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ON CONVERGENCE OF SPECTRAL SEQUENCES

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1. Introduction.

We will study spectral sequences which arise from graded exact couples and we will give criteria for these spectral sequences to converge. There are several different concepts of convergence in the literature; we will use Cartan and Eilenberg's concepts of weak convergence and strong convergence [3]. A condition on the spectral sequence which is particularly interesting is the following:

For each p there is an integer N such that $d_r{}^p = 0$ for all $r \geqq N$.

If that condition is satisfied we will say that the spectral sequence has nice differentials. We will then show this theorem:

THEOREM 1. *Given an exact couple* $(D^p, E^p, i^p, j^p, k^p)$ *such that the resulting spectral sequence has nice differentials and* $\varinjlim \{D^p, i^p\} = 0$. *Then the spectral sequence converges strongly to* $\varprojlim \{D^p, i^p\}$.

As to convergence to $\varinjlim \{D^p, i^p\}$ it is in general very difficult to show that the filtration $(F^p)_{p \in Z}$ on $\varinjlim \{D^p, i^p\}$ is complete. It is always the case that $\bigcup F^p = \varinjlim \{D^p, i^p\}$, but $\bigcap F^p$ need not be zero and $\varprojlim^{(1)} \{F^p\}$ need not be zero.

THEOREM 2. *Given an exact couple* $(D^p, E^p, i^p, j^p, k^p)$ *such that the resulting spectral sequence has nice differentials and* $\varprojlim \{D^p, i^p\} = 0$. *Then the spectral sequence converges weakly to* $\varinjlim \{D^p, i^p\}$.

THEOREM 3. *Given an exact couple* $(D^p, E^p, i^p, j^p, k^p)$ *such that the resulting spectral sequence has nice differentials and* $\varprojlim \{D^p, i^p\} = 0$.

Suppose that for some p there is an integer N such that $\ker \{D^p \to D^{p-N}\} = \ker \{D^p \to D^{p-r}\}$ *for all $r \geqq N$ (where we have assumed that* degree $(i^p) = -1$). *Then the spectral sequence converges weakly to* $\varinjlim \{D^p, i^p\}$ and $\bigcup F^p = \varinjlim \{D^p, i^p\}$ *and* $\bigcap F^p = 0$.

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Finally we will study the special case when for each p there is an integer N such that $E_{r+1}^{p} \subset E_r^p$ for all $r \geqq N$ (or equivalently: all differentials ending at E_r^p are zero if $r \geqq N$). Then, instead of assuming that the spectral sequence has nice differentials, which in this case means that for each p there is an integer N such that $E_N^p = E_{N+1}^p = \ldots = E_\infty^p$, it suffices to assume that for each p

$$\varprojlim_r{}^{(1)}\{E_r^p\} = 0 .$$

There is some overlap with a recent article of Boardman: *Stable Homotopy Theory*, Appendix B *Spectral sequence and images*, John Hopkins University Nov. 1970, where the author studies convergence to $\varinjlim\{D^p, i^p\}$. Theorem 3 is proved there with the assumption that

$$\ker\{D^p \to D^{p-m}\} = \ker\{D^p \to D^{p-n}\}$$

for all sufficiently big m and n replaced by the following condition on the differentials:
There are integers M and N such that $d_{m+n}^{-m}: E_{m+n}^{-m} \to E_{m+n}^{s+t+n-1}$ is zero for all $m > M$ and $n > N$.

2. Review of exact couples and spectral sequences.

The construction of a spectral sequence from an exact couple was done by Massey [9]. Eckmann and Hilton have generalized this construction to an arbitrary abelian category which has countable unions and intersections [4]. We will assume that the category is the category of all graded (or bigraded) R-modules or a category of functors into one of those categories. In the last case we talk about functorial exact couples, functorial spectral sequences etc. Without loss of generality we will always assume that the exact couples $(D^p, E^p, i^p, j^p, k^p)$ are such that degree$(i^p) = -1$. We put degree$(j^p) = s$ and degree$(k^p) = t$, so

$$i^p : D^p \to D^{p-1},\ j^p : D^p \to E^{p+s} \quad \text{and} \quad k^p : E^p \to D^{p+t} .$$

Definition. Given an exact couple $(D^p, E^p, i^p, j^p, k^p)$. The *derived couple* $(D_2^p, E_2^p, i_2^p, j_2^p, k_2^p)$ is defined by:

$$D_2 = i(D), \quad E_2 = k^{-1}(i(D))/j(i^{-1}(0)) ,$$
$$i_2 = i, \quad j_2 = j \circ i^{-1}, \quad k_2 = k .$$

D_2 is graded such that the degree of the inclusion $D_2 \to D$ is zero and E_2 is graded such that degree$(k_2) = $ degree(k).

