ON THE REALIZATION OF CERTAIN MODULES
OVER THE STEENROD ALGEBRA

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1.
Let $\mathfrak{a}$ denote the $(\text{mod } p)$ Steenrod algebra, $p$ any prime and let $Q_i$ be the unique non-zero primitive in degree $2p^i - 1$. In the present note we show that for any string of integers $0 \leq j_0 < j_1 < \ldots < j_q$ there exists a spectrum whose cohomology (as an $\mathfrak{a}$-module) is isomorphic to $\mathfrak{a}/\mathfrak{a}(Q_{j_0}, \ldots, Q_{j_q})$. Here $\mathfrak{a}(Q_{j_0}, \ldots, Q_{j_q})$ denotes the left ideal generated by $Q_{j_0}, \ldots, Q_{j_q}$. Previously, Margolis [5] has realized the modules $\mathfrak{a}/\mathfrak{a}(Q_i)$ and $\mathfrak{a}/\mathfrak{a}(Q_0, Q_i)$ by other (and quite complicated) methods.

Let $MU$ denote the Thom spectrum of the universal unitary bundle over $BU$. It is well-known that

$$\pi_\ast(MU) = \mathbb{Z}[x_1, x_2, \ldots],$$

where $\deg x_i = 2i$.

For any string of integers $0 < n_1 < n_2 < \ldots < n_q$ there exists a spectrum $MU\langle n_1, \ldots, n_q \rangle$ such that

$$\pi_\ast(MU\langle n_1, \ldots, n_q \rangle) = \mathbb{Z}[x_{n_1}, \ldots, x_{n_q}]$$

and a map

$$\nu: MU \to MU\langle n_1, \ldots, n_q \rangle.$$ 

The Thom spectrum $MU$ is a ring spectrum and $MU\langle n_1, \ldots, n_q \rangle$ is a “module” spectrum over $MU$, that is there is a map

$$MU \wedge MU\langle n_1, \ldots, n_q \rangle \to MU\langle n_1, \ldots, n_q \rangle.$$ 

Further, $\nu: MU \to MU\langle n_1, \ldots, n_q \rangle$ is a “module” map. The induced map $\nu_\ast$ on homotopy groups is just

$$\nu_\ast(x_i) = 0 \quad \text{if } i \notin \{n_1, \ldots, n_q\}$$
$$= x_i \quad \text{if } i \in \{n_1, \ldots, n_q\}$$

The construction of $MU\langle n_1, \ldots, n_q \rangle$ is based on ideas of Sullivan and was carried out in detail in [2]. The spectrum $MU\langle n_1, \ldots, n_q \rangle$ appears

Received December 21, 1970; in revised form February 20, 1972.
as representing object for a bordism theory of manifolds with certain cone singularities.

We shall calculate $H^\ast(MU\langle n_1, \ldots, n_q \rangle; \mathbb{Z}_p)$ as a module over the Steenrod algebra in case all the $n_i$’s are of the form $p^j - 1$. The calculation is based on a simple application of the Atiyah–Hirzebruch spectral sequence for a generalized homology theory. If $X$ and $Y$ are spectra (with zero homotopy in negative degrees) then the spectral sequence, denoted $E^r_{\ast, \ast}(X, Y)$ is a first quadrant, homology type, spectral sequence with

$$E^2_{\ast, \ast} = H_\ast(X; \pi_\ast(Y))$$
$$E^\infty_{\ast, \ast} = E^0 Y_\ast(X),$$

where $Y_\ast(X) = \pi_\ast(Y \wedge X) \cong X_\ast(Y)$. We shall only use the spectral sequence in the case where $X$ is the Eilenberg–MacLane spectrum and we write $E^r_{\ast, \ast}(Y)$ instead of $E^r_{\ast, \ast}(K(Z_p), Y)$. If $Y$ is torsion free then

$$E^2_{\ast, \ast}(Y) = H_\ast(K(Z_p); \mathbb{Z}) \otimes \pi_\ast(Y)$$

and the spectral sequence converges to $Y_\ast(K(Z_p)) \cong H_\ast(Y; \mathbb{Z}_p)$.

