MATH. SCAND. 31 (1972), 201–208

CHARACTERIZATIONS OF CONJUGATE LOCALLY CONVEX SPACES[1]

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1. Introduction.

Various characterizations of conjugate Banach spaces have been given by Dixmier [2], Ruston [7] and Singer [9]–[13]. Krishnamurthy [3] and Lohman [4] have considered conjugate locally convex topological vector spaces (l.c. spaces). The work of the latter authors is an extension to l.c. spaces of the initial work by Dixmier. In this paper we give additional characterizations of both conjugate l.c. spaces and conjugate Banach spaces. Theorem 1 is a general criterion for V-semi-reflexivity and is similar to the Šmulian-like criterion given in [4, Theorem 2]. Theorem 2 is a natural extension to l.c. spaces of a result of De Vito [1, Corollary 2], where the criterion for conjugacy is given in terms of linear functionals attaining their suprema on a certain weak closure of bounded sets. De Vito's theorem is itself an extension of the deep result of R. C. James in which weak compactness is characterized in terms of linear functionals attaining their suprema. Theorem 3 is a partial extension to l.c. spaces of a result of Ruston [7, Theorem 4]. Theorem 4 is a Petunin-like characterization (see [5, Theorem 1]) of conjugate Banach spaces.

2. Definitions and Notations.

We use the notation set forth in [8]. E_τ or simply E denotes a vector space E endowed with a locally convex Hausdorff vector topology τ. E' denotes the dual of E. V always denotes a linear subspace of E' and we assume $\langle E, V\rangle$ is the natural pairing. Notations for the various topologies (weak, strong, Mackey, etc.) which arise from $\langle E, V\rangle$ are as in [8]. In addition, the topology $\beta^*(E, V)$ on E denotes the topology of uniform convergence on $\beta(V,E)$-bounded sets.

If A is a subset of E_τ, A_τ represents the set A with the topology τ, and the latter topology is denoted by $\tau|A$. V' denotes the dual of V_β.

[1] These results are part of the author's doctoral dissertation written at The University of Iowa under the supervision of Professor Bor-Luh Lin.

Received March 1, 1971.

The linear mapping $\varphi: E \to V'$, defined by $\varphi(f)(x)=f(x)$ for all $f \in V$ and all $x \in E$ is called the canonical embedding. When $V=E'$, the canonical embedding is denoted by φ_0. E is said to be V-semi-reflexive in case φ is a surjection. We say E is V-reflexive in case φ is a topological isomorphism of E_τ onto V_β'.

V is minimal in case V is closed in E_β', dense in E_σ' and no proper linear subspace of V has both preceding properties.

As in [4] we say V is τ-p.c. if for each τ-equicontinuous subset B of E' there is a $\beta(E',E)$-bounded set F such that B is contained in the $\sigma(E',E)$-closure of $V \cap F$.

If E is a Banach space, S_E denotes the unit ball of E.

3. Conjugate Locally Convex Spaces.

Let V be $\beta(E',E)$-closed, total, $\sigma(V,E)$-separable and such that V_β is barrelled. In [4], Lohman proved that E is V-semi-reflexive if and only if each decreasing sequence of nonempty $\sigma(E, V)$-bounded, $\sigma(E, V)$-closed convex sets has a nonempty intersection. If we remove the separability condition on V, it is possible that the Šmulian-like intersection property holds and yet E may fail to be V-semi-reflexive (see [4, example 3]). We can, however, remove the separability condition in order to obtain a more general criterion for V-semi-reflexivity, provided we consider families of sets which are directed by inclusion.

THEOREM 1. *Let V be $\beta(E',E)$-closed, total and such that V_β is barrelled. Then E is V-semi-reflexive if and only if each family of nonempty $\sigma(E, V)$-bounded, $\sigma(E, V)$-closed convex subsets of E which is directed by inclusion has a nonempty intersection.*

PROOF. Assume that E is V-semi-reflexive. Then φ is a topological isomorphism of $E_{\sigma(E,V)}$ onto V_σ'. Let B be an absolute convex $\sigma(E, V)$-bounded set in E. If $\bar{B}$ denotes the $\sigma(E, V)$-closure of B, $\varphi(\bar{B})$ is the bipolar (for the pairing $\langle V, V' \rangle$) of the $\sigma(V', V)$-bounded set $\varphi(B)$. By the barrelledness of V_β, $\varphi(\bar{B})$ is $\sigma(V', V)$-compact. Thus $\bar{B}$ is $\sigma(E, V)$-compact. The intersection property now follows easily.

