UNIFORM APPROXIMATION ON MANIFOLDS

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1. Introduction.

Suppose that $M$ is a real $C^1$-manifold of dimension $m$, and that $\Phi$ is a family of complex-valued $C^1$-functions on $M$. Then the exceptional set, $E(\Phi)$, is the set

$$ \{ x \in M ; \quad df_1 \wedge \ldots \wedge df_m(x) = 0, \quad \forall (f_1, \ldots, f_m) \in \Phi^m \}.$$  

We fix a compact subset $X$ of $M$, and we shall often write $E$ instead of $E(\Phi) \cap X$.

Let $A \subset C(X)$ denote the closed Banach-algebra generated by the restriction to $X$ of the elements of $\Phi$. Assume that $A$ separates points in $X$ and that $M_A = X$, where $M_A$ is the maximal ideal space of $A$. It is an open problem, see [1, pp. 348–349], if $A$ includes all continuous functions on $X$ which vanish identically on $E$. Michael Freeman proved this in [2] under the additional hypothesis that both $M$ and the functions in $\Phi$ are real-analytic. In this work we will solve the problem if $M$ and the functions in $\Phi$ are of class $C^r$, for some sufficiently large real $r$.

Our result will be proved via the following corollary of theorem 3.1: If $\Sigma$ is a $C^r$-manifold in $\mathbb{C}^n$ without complex tangents (see [4] for the precise meaning of the last term) and $K = \sigma(f_1, \ldots, f_m)$ is the spectrum of some members of $A$, then all continuous functions on $K$ which vanish on $K - \Sigma$ operate on $A$.

The proof will follow by adaptation of a technique developed in the work of Hörmander and Wermer [4].

2. Fundamental constructions.

Assume that $r \geq 1$ and that $\Sigma$ is a closed, real $C^r$-sub-manifold, without complex tangents, of an open set $\Omega$ in $\mathbb{C}^n$. Let $N_1$ and $N_2$ be some open sets in $\mathbb{C}^n, \bar{N}_2 \subset N_1$.

The Euclidean distance between the point $x$ and the set $A$ will be denoted $d(x, A)$.

**Lemma 2.1.** Suppose that $u \in C^r(\Omega \cup N_1)$ is holomorphic in $N_1$. Then there exists a $v \in C^r(\Omega \cup N_2)$ with $v = u$ on $\Sigma \cup N_2$ and such that:

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For every compact \( F \subset \Omega \cup N_2 \) and every \( \eta > 0 \) we can find a \( \delta > 0 \) with the property:

If \( z \in F \) and \( d(z, \Sigma) < \delta \), then \( |\bar{\partial}v(z)| \leq \eta \ d(z, \Sigma)^{r-1} \).

**Proof.** This lemma is a restatement of lemma 4.3 in [4].

The next result is similar to theorem 3.1 in [4]. However, since the proof is a bit different, we will carry it out in some detail.

Consider \( \Sigma, \Omega, N_1 \) and \( N_2 \) as above with \( r = 1 \). Suppose \( A \) is a commutative Banach algebra with unit, and let \( f_1, \ldots, f_n \) be elements of \( A \). Define \( K \) to be the joint spectrum \( \sigma(f_1, \ldots, f_n) \).

**Lemma 2.2.** Assume \( K - N_2 \subset \Sigma \). Then there exist real numbers \( \epsilon_0 > 0 \) and \( t \in (0, 1) \), elements \( f_{n+1}, \ldots, f_m \in A \), a compact set \( F \subset \Sigma \), and, for every \( \epsilon \in (0, \epsilon_0) \), a domain of holomorphy \( \omega_\epsilon \Subset \mathbb{C}^m \) such that

(i) \( \sigma(f_1, \ldots, f_m) \subset \omega_\epsilon \subset \mathbb{C}^n \times \{(z_{n+1}, \ldots, z_m) | |(z_{n+1}, \ldots, z_m)| < 1/t\}, \)

(ii) if \( z \in \mathbb{C}^m \) and

\[
d((z_1, \ldots, z_n, \epsilon z_{n+1}, \ldots, \epsilon z_m), \sigma(f_1, \ldots, f_n, \epsilon f_{n+1}, \ldots, \epsilon f_m)) < t \epsilon,
\]

then \( z \in \omega_\epsilon \),

(iii) if \( z \in \omega_\epsilon - (N_1 \times \mathbb{C}^{m-n}) \), then \( d((z_1, \ldots, z_n), F) < \epsilon/t \).

**Proof.** Let \( N_3 \) be an open set such that \( \tilde{N}_2 \subset N_3 \subset \tilde{N}_3 \subset N_1 \). By applying the proof of theorem 3.1 in [4], we get:

An open set \( V \) such that \( K \cap (\tilde{N}_3 - N_2) \subset V \Subset N_1 \), real numbers \( \epsilon_0 > 0 \) and \( t_1 \in (0, 1) \), a compact set \( F \subset \Sigma \), and, for every \( \epsilon \in (0, \epsilon_0) \), a domain of holomorphy \( v_\epsilon \) such that

(i) \( (\sigma(f_1, \ldots, f_n) - N_2) \cup V \subset v_\epsilon \),

(ii) if \( z \in \mathbb{C}^n \) and

\[
d((z_1, \ldots, z_n), \sigma(f_1, \ldots, f_n) - N_2) < t_1 \epsilon,
\]

then \( z \in v_\epsilon \),

(iii) if \( z \in v_\epsilon - N_1 \), then \( d((z_1, \ldots, z_n), F) < \epsilon/t_1 \).