Suppose that $Y$ is connected ($\pi_0(Y) = \mathbb{Z}$, $\pi_1(Y) = 0$ for $i < 0$) and let $U: Y \to K(\mathbb{Z})$ be the map which induces the identity on $\pi_0$. The edge homomorphisms are then

$$h : \pi_\ast(Y) \to H_\ast(Y; \mathbb{Z}_p)$$
$$\chi \circ U : H_\ast(Y; \mathbb{Z}_p) \to H_\ast(K(\mathbb{Z}); \mathbb{Z}_p) \cong H_\ast(K(Z_p); \mathbb{Z}),$$

where $h$ is the Hurewicz homomorphism and $\chi$ the isomorphism $H_\ast(K(\mathbb{Z}); \mathbb{Z}_p) \cong H_\ast(K(Z_p); \mathbb{Z})$.

If $Y$ is a ring spectrum then $E^r_{\ast, \ast}(Y)$ is a spectral sequence of algebras over $\mathbb{Z}_p$. If $M$ is a “module” spectrum over the ring spectrum $Y$ then $E^r_{\ast, \ast}(M)$ is a differential module over $E^r_{\ast, \ast}(Y)$ (that is

$$d_i(y \cdot m) = d_{i+1}y \cdot m + (-1)^{\deg y} v \cdot d_{i+1}m).$$

2.

We first consider the case $Y = MU$. Then

$$E^2_{\ast, \ast}(MU) = \mathbb{Z}_p[\xi_1, \xi_2, \ldots] \otimes E\{\tau_1, \tau_2, \ldots\} \otimes \mathbb{Z}_p[x_1, x_2, \ldots],$$

where $E\{\}$ denotes the exterior algebra and $\bideg(x_i) = (0, 2i)$, $\bideg(\xi_i) = (2(p^i - 1), 0)$ and $\bideg(\tau_i) = (2p^i - 1, 0)$. (If $p = 2$ then $\tau_i$ and $\xi_i$ above should be interpreted as $\xi_i$ and $\xi_i^2$, respectively.)

J. Cohen in [4] has examined the structure of $E^r_{\ast, \ast}(MU)$: The elements $\xi_i$ all survives to $E^\infty$ and there are elements $\overline{\tau}_j$ (where $\overline{\tau}_j \equiv \tau_j$ modulo decomposable elements) such that
\[ d_{2p^j-1}(\bar{x}_j) = \bar{x}_k, \quad k = p^j - 1, \]

where \( \bar{x}_k \equiv x_k \) modulo decomposable elements.

This is all easily implied by the remark that \( \tau_j \in E^{2, *}_0 \) cannot survive to \( E^\infty \) since \( H_*(MU; \mathbb{Z}_p) \) is a polynomial algebra with one generator in each even dimension and the fact (Milnor [6]) that the composite map

\[ \tau_*(MU) \rightarrow H_*(MU; \mathbb{Z}_p) \rightarrow QH_*(MU; \mathbb{Z}_p) \]

is zero in dimensions \( 2(p^j - 1) \).

The structure of \( E^{r, *}_*(MU) \) is now computed by a standard argument using the comparison theorem for spectral sequences

\[
E^{r, *}_*(MU) = \mathbb{Z}_p[\xi_1, \xi_2, \ldots] \otimes E\{\bar{\xi}_j \mid 2p^j - 1 \geq r - 1\} \\
\otimes \mathbb{Z}_p[\{\bar{x}_k \mid k \equiv p^j - 1, 2p^j - 1 \leq r - 1\}].
\]

3.

Next we consider \( E^{r, *}_*(MU^{n_1, \ldots, n_q}) \) where all the \( n_i \)'s are of the type \( p^j - 1 \), say \( n_i = p^{j_i} - 1 \). We have

\[
E^{2, *}_*(MU^{n_1, \ldots, n_q}) = \mathbb{Z}_p[\xi_1, \xi_2, \ldots] \otimes E\{\bar{\xi}_1, \bar{\xi}_2, \ldots\} \\
\otimes \mathbb{Z}[x_{n_1}, \ldots, x_{n_q}].
\]

The map \( \nu: MU \rightarrow MU^{n_1, \ldots, n_q} \) induces an \( E^{r, *}_*(MU) \)-module map of spectral sequences

\[ \nu_*: E^{r, *}_*(MU) \rightarrow E^{r, *}_*(MU^{n_1, \ldots, n_q}) \]

which on the \( E^2 \)-level is just

\[ \nu_*(\xi_i \otimes \bar{\xi}_j \otimes x_k) = \xi_i \otimes \bar{\xi}_j \otimes x_k \quad \text{if} \quad k \in \{n_1, \ldots, n_q\} \]

\[ = 0 \quad \text{otherwise}.
\]