For the converse, assume that the stated intersection property holds. Given a closed, bounded absolute convex subset A of E we let $\bar{A}$ denote the $\sigma(E, V)$-closure of A. Then $\bar{A}$ is $\sigma(E, V)$-totally bounded. Let $\{x_\alpha\}$ be a net in $\bar{A}$ which is Cauchy for $\sigma(E, V)$. Denote the $\sigma(E, V)$-closed convex hull of $\{x_\beta: \beta \geqq \alpha\}$ by B_α. Then $\{B_\alpha\}$ is a family of nonempty $\sigma(E, V)$-bounded, $\sigma(E, V)$-closed convex subsets of E which is directed by inclusion. By hypothesis, there exists $x \in \bigcap_\alpha B_\alpha$. Now $x \in \bar{A}$ and it is routine to show that $x_\alpha \to x$ relative to $\sigma(E, V)$. Consequently, $\bar{A}$ is also $\sigma(E, V)$-complete. It follows that $\bar{A}$ is $\sigma(E, V)$-compact. By [3, Theorem 1], V is minimal so that E is V-semi-reflexive.

Let E be a real, separable Banach space and let V be a closed, total subspace of positive characteristic (as in [2, Definition 2]) in E'. In 1968 De Vito [1, Corollary 2] proved the following extension of the famous theorem of R. C. James: E is canonically isomorphic to the strong dual of V if and only if each member of V attains its supremum on the $\sigma(E, V)$-closure of S_E. The next result is a natural extension of De Vito's theorem to l.c. spaces.

THEOREM 2. *Let E_τ be a real, complete, barrelled and separable* l.c. *space. Assume that V is a total linear subspace of E' such that V_β is barrelled, complete and τ-*p.c. *Then*

(a) $\beta(V,E) = \beta(E',E) \mid V$

(b) $\tau = \beta(E, V) = \beta^*(E, V)$

(c) $E_{\tau(E,V)}$ *is quasi-complete*

(d) E_τ *is V-reflexive if and only if each member of V attains its supremum on the $\sigma(E, V)$-closure of each τ-bounded subset of E.*

PROOF. First, note that V is closed in E_β'. Since V is total and τ-p.c., the canonical embedding $\varphi: E_\tau \to V_\beta'$ is one-one and relatively open.

(a) Let A be a $\sigma(E, V)$-bounded subset of E. Then $\varphi(A)$ is pointwise bounded on V_β. By the uniform boundedness theorem, $\varphi(A)$ is $\beta(V', V)$-bounded. Since $\varphi: E_\tau \to V_\beta'$ is relatively open, A is τ-bounded. It follows that a subset A of E is $\sigma(E, V)$-bounded if and only if A is τ-bounded. This implies $\beta(V,E) = \beta(E',E) \mid V$.

(b) Let B be a $\beta(V,E)$-bounded subset of V. Then by (a), B is $\beta(E',E)$-bounded. By the quasi-barrelledness of E_τ, B_0 is a τ-neighborhood of 0. Consequently, $\beta^*(E, V) \subset \tau$. On the other hand, let B be a τ-equicontinuous subset of E'. Since V is τ-p.c., there exists a $\beta(E',E)$-bounded set F such that $B \subset \overline{V \cap F}$, where this closure is taken with respect to $\sigma(E',E)$. Therefore, $(\overline{V \cap F})_0 = (V \cap F)_0 \subset B_0$. By (a), $V \cap F$ is $\beta(V,E)$-bounded so that $(V \cap F)_0$ is a $\beta^*(E, V)$-neighborhood of 0. Thus $\tau \subset \beta^*(E, V)$ which yields $\tau = \beta^*(E, V)$. Since E_τ is barrelled, we have $\beta^*(E, V) \subset \beta(E, V) \subset \tau$, whence all three topologies are equal.

(c) We have $(E_{\tau(E,V)})' = V$. By (a) and (b), $E_{\beta^*(E,V)}$ is both complete and separable. Also, $V_{\beta(E,V)}$ is complete. By another result of De Vito [1, Theorem 1], $E_{t(E,V)}$ is quasi-complete.

(d) If E_τ is V-reflexive, then E is V-semi-reflexive. Consequently, V is minimal [4, Lemma 2]. Therefore, the τ-bounded sets are relatively $\sigma(E, V)$-compact. Hence the necessity easily follows.

