ON THE POSSIBILITY OF FINDING
CERTAIN CRITERIA FOR THE IRRATIONALITY
OF A NUMBER DEFINED AS A LIMIT OF A SEQUENCE
OF RATIONAL NUMBERS

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1.

In 1910 I (Viggo Brun) put forth the following theorem in an article entitled “Ein Sats über Irrationalität” (See [1]).

If the sequence

\[
\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots, \frac{a_n}{b_n}, \frac{a_{n+1}}{b_{n+1}}, \ldots
\]

is composed of strictly increasing positive rational numbers which are converging towards \( c \), while the sequence

\[
\frac{a_2-a_1}{b_2-b_1}, \frac{a_3-a_2}{b_3-b_2}, \ldots, \frac{a_{n+1}-a_n}{b_{n+1}-b_n}, \frac{a_{n+2}-a_{n+1}}{b_{n+2}-b_{n+1}}, \ldots
\]

is composed of strictly decreasing numbers, then \( c \) is irrational. Here \( a_1, a_2, \ldots, a_n, \ldots \) and \( b_1, b_2, \ldots, b_n, \ldots \) are supposed to be positive integers such that \( b_{n+1} > b_n \).

The geometrical considerations that led me to this theorem, but which are not mentioned in [1], are the following:

![Graph](attachment:image.png)

Fig. 1.

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Let $x$ and $y$ be coordinates in the plane. The points in the plane with integral coordinates will be called lattice points. Let $c$ be the limit of the given sequence. Then the lattice points $(b_i, a_i)$ form a configuration as shown in figure 1.

The characteristic feature of this configuration is that the polygon formed by the lattice points $(b_i, a_i)$ shows its convex side towards the limit line $y = cx$. If $c$ is rational, the line $y = cx$ will contain infinitely many equidistant lattice points. On each side of the line there will exist a stripe which does not contain any lattice points. This is impossible when the configuration of lattice points is as in fig. 1. For a non-geometrical proof see [1]. The above theorem is simple but unfortunately not very useful since the picture very often will look as in fig. 2.

![Fig. 2.](image)

Here the polygon shows its concave side towards the limit line. Since the situation in fig. 2 occurs "much more frequently" than the situation in fig. 1, the value of my theorem is very limited.

In 1963 Al Froda published a generalization of the theorem in this journal [2].

I myself have made many efforts to find useful generalizations. My attempts have been to determine lattice points in the triangles

$$\langle(0,0),(b_n,a_n),(b_{n+1},a_{n+1})\rangle$$

which lie closer to the origin than $(b_n,a_n)$ and $(b_{n+1},a_{n+1})$ do. Possibly I will give some results in this direction in a future article.

During my work I made some temporary hypotheses. The first one was the following:

**Hypothesis 1.** Let

$$c = u_1 + u_2 + \ldots + u_n + \ldots$$
be the sum of a convergent series, where the \( u_i \)'s are positive rational numbers. If

\[
\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = 0
\]

then \( c \) is irrational.

This condition at least excludes all geometric series. I mentioned this temporary hypothesis to my young friend Finn Faye Knudsen. He recognized very quickly that the hypothesis was wrong and gave a counterexample.

**Counterexample 1.** Put

\[
u_n = \frac{n}{(n+1)!}.
\]

Here the partial sums are

\[
s_n = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.
\]

Therefore

\[
c = \lim_{n \to \infty} s_n = 1
\]

whereas

\[
\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = 0.
\]

After a while I mentioned another possible hypothesis for irrationality (which excluded counterexample 1).

**Hypothesis 2.** Let again

\[
c = u_1 + u_2 + u_3 + \ldots + u_n + \ldots
\]

be the sum of a convergent series whose terms are positive rational numbers such that

(1) \[
\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = 0,
\]

(2) \[
u_n(u_{n+2} + u_{n+3}) - u_{n+1}^2 > 0.
\]

Then \( c \) is irrational.

When I showed this hypothesis to my friend, he proved that also these conditions are insufficient and he gave a

**Counterexample 2.** Put

\[
u_n = \frac{n2^n}{(n+2)!} = \frac{2^n}{(n+1)!} - \frac{2^{n+1}}{(n+2)!}.
\]
The rest of this paper, which is written by Finn Faye Knudsen, will be devoted to a general theorem which shows that criteria of "this type" can never lead to the goal. Even though this is a negative result, it gives in my opinion a valuable contribution to the study of this fundamental problem.

2.

Let \( k \) be a natural number, and let \( E \subset \mathbb{R}^k \) be the subset of real \( k \)-dimensional space defined by \( x_1 > 0, x_2 > 0, \ldots, x_k > 0 \).

**Definition 1.** By a criterion we shall mean a finite number of real-valued functions, \( F_1, F_2, \ldots, F_m \) and \( G \), defined on \( E \).

Let \( u_1 + u_2 + u_3 + \ldots + u_n + \ldots \) be a series with real positive terms.

**Definition 2.** We shall say that the series \( u_1 + u_2 + u_3 + \ldots + u_n + \ldots \) satisfies the criterion \( (F_1, F_2, \ldots, F_m, G) \) if there exists a natural number \( N \) with the property that for all \( n \geq N \) we have:

1. \( F_i(u_n, u_{n+1}, u_{n+2}, \ldots, u_{n+k-1}) > 0, \quad 1 \leq i \leq m, \)
2. \( \lim_{n \to \infty} G(u_n, u_{n+1}, u_{n+k-1}) = 0. \)

**Definition 3.** We shall say that a criterion \( (F_1, F_2, \ldots, F_m, G) \) is an irrationality criterion if the following is true.