Comparing the two spectral sequences via \( \nu_* \) we see that the only non-zero transgressive differentials in \( E^{r, *}_*(MU^{n_1, \ldots, n_q}) \) are \( d_{2n_1+1}, d_{2n_2+1}, \ldots, d_{2n_q+1} \), and that

\[ d_{2n_i+1}(\bar{x}_j) = \bar{x}_{n_i} \quad (n_i = p^j - 1).
\]

It follows that

\[
E^{\infty, *}_*(MU^{n_1, \ldots, n_q}) = \mathbb{Z}_p[\xi_1, \xi_2, \ldots] \\
\otimes E\{\bar{\xi}_j \mid \ p^j - 1 \notin \{n_1, \ldots, n_q\}, j > 0\}.
\]

The map

\[ U_*: H_*(MU^{n_1, \ldots, n_q}; \mathbb{Z}_p) \rightarrow H_*(K(Z); \mathbb{Z}_p) \]

is therefore an injection with image
where $\chi$ is the canonical anti-automorphism of $\hat{a}$.

The primitive element $Q_j$ (of degree $2p^j - 1$) in $\hat{a}$ is dual to $\tau_j$. Since $\chi(\tilde{\tau}_j) = \tau_j$ modulo decomposable elements,

$$U^* : H^*(K(Z); Z_p) \to H^*(MU\langle n_1, \ldots, n_q \rangle; Z_p)$$

maps $Q_{i_1}, \ldots, Q_{i_q}$ ($n_i = p^{j_i} - 1$) to zero so that $U^*$ factors to give an epimorphism

$$v^* : \hat{a}/\hat{a}(Q_0, Q_{i_1}, \ldots, Q_{i_q}) \to H^*(MU\langle n_1, \ldots, n_q \rangle; Z_p).$$

The two vector spaces have the same dimension and $U^*$ is therefore an isomorphism. This proves

**Theorem A.** Let $0 < i_1 < \ldots < i_q$ and set $n_i = p^{j_i} - 1$. There is an isomorphism of left $\hat{a}$-modules

$$\hat{a}/\hat{a}(Q_0, Q_{i_1}, \ldots, Q_{i_q}) \cong H^*(MU\langle n_1, \ldots, n_q \rangle; Z_p).$$

Let $L$ be the $\mathbb{Z}_p$-Moore spectrum, ($L$ is the cofiber of a map of degree $p : S^0 \to S^0$) and write $MU_p\langle n_1, \ldots, n_q \rangle$ for $MU\langle n_1, \ldots, n_q \rangle \wedge L$. Then

$$\pi_* (MU_p\langle n_1, \ldots, n_q \rangle) = \mathbb{Z}_p[x_{n_1}, \ldots, x_{n_q}]$$

and again there is a $MU$-"module" map

$$v_p : MU \to MU_p\langle n_1, \ldots, n_q \rangle$$

which on homotopy is $v_*$ composed with reduction modulo $p$. Let

$$U_p : MU_p\langle n_1, \ldots, n_q \rangle \to K(Z_p)$$

be a map which induces the identity on $\pi_0$. Arguing as before we get

**Theorem B.** There is a left $\hat{a}$-module isomorphism

$$\hat{a}/\hat{a}(Q_{i_1}, \ldots, Q_{i_q}) \cong H^*(MU_p\langle n_1, \ldots, n_q \rangle; Z_p).$$

**Remarks:**

(a) The spectrum $MU_p\langle n_1, \ldots, n_q \rangle$ is again a representing object for a bordism theory of manifolds with singularities. In fact one just adds one more singularity to the singularities determining $MU\langle n_1, \ldots, n_q \rangle$, namely the manifold consisting of $p$ distinct points.

(b) Let $0 < m_1 < \ldots < m_r$ be a string of integers not all of the form $p^j - 1$. In order to compute even the additive structure of $H^*(MU\langle m_1, \ldots, m_r \rangle)$ one would need more precise information on the structure of $E^r_*, * (MU)$, than provided by Cohen [4].
REFERENCES


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