In order to prove sufficiency, assume that each member of V attains its supremum on the $\sigma(E, V)$-closure of each τ-bounded subset of E. Since $\tau=\beta(E, V)$, φ is a topological isomorphism of E_τ into V_β'. Thus it suffices to show φ is onto. That is, it suffices to show V is minimal. Let A be a τ-bounded set in E and let $\bar{A}$ denote the $\sigma(E, V)$-closure of A. By (c), $\bar{A}$ is $\tau(E, V)$-complete. Therefore, by the extension to l.c. spaces of the theorem of James as stated in [6], $\bar{A}$ is $\sigma(E, V)$-compact. Hence V is minimal.

Ruston, [7, Theorem 4] has shown that a Banach space E is topologically isomorphic to the strong dual of a Banach space if and only if there exists a continuous projection of E'' onto E which annihilates a weak*-closed linear subspace of E''. Recall from [4] that a l.c. space F is an MSD space in case F is $\beta(F'',F')$-closed in F'', F is quasi M-barrelled and F is semi-distinguished. We can now state a partial generalization of Ruston's result.

THEOREM 3. *Let E_τ be a quasi-barrelled* l.c. *space. If E_τ is topologically isomorphic to the strong dual of an* MSD *space, then there exists a $\beta(E'',E')$-continuous projection of E'' onto E which annihilates a weak*-closed linear subspace of E''.*

PROOF. By hypothesis, E_τ is a Mackey space and is topologically isomorphic to the strong dual of an MSD space. By [3, Theorem 6], there exists a minimal subspace V of E' such that E_τ is V-reflexive. Let $\varphi: E_\beta \to V_\beta'$ and $\varphi_0: E_\beta \to E_\beta''$ denote the canonical embeddings. φ is a topological isomorphism and φ_0 is continuous. Let $R: E_\beta'' \to V_\beta'$ be the continuous linear mapping defined by $R(f)=f|V$ for all $f \in E''$. Define $P: E_\beta'' \to E_\beta''$ by

$$P(f) = (\varphi_0 \circ \varphi^{-1} \circ R)(f) \quad \text{for all } f \in E'' .$$

Then, using the same argument as in [7, Theorem 4], it is easily verified that P is a continuous projection of E'' onto E which annihilates a weak*-closed linear subspace of E''.

4. Conjugate Banach Spaces.

We now restrict our considerations to Banach spaces, where the characterizations of conjugate spaces become sharper. In the Banach space setting, we say E is V-reflexive if and only if φ is an isometric isomorphism of E onto V'. Petunin [5, Theorem 1] has shown that a Banach space is reflexive if and only if its unit ball is closed in every Hausdorff locally convex topology that is comparable with the norm topology. Our next result is a Petunin like criterion for V-reflexivity.

THEOREM 4. *If E is a Banach space and V is a total subspace of E', then E is V-reflexive if and only if S_E is closed in every Hausdorff locally convex topology on E that is comparable with the topology $\sigma(E, V)$.*

PROOF. If E is V-reflexive, then S_E is $\sigma(E, V)$-compact (by [9, Theorem 2]) and therefore $\sigma(E, V)$-closed. Thus if τ is any Hausdorff topology on E comparable with $\sigma(E, V)$, then S_E is τ-closed.

Suppose, on the other hand, that S_E is closed in every Hausdorff locally convex topology on E that is comparable with $\sigma(E, V)$. Then S_E is $\sigma(E, V)$-closed so that φ is an isometry (by [11, lemma 2]). Assume E is not V-reflexive. Then $\varphi(E)$ is a proper subspace of V'. Let $x \in E$ such that $\|x\| = 1$ and consider the open set

$$U = \{f \in V' : \|f - \varphi(x)\| < 1\}$$

of V'. Since $\varphi(E) \neq V'$, there exists $f_0 \in U$ such that $f_0 \notin \varphi(E)$. Consequently, $\|f_0 - \varphi(x)\| < 1$. Let f be a norm-preserving linear extension of $f_0 - \varphi(x)$ to E' and let $g = f + \varphi_0(x)$. Then $\|g - \varphi_0(x)\| < 1$ and $g \notin \varphi_0(E)$.

Let Z denote the one dimensional linear subspace generated by g and let $W = V \cap Z_0 = V \cap (\ker g) = \ker f_0$. If $N = Z_0$, then $Z = Z_0{}^0 = N^0$ and, since $W \subset N$, we have $N^0 \subset W^0$. Therefore

$$\begin{aligned} 1 &> \inf\{\|h - \varphi_0(x)\| : h \in N^0,\ x \in E,\ \|x\| = 1\} \\ &\geqq \inf\{\|h - \varphi_0(x)\| : h \in W^0,\ x \in E,\ \|x\| = 1\}. \end{aligned}$$

The latter number, call it r, is the characteristic [2, Theorem 9] of W. If $\tau = \sigma(E, W)$, then S_E is not τ-closed because $1 > r$. Clearly $\tau \subset \sigma(E, V)$. W is $\sigma(V, E)$-dense in V because $f_0 \notin \varphi(E)$. Since V is $\sigma(E', E)$-dense in E', W is $\sigma(E', E)$-dense in E', implying τ is Hausdorff. The contradiction shows E is V-reflexive.