So E_2 is constructed by first making a pull back construction and then a push out:

$$\begin{array}{ccccc}
D & \xrightarrow{j} & E & \xrightarrow{k} & D \\
\Big| & & \uparrow & & \uparrow \\
D & \longrightarrow & k^{-1}(i(D)) & \longrightarrow & i(D)=D_2 \\
{\scriptstyle i}\downarrow & & \downarrow & & \Vert \\
i(D)=D_2 & \xrightarrow{j_2} & E_2 & \xrightarrow{k_2} & D_2
\end{array}$$

Repeating the construction and putting $d_r = j_r \circ k_r$ we get a spectral sequence $(E_r{}^p, d_r{}^p)_{r \geqq 1}$. For $r=1$ we have $E_1 = E$ and $d_1 = j \circ k$. Furthermore

$$\operatorname{degree}(d_r) = \operatorname{degree}(j) + \operatorname{degree}(k) + r - 1 ,$$

so $d_r{}^p : E_r{}^p \to E_r^{p+s+t+r-1}$.

Definition. We put

$$\begin{aligned}
U &= D/\textstyle\bigcup i^{-n}(0) \text{ graded such that } D \to U \text{ has degree } 0 , \\
I &= \textstyle\bigcap i^n(D) \text{ graded such that } I \to D \text{ has degree } 0 , \\
E_\infty &= k^{-1}(I)/j(\ker\{D \to U\}) .
\end{aligned}$$

So E_∞ is constructed in the same way as the derived couple and graded such that $\operatorname{degree}(k_\infty) = \operatorname{degree}(k)$ and $\operatorname{degree}(j_\infty) = \operatorname{degree}(j)$:

$$\begin{array}{ccccc}
D & \xrightarrow{j} & E & \xrightarrow{k} & D \\
\downarrow & & & & \uparrow \\
U & \xrightarrow{j_\infty} & E_\infty & \xrightarrow{k_\infty} & I
\end{array}$$

The homomorphism $i : D \to D$ induces homomorphisms $I \to I$ and $U \to U$ of degree -1. The next two propositions are due to Eckmann and Hilton [4].

Proposition 2.1. *There is a short exact sequence*

$$0 \to U/\operatorname{im}\{U \to U\} \longrightarrow E_\infty \longrightarrow \ker\{I \to I\} \to 0$$

where $U/\operatorname{im}\{U \to U\} \to E_\infty$ *is induced by* j_∞ *and* $E_\infty \to \ker\{I \to I\}$ *is* k_∞.

The canonical homomorphism $D^p \to \varinjlim\{D^p, i^p\}$ factors through U^p so we have a homomorphism $\bar{\alpha}^p : U^p \to \varinjlim\{D^p, i^p\}$, and the canonical projection $\varprojlim\{D^p, i^p\} \to D^p$ maps into I^p so we have a homomorphism

$\bar{\beta}^p:\varprojlim\{D^p,i^p\}\to I^p$. We define filtrations $(F^p)_{p\in\mathsf{Z}}$ and $(G^p)_{p\in\mathsf{Z}}$ on $\varinjlim\{D^p,i^p\}$ and $\varprojlim\{D^p,i^p\}$ respectively by:

$$F^p = \operatorname{im}\{D^p\to\varinjlim\{D^p,i^p\}\},\qquad G^p = \ker\{\varprojlim\{D^p,i^p\}\to D^p\}\,.$$

Then $\bar{\alpha}^p$ induces

$$\alpha^p: U^p/\operatorname{im}\{U^{p+1}\to U^p\}\to F^p/F^{p+1}$$

and $\bar{\beta}^p$ induces

$$\beta^p: G^{p-1}/G^p\to\ker\{I^p\to I^{p-1}\}\,.$$

Proposition 2.2. *For each p, α^p is an isomorphism and β^p is a monomorphism.*

In general it is not the case that all β^p are isomorphisms:

Example 2.3. Let (D^p,E^p,i^p,j^p,k^p) be an exact couple with $D^p=\bigoplus_p^\infty\mathsf{Z}$ for $p>0$, $D^0=\mathsf{Z}$ and $D^p=0$ for $p<0$. Let $i^p:D^p\to D^{p-1}$ be the inclusion if $p>1$, $i^p=0$ if $p<1$ and $i^1:D^1\to D^0$ sends (a_n) to $\sum_n a_n$. Then we get $I^0=\mathsf{Z}$ and $I^{-1}=0$ so $\ker\{I^0\to I^{-1}\}=\mathsf{Z}$ but $\varprojlim\{D^p,i^p\}=0$ so β^0 is not an isomorphism.

3. The concept of convergence.

The following terminology is introduced by Cartan and Eilenberg [3].

Definition. We say that the spectral sequence *converges weakly* to $\varinjlim\{D^p,i^p\}$ if the homomorphisms

$$F^p/F^{p+1}\xleftarrow{\cong} U^p/\operatorname{im}\{U^{p+1}\to U^p\}\to E_\infty{}^{p+s}$$

are isomorphisms for all p. We say that it *converges weakly* to $\varprojlim\{D^p,i^p\}$ if the additive relations

$$G^{p-1}/G^p\to\ker\{I^p\to I^{p-1}\}\leftharpoondown E_\infty{}^{p-t}$$

are isomorphisms for all p.

There is a necessary and sufficient condition for weak convergence to $\varinjlim\{D^p,i^p\}$ which is due to Cartan and Eilenberg [3]: The spectral sequence converges weakly to $\varinjlim\{D^p,i^p\}$ if and only if $I^p\to I^{p-1}$ is a monomorphism for each p. The proof is immediate from proposition 2.1. As to convergence to $\varprojlim\{D^p,i^p\}$ there is no equally simple condition, since the homomorphisms β^p are not in general isomorphisms.

If the spectral sequence converges weakly one may expect the $E_\infty{}^p$-

terms to determine $\varinjlim\{D^p, i^p\}$ (or $\varprojlim\{D^p, i^p\}$) uniquely up to extensions. This is in general not the case unless one puts certain conditions on the filtrations $(F^p)_{p\in \mathsf{Z}}$ (or $(G^p)_{p\in\mathsf{Z}}$):

Definition. Given a decreasing filtration $(F^p)_{p\in\mathsf{Z}}$ on an object F, i.e. $\ldots \subset F^{p+1} \subset F^p \subset \ldots \subset F$. We say that the filtration is *complete* if $\bigcup F^p = F$ and if the homomorphism $F \to \varprojlim\{F/F^p\}$ is an isomorphism.

The condition that $F \to \varprojlim\{F/F^p\}$ should be an isomorphism is equivalent to the following condition: $\bigcap F^p = 0$ and $\varprojlim^{(1)}\{F^p\} = 0$. It is clear that if the filtration is finite and if $\bigcap F^p = 0$ then it is complete, however, the standard filtration on the p-adic integers is an example of a filtration which is complete but which is not finite. If F is a countable module, then every complete filtration on F must satisfy Mittag–Leffler's condition (see proposition 4.1) and so $F^p = 0$ for some p. We recall the following result due to Eilenberg and Moore [5].

Proposition 3.1. *Let $(F^p)_{p\in\mathsf{Z}}$ be a complete filtration on F and let $(G^p)_{p\in\mathsf{Z}}$ be a complete filtration on G. Suppose that $f: F \to G$ is a filtration preserving homomorphism which induces isomorphisms $F^p/F^{p+1} \to G^p/G^{p+1}$ for all p. Then f is an isomorphism.*

We remark that it suffices to assume that the filtration $(G^p)_{p\in\mathsf{Z}}$ is such that $\bigcup G^p = G$ and $\bigcap G^p = 0$. In fact, if $\bigcup F^p = F$ and $\bigcap F^p = 0$ then f is a monomorphism, and if $\varprojlim^{(1)}\{F^p\} = 0$ and $\bigcup G^p = G$ and $\bigcap G^p = 0$ then f is an epimorphism. Following Cartan and Eilenberg [3] we use the following terminology.

Definition. We say that the spectral sequence *converges strongly* to $\varinjlim\{D^p, i^p\}$ if it converges weakly to $\varinjlim\{D^p, i^p\}$ and if the filtration $(F^p)_{p\in\mathsf{Z}}$ on $\varinjlim\{D^p, i^p\}$ is complete. We say that it *converges strongly* to $\varprojlim\{D^p, i^p\}$ if it converges weakly to $\varprojlim\{D^p, i^p\}$ and if the filtration $(G^p)_{p\in\mathsf{Z}}$ on $\varprojlim\{D^p, i^p\}$ is complete.

Then, if $(E_r{}^p, d_r{}^p)_{r>1}$ is a functorial spectral sequence which converges strongly to $\varinjlim\{D^p, i^p\}$ (or to $\varprojlim\{D^p, i^p\}$) and if $f: X \to Y$ is such that $E_\infty{}^p(X) \to E_\infty{}^p(Y)$ is an isomorphism for each p, then $\varinjlim D^p(X) \to \varinjlim D^p(Y)$ ($\varprojlim D^p(X) \to \varprojlim D^p(Y)$) is an isomorphism.

Example 3.2. Let X_p be the unit circle S^1 and let $f_p: X_p \to X_{p-1}$ be the identity if $p \leqq 0$ and be $f_p(x) = x^2$ if $p > 0$. Put $D^p = H_*(X_p; \mathsf{Z})$, $E^p = H_*(C_{f_p}; \mathsf{Z})$ and define the homomorphisms i^p, j^p and k^p using the exact homology sequences of the maps f_p. Then we get an exact couple $(D^p, E^p, i^p, j^p, k^p)$ with $D^p = \mathsf{Z}$ and $i^p : D^p \to D^{p-1}$ is the identity for $p \leqq 0$ and is multiplication by 2 if $p > 0$. So $I^p = 0$ for all p, and so the spectral sequence converges weakly to $\varinjlim\{D^p, i^p\} = \mathsf{Z}$ according to the criterion of Cartan and Eilenberg. $F^p = \mathsf{Z}$ for $p \leqq 0$ and $F^p = 2^p\mathsf{Z}$ for $p > 0$ so $\bigcup F^p = F$ and $\bigcap F^p = 0$ but $\varprojlim^{(1)}\{F^p\} \neq 0$ so the filtration is not complete. The map $f : S^1 \to S^1$ given by $f(x) = x^3$ induces a homomorphism of the spectral sequence into itself (in fact, the construction above is functorial on the category of topological groups) which induces isomorphisms of the $E_\infty{}^p$-terms. But the homomorphism $\varinjlim\{D^p, i^p) \to \varinjlim\{D^p, i^p\}$ is multiplication by $3 : \mathsf{Z} \to \mathsf{Z}$ so it is not onto. This example shows that even in the category of finitely generated abelian groups, it does not suffice to assume $\bigcup F^p = F$ and $\bigcap F^p = 0$ to assure that the $E_\infty{}^p$-terms determine $\varinjlim\{D^p, i^p\}$ up to extensions.

In general it is difficult to say anything concerning strong convergence to $\varinjlim\{D^p, i^p\}$. We have defined $F^p = \operatorname{im}\{D^p \to \varinjlim\{D^p, i^p\}\}$, so it is always the case that $\bigcup F^p = F$. Example 3.2 shows that $\varprojlim^{(1)}\{F^p\}$ need not be zero, even if the spectral sequence is weakly convergent and $\bigcap F^p = 0$. It may also be the case that $\bigcap F^p \neq 0$ as the following example shows:

Example 3.3. Let $(D^p, E^p, i^p, j^p, k^p)$ be an exact couple such that $D^p = \bigoplus_p^\infty \mathsf{Z}$ for $p > 0$ and $D^p = \bigoplus_0^\infty \mathsf{Z}$ for $p \leqq 0$ and $i^p : D^p \to D^{p-1}$ is defined as follows:

i^p is the inclusion for $p > 0$ and

$$i^p : \bigoplus_0^\infty \mathsf{Z} \to \bigoplus_0^\infty \mathsf{Z}$$

for $p \leqq 0$ is defined by sending (a_n) to (b_n), where $b_0 = a_0 - a_1$ and $b_n = a_{n+1}$ for $n > 0$.

Then $I^p = 0$ so the spectral sequence converges weakly to $\varinjlim\