To proceed, we must now study \( K \cap \tilde{N}_2 \). This was done in [4] by imposing a holomorphic convexity condition on this part of \( K \). Instead, we will use the fact that \( K \) is the spectrum of elements from a Banach algebra; therefore the well known Arens–Calderón theorem applies here.

Obviously, \( V_1 = (C^n - \tilde{N}_3) \cup V \cup N_2 \) is an open cover of \( K \). Consequently, we can find an open holomorphically convex set \( U \Subset \mathbb{C}^m \) and elements \( f_{n+1}, \ldots, f_m \in A \) such that

(ii) \( \sigma(f_1, \ldots, f_m) \subset U \subset V_1 \times \mathbb{C}^{m-n} \).
We can now define the required $\omega_\varepsilon$ for every $\varepsilon \in (0, \varepsilon_0)$. Define

$$O_1 = (\mathbb{C}^n - \overline{N_2}) \times \mathbb{C}^{m-n} \quad \text{and} \quad O_2 = N_3 \times \mathbb{C}^{m-n}.$$ 

Since $O_1 \cup O_2 = \mathbb{C}^m$, it is enough to specify $\omega_\varepsilon \cap O_1$ and $\omega_\varepsilon \cap O_2$. We define

(i) $\omega_\varepsilon \cap O_1 = (V_\varepsilon \times \mathbb{C}^{m-n}) \cap U \cap O_1$,

(ii) $\omega_\varepsilon \cap O_2 = U \cap O_2$.

Now we must prove that (i) and (ii) agree on $O_1 \cap O_2$. It suffices to prove that $U \cap O_1 \cap O_2 \subseteq V_\varepsilon \times \mathbb{C}^{m-n}$.

Since

$$U \cap O_1 \cap O_2 \subseteq (V_1 \cap (N_3 - \overline{N_2})) \times \mathbb{C}^{m-n}$$

by (2.2) and since $(V_1 \cap (N_3 - \overline{N_2})) \subseteq V$ by the definition of $V_1$, we have

$$U \cap O_1 \cap O_2 \subseteq V_\varepsilon \times \mathbb{C}^{m-n}$$

by (i) in (2.1).

It is easy to check that (i), (ii), and (iii) of the lemma now hold, if we choose $t > 0$ small enough.

By applying a technique introduced by Nachbin in [5], we shall now determine a class of mappings of a manifold into $\mathbb{C}^n$. More precisely, let $M$ be a $k$-dimensional real $C^r$-manifold, $r \geq 1$. Suppose $X$ is a compact subset of $M$. Assume further that $\Phi \subset C^r_c(M)$ and separates points in $X$. Let $E$ denote the exceptional set $E(\Phi) \cap X$.

**Lemma 2.3.** For every compact set $X_0 \subset X - E$ we can find an open neighbourhood $V$ of $X_0$, a finite number of functions, $f_1, \ldots, f_n \in \Phi$, and an open $\Omega \subset \mathbb{C}^n$ such that

(i) $(f_1, \ldots, f_n)(V)$ is a $\omega$-closed $C^r$-submanifold of $\Omega$ of dimension $k$ and without complex tangents, and

(ii) $(f_1, \ldots, f_n)(X - V) \subset \mathbb{C}^n - \Omega$.

**Proof.** Choose a finite number of functions $f_1, \ldots, f_n \in \Phi$ and an open neighbourhood $V$ of $X_0$ with $E(\{f_1, \ldots, f_n\}) \cap V = \emptyset$. It follows from the inverse mapping theorem that the multiple function $(f_1, \ldots, f_n): M \rightarrow \mathbb{C}^n$ is locally 1-1 on $V$. Obviously, the set

$$\{(x, y) \in (X_0 \times X_0) - \Delta ; f_i(x) = f_i(y), \forall i = 1, \ldots, n\}$$

is a compact subset of $X_0 \times X_0$, where $\Delta$ denotes the diagonal in $X_0 \times X_0$. Consequently, by adding some more functions if necessary, we may assume that $\{f_1, \ldots, f_n\}$ separate points in $X_0$. Shrinking $V$ if necessary, we then get that $(f_1, \ldots, f_n)$ is 1-1 on $\overline{V}$ and that $\overline{V}$ is compact.
Since $\Phi$ separates points in $X$, we can also suppose, after further modifications, that $(f_1, \ldots, f_n)(X - V)$ and $(f_1, \ldots, f_n)(V)$ are disjoint.

The choice $\Omega = C^n - (f_1, \ldots, f_n)(X - V)$ finishes the proof.