The following characterization of reflexive Banach spaces is an immediate consequence of the preceding theorem.

COROLLARY 1. *A Banach space E is reflexive if and only if S_E is closed in every Hausdorff locally convex topology on E that is comparable to the weak topology of E.*

Recall that a subspace W of E' is called duxial in case its characteristic equals one.

COROLLARY 2. *Let E be a Banach space and V be a total subspace of E'. E is V-reflexive if and only if each total subspace of V is duxial.*

PROOF. Assume E is V-reflexive and let W be a total closed subspace of V. Then S_E is $\sigma(E, W)$-closed, implying W is duxial.

On the other hand, assume each total subspace of V is duxial. Let τ be a Hausdorff locally convex topology comparable with $\sigma(E. V)$. V is itself total so that V is duxial. Therefore S_E is $\sigma(E, V)$-closed. Hence we may assume $\tau \subset \sigma(E, V)$. Letting $W = E_\tau'$, W is total and therefore duxial subspace of E'. Consequently, S_E is $\sigma(E, W)$-closed and hence τ-closed.

COROLLARY 3. *A Banach space E is reflexive if and only if each total subspace of E' is duxial.*

The following internal characterizations of conjugate Banach spaces are stated in terms of the existence of a Hausdroff locally convex topology for which S_E must satisfy some well known compactness conditions. Unfortunately, an analogous statement fails to hold for the Šmulian compactness condition (see Remark after Theorem 5).

THEOREM 5. *Let E be a separable Banach space. The following are equivalent:*

(a) *E is isometrically isomorphic (respectively, topologically isomorphic) to the strong dual of a Banach space.*

(b) *S_E is compact (respectively, relatively compact) for some Hausdorff locally convex topology on E.*

(c) *S_E is sequentially compact (respectively, relatively sequentially compact) for some Hausdorff locally convex topology on E.*

(d) *S_E is countably compact (respectively, relatively, countably compact) for some Hausdorff locally convex topology on E.*

PROOF. It is well-known that (a) and (b) are equivalent (see [7, Theorems 1 and 2] and [13, Proposition 1]). Likewise, the fact that (a) implies (c) is known (see [11, Theorem 4]). Clearly (c) implies (d).

We now show that (d) implies (a). Assume S_E is relatively countably compact for the Hausdorff locally convex topology τ. Let U be a convex, circled and τ-open neighborhood of 0. Assume U does not absorb S_E.

Then there exists a sequence $\{x_n\}$ in S_E such that $x_n \notin nU$ for every n. By hypothesis, $\{x_n\}$ has a τ-cluster point x. Let m be a positive integer such that $x \in mU$. Since $x_n \in x + U \subset (m+1)U$ for infinitely many n, we have a contradiction. It follows that U absorbs S_E, implying that τ is weaker than the norm topology. If $V = E_\tau'$, then V is a total subspace of E and S_E is relatively $\sigma(E, V)$-countably compact. It follows from [4, Theorem 1] that E is topologically isomorphic to the strong dual of V under the canonical embedding. If the stronger condition holds, namely S_E is τ-countably compact, then φ is also an isometry.

Remark. In view of Theorem 5, it is natural to ask if there is an internal characterization of conjugacy in terms of a Šmulian intersection property for S_E. That is, one might conjecture that a separable Banach space E is isometrically isomorphic to the strong dual of a Banach space if and only if there exists a Hausdorff locally convex topology τ on E such that each decreasing sequence of nonempty, τ-bounded, τ-closed convex subsets of S_E has a nonempty intersection. This, however, is not the case. For example, it is well-known that $E = c_0$ is not topologically isomorphic to the strong dual of a Banach space. Let τ be the strongest locally convex topology on E. Suppose $\mathscr{F}$ is any family of nonempty, τ-bounded, τ-closed convex subsets of E directed by inclusion. If $A \in \mathscr{F}$, then A is contained in a finite dimensional subspace of E. Since A is τ-closed, A is τ-compact. Therefore, $\bigcap_{A \in \mathscr{F}} A \neq \emptyset$.

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