Whenver \( u_1 + u_2 + u_3 + \ldots + u_n + \ldots \) is a convergent series whose terms are positive rational numbers and satisfies the criterion \( (F_1, F_2, \ldots, G) \), its sum \( c \) is an irrational number.

**Definition 4.** A criterion \( (F_1, F_2, \ldots, F_m, G) \) will be said to be of continuous type if all the functions \( F_1, F_2, \ldots, F_m \) and \( G \) are continuous functions.

We shall now prove that there exists no irrationality criterion of continuous type or more precisely:

**Theorem.** Let \( (F_1, F_2, \ldots, F_m, G) \) be a criterion of continuous type and suppose that a series \( u_1 + u_2 + u_3 + \ldots + u_n + \ldots \) with rational positive terms satisfies the criterion. Also suppose \( c = \sum u_n < \infty \). Then we can find another convergent series of positive rational terms \( v_1 + v_2 + \ldots + v_n + \ldots \) which also satisfies the criterion and whose sum is rational.
The method of proof will be to change the terms in the original series \( u_i \) by sufficiently small positive rational numbers \( \delta_i \). Then define \( v_i = u_i + \delta_i \). We will divide the proof into several parts.

**Lemma 1.** Let \( F \) be a real-valued continuous function on \( E \) and let \( u_1, u_2, \ldots, u_n, \ldots \) be a sequence of positive real numbers such that

1. \( \sum u_i < \infty \),
2. \( F(u_n, u_{n+1}, \ldots, u_{n+k-1}) > 0 \) for all \( n > N \).

Then there is a sequence \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \ldots \) of positive real numbers such that

3. \( \sum \varepsilon_i < \infty \),
4. for all sequences \( \delta_1, \delta_2, \ldots, \delta_n, \ldots \) with \( 0 \leq \delta_i \leq \varepsilon_i \) we have

\[
F(u_n + \delta_n, u_{n+1} + \delta_{n+1}, \ldots, u_{n+k-1} + \delta_{n+k-1}) > 0 \quad \text{for } n > N.
\]

**Proof.** We can choose the first \( N \) \( \varepsilon \)'s quite arbitrarily. For all \( n > N \) we can choose a neighbourhood \( V_n \) of the point \( (u_n, u_{n+1}, \ldots, u_{n+k-1}) \) in \( E \) such that

\[
F(v_n, v_{n+1}, \ldots, v_{n+k-1}) > 0
\]

for all points \( (v_n, \ldots, v_{n+k-1}) \) in \( V_n \). The set \( V_n \) certainly contains a closed box of type:

\[
\{(\delta_n, \ldots, \delta_{n+k-1}) ; 0 \leq |u_i - \delta_i| \leq \varepsilon^n_i \}.
\]

If we put

\[
\varepsilon'_n = \min(\varepsilon^n_{n-k+1}, \varepsilon^n_{n-k+2}, \ldots, \varepsilon^n_n)
\]

and define \( \varepsilon_n = 2^{-n} \min(1, \varepsilon'_n) \), the sequence \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n, \ldots \) has the required property.

**Lemma 2.** Let \( G \) be a real-valued continuous function on \( E \) and let \( u_1, u_2, \ldots, u_n, \ldots \) be a sequence of positive real numbers such that

1. \( \sum u_i < \infty \)
2. \( \lim_{n \to \infty} G(u_n, u_{n+1}, \ldots, u_{n+k-1}) = 0 \).

Then there is a sequence of positive real numbers \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \ldots \) such that

3. \( \sum \varepsilon_i < \infty \)
4. for all sequences \( \delta_1, \delta_2, \ldots, \delta_n, \ldots \) with \( 0 \leq \delta_i \leq \varepsilon_i \) we have

\[
\lim_{n \to \infty} G(u_n + \delta_n, \ldots, u_{n+k-1} + \delta_{n+k-1}) = 0.
\]
Proof. Let \( V_k \) be a neighbourhood of the point \( (u_n, \ldots, u_{n+k-1}) \) in \( E \) such that
\[
|G(u_n, \ldots, u_{n+k-1}) - G(v_n, \ldots, v_{n+k-1})| < 2^{-n}.
\]
Again \( V_n \) contains a box
\[
\{(v_n, \ldots, v_{n+k-1}) ; 0 \leq |u_i - v_i| \leq \varepsilon_i^n\},
\]
and we see that the sequence \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \ldots \) has the required property if we choose
\[
\varepsilon_n = 2^{-n}[\min(1, \varepsilon_n^{n-k+1}, \varepsilon_n^{n-k+2}, \ldots, \varepsilon_n^n)].
\]

Lemma 3. Let \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n, \ldots \) be a sequence of positive real numbers such that \( \sum \varepsilon_i = \varepsilon < \infty \). Then given any number \( \beta \), with \( 0 < \beta < \varepsilon \), we can find a sequence \( \delta_1, \delta_2, \ldots, \delta_n, \ldots \) of positive rational numbers such that
\[
(1) \quad 0 \leq \delta_i < \varepsilon_i,
\]
\[
(2) \quad \sum \delta_i = \beta.
\]

Proof. Let \( \varepsilon'_i = \varepsilon_i \varepsilon / \beta \) and let \( s_n \) be the partial sum,
\[
s_n = \sum_{i=1}^{n} \varepsilon'_i.
\]
Inductively we define a sequence \( k_0, k_1, \ldots, k_n, \ldots \) of rational numbers as follows:
\[
k_0 = 0
\]
\[
k_n = \text{some rational number in the interval}
\]
\[
\langle \max(\langle k_{n-1}, s_{n+1} - \varepsilon_{n+1}\rangle, s_n) \rangle.
\]
If we define \( \delta_n = k_n - k_{n-1} \) for \( n = 1, 2, \ldots \) we have (1) and (2).

This completes the proof of the theorem.

References


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