sum_{n=1}^\infty \gamma_{\varepsilon,0}(n)x^n,
\]

thus \( H_{\varepsilon}(\tau) \) has a zero at \( \tau = 0 \).

Hence

\[
H_{\varepsilon}(\tau) \in C^0(\Gamma_0(p^{k_0}), -k, 1).
\]

The coefficients \( \gamma_{\varepsilon,P}(n) \) in the expansion of \( H_{\varepsilon}(\tau) \) at the cusp \( P \) are estimated by (cf. Hecke [4, pp. 644–707])

\[
\gamma_{\varepsilon,P}(n) = O(n^{1k}).
\]

Thus the proof is completed when equating coefficients of (4.1) at the cusp \( \tau = i\infty \).

Similar results are obtained for \( a^*(n) \) when equating coefficients of (4.1) at the cusp \( \tau = 0 \).

The rest of Theorem 1, Theorem 2 and 3 are proved in quite a similar way.

5.

As an application of the results given in section 2, we will find asymptotic formulae for \( r_k(n) \) and \( t_k(n) \), \( k \) even. \( r_k(n) \) is the number of representations of \( n \) as a sum of \( k \) squares (with the usual convention as to sign and order). Similarly, \( t_k(n) \) is the number of representations of \( n \) as a sum of \( k \) triangular numbers.
In particular we put \( r_k(0) = t_k(0) = 1 \). Then we have

\[
\sum_{n=0}^{\infty} r_k(n)x^n = \prod_{n=1}^{\infty} (1 - (-x)^n)^k (1 + (-x)^n)^{-k},
\]

\[
\sum_{n=0}^{\infty} t_k(n)x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-k} (1 - x^{2n})^{2k}.
\]

Dedekind’s eta function is defined by

\[
\eta(\tau) = x^{1/24} \prod_{n=1}^{\infty} (1 - x^n).
\]

If we put

\[
E_k(\tau) = \eta(\tau)^{2k}/\eta(2\tau)^k; \quad E_k^*(\tau) = \eta(2\tau)^{2k}/\eta(\tau)^k,
\]

then

\[
E_k(\tau) = \sum_{n=0}^{\infty} (-1)^n r_k(n)x^n; \quad E_k^*(\tau) = x^{i\tau} \sum_{n=0}^{\infty} t_k(n)x^n.
\]

From Weber [8, § 38] and Kolberg [5] we have

\[
E_k(\tau) \in (\Gamma_0(2), -\frac{1}{2}k, \nu), \quad \nu(M) = \epsilon(\frac{1}{2}k(\frac{1}{2}dc - d - c + 1))
\]

and

\[
E_k(-\frac{1}{2}\tau^{-1}) = (-i)^{\frac{1}{2}k} \tau^{i\frac{1}{2}k} 2^k E_k^*(\tau).
\]

Hence

\[
E_k(\tau) \in \left( \Gamma_0(16)/(4, k) \right), \quad -\frac{1}{2}k, \nu_k,
\]

where

\[
\nu_k(M) = 1 \quad \text{if} \ 4 | k, \\
= \chi(d) \quad \text{if} \ 4 \nmid k.
\]

From the Theorems 1 and 3 we then obtain the following known asymptotic formulae (see for example Hilfssatz 7 in Suetuna [7] or Hecke [4, pp. 644–707]).

If \( 4 | k, k > 4 \) then

\[
(5.1) \quad (2^{ik} - 1)k^{-1} A_{\frac{1}{2}k} r_k(n) = \left\{ \ell_k, n \sigma_{\frac{1}{2}k - 1}(n) - 2^{ik} \sigma_{\frac{1}{2}k - 1}(n/2\ell_k) \right\} \{1 + O(n^{-\frac{1}{2}k + 1})\},
\]

and if \( 4 \nmid k, k > 2 \)

\[
(5.2) \quad A_{\frac{1}{2}k, 2} r_k(n) = \left\{ v_{\frac{1}{2}k, 2}(n) + (-1)^{\frac{i(k - 2)}{4} + n - 2^{ik - 1} v_{\frac{1}{2}k, 2}(n)} \right\} \{1 + O(n^{-\frac{1}{2}k + 1})\}
\]

where

\[
\ell_k = (\frac{-1)^{n}}{k} \quad \text{if} \ 8 | k, \quad \ell_k = 1 \quad \text{if} \ 8 \nmid k, \\
= 1 \quad \text{if} \ 8 \nmid k, \quad \lambda_k = 2 \quad \text{if} \ 8 \nmid k.
\]

The technique used in proving Theorems 1–3 is to construct a cuspform from a given modular form. Hence, to each result in these theorems there corresponds a certain cuspform whose explicit form is seen immediately hereof. Thus from Theorems 1 and 3 we obtain:
If $4 \mid k$, $k > 4$ then

\begin{equation}
(5.3) \quad (2^{ik} - 1)k^{-1}B_{\frac{1}{2}, k}E_{k}(\tau) + \mu_{\frac{1}{2}, 2}L_{2, 1}F_{k}(\tau) \in C^0(\Gamma_0(4), -\frac{1}{2}k, 1),
\end{equation}

and if $4 \nmid k$, $k > 2$

\begin{equation}
(5.4) \quad A_{\frac{1}{2}, 2}L_{2, 1}E_{k}(\tau) + \frac{1}{2}r_{k}(0)\delta L_{2, 1}V_{\frac{1}{2}, 2}(\tau) + 2^{ik-1}\mu_{\frac{1}{2}, 2}L_{2, 1}U_{\frac{1}{2}, 2}(\tau) \in C^0(\Gamma_0(8), -\frac{1}{2}k, \nu),
\end{equation}

where $\nu(M) = \chi(d)$, $\mu_{\frac{1}{2}, 2, 1} = \begin{cases} (-(1)^{\frac{1}{2}k+1} & \text{if } 4 \mid k \\ (-(1)^{\frac{1}{2}(k-2)} & \text{if } 4 \nmid k \end{cases}$ and $\delta = 2$.

Now, subjecting (5.3) and (5.4) to the transformations $\tau \rightarrow -1/4\tau$, $\tau \rightarrow -1/8\tau$, respectively, and using (5.1) and (5.2) we obtain after some calculation:

If $4 \mid k$, $k > 4$ then

\[ 2^{k-1}(2^{ik} - 1)k^{-1}B_{\frac{1}{2}, k}t_{k}(n) = \left\{ (\lambda_{k} - 1)\sigma_{\frac{1}{2}, k-1}(2n + \frac{1}{4}k) + 2^{ik-1}(\sigma_{\frac{1}{2}, k-1}(\frac{3}{2}(2n + \frac{1}{4}k)) - 
\quad - \sigma_{\frac{1}{2}, k-1}(2n + \frac{1}{4}k)/2\lambda_{k}) \right\} \{ 1 + O(n^{-\frac{1}{4}k+1}) \}, \]

and if $4 \nmid k$, $k > 2$

\[ 2^{k-1}A_{\frac{1}{2}, 2}t_{k}(n) = v_{\frac{1}{2}, 2}(4n + \frac{1}{3}k)\{ 1 + O(n^{-\frac{1}{4}k+1}) \}. \]

In this paper we have studied modular forms on $\Gamma_0(pk^0)$. Carrying on by the same method which led us to Theorems 1–3, we may obtain analogous results for modular forms on $\Gamma_0(q)$, $q$ an integer $\geq 1$.

REFERENCES


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