3. Approximation theorems.

Let $X$ be a compact Hausdorff space, and let $C(X)$ denote the Banach space under the supremum norm of continuous complex-valued functions. The notation $A \subset C(X)$ means that $A$ is a closed linear subspace which is closed under pointwise multiplication, separates points and contains the constant functions.

If $K$ is a compact subset of $C^n$, then $A(K)$ is defined to be the class of continuous, complex-valued functions on $K$ which can be uniformly approximated on $K$ by functions holomorphic in a neighbourhood of $K$.

**Theorem 3.1.** Suppose $A \subset C(X)$, where $X$ is a compact Hausdorff space. Let $\Sigma$ be a closed $k$-dimensional submanifold of an open set $\Omega \subset C^n$, without complex tangents, and of class $C^r$, $r = \frac{1}{2}k + 1$. Choose $f_1, \ldots, f_n \in A$ and define $K = \sigma(f_1, \ldots, f_n)$, $K_0 = \overline{K - \Sigma}$.

If $u \in C(K)$ with $u|_{K_0} \in A(K_0)$, then $u \circ (f_1, \ldots, f_n) \in A$.

**Remark.** We can replace the condition $r = \frac{1}{2}k + 1$ with $r = \max\{\frac{1}{2}k, 1\}$, but the proof will then be more involved. More specifically, we need a stronger version of lemma 2.2.

**Proof of Theorem 3.1.** Evidently, we may assume that $u \in C^r(\Omega \cup N^1)$ for some open neighbourhood $N_1$ of $K_0$, and also that $u$ is holomorphic in $N_1$. We will further suppose that $u$ has been modified as described in lemma 2.1. If $N_2$ is chosen as an open set with $K_0 \subset N_2 \subset N_1$, we can apply lemma 2.2.

With notations as in lemma 2.2, we will define

$$
\omega'_\varepsilon = \{(z_1, \ldots, z_n, \varepsilon z_{n+1}, \ldots, \varepsilon z_m) ; (z_1, \ldots, z_m) \in \omega_\varepsilon\},
$$

and also

$$
v_\varepsilon : \omega'_\varepsilon \to C \text{ by } (z_1, \ldots, z_m) \mapsto u(z_1, \ldots, z_n).
$$

Condition (iii) in lemma 2.2 ensures that $v_\varepsilon$ is well-defined in $\omega'_\varepsilon$ for all small enough $\varepsilon > 0$.

With this notation, lemma 2.2 says:

(i) $\sigma(f_1, \ldots, f_n, \varepsilon f_{n+1}, \ldots, \varepsilon f_m) \subset \omega'_\varepsilon \subset C^n \times \{(z_{n+1}, \ldots, z_m)| < \varepsilon/t\};$
(ii) if \( z \in \mathbb{C}^m \) and \( d((z_1, \ldots, z_m), \sigma(f_1, \ldots, f_n, \varepsilon f_{n+1}, \ldots, \varepsilon f_m)) < t\varepsilon \), then \( z \in \omega'_\varepsilon \),

(iii) if \( z \in \omega'_\varepsilon - (N_1 \times \mathbb{C}^{m-n}) \), then \( d((z_1, \ldots, z_n), F) < \varepsilon/t \).

Then proceeding exactly as in the proof of theorem 4.1 in [4], we obtain a function \( v'_\varepsilon \) which is holomorphic in \( \omega'_\varepsilon \) and which satisfies

\[
||v'_\varepsilon - v_\varepsilon||_{\omega'_\varepsilon} \to 0 \quad \text{as} \quad \varepsilon \to 0 .
\]

Then, since holomorphic functions operate on \( A \),

\[
v'_\varepsilon \circ (f_1, \ldots, f_n, \varepsilon f_{n+1}, \ldots, \varepsilon f_m) \in A .
\]

Since \( v_\varepsilon \circ (f_1, \ldots, f_n, \varepsilon f_{n+1}, \ldots, \varepsilon f_m) = u \circ (f_1, \ldots, f_n) \), the completeness of \( A \) implies \( u \circ (f_1, \ldots, f_n) \in A \).

We are now able to prove a generalization of a result by Freeman [2].

**Theorem 3.2.** Let \( M \) be a \( k \)-dimensional real manifold of class \( C^r \), \( r = \frac{1}{2}k + 1 \). Suppose that \( \Phi \subset C^r(M) \) separates points on a compact subset \( X \) of \( M \). Define \( E = E(\Phi) \cap X \) and \( A = \text{the sup norm algebra in } C(X) \) generated by \( \Phi \). If \( M_A = X \), then

\[
A = \{ g \in C(X); \ g_{|E} \equiv 0 \} .
\]

**Proof.** Choose any compact subset \( X_0 \subset X - E \) and use lemma 2.3. It follows from theorem 3.1 that the family

\[
\{ g \in A \cap C_R(X); \ g_{|E} \equiv 0 \text{ and } 0 \notin g(X_0) \}
\]

is non-empty and separates points in \( X_0 \). The theorem now is a consequence of Stone–Weierstrass.

**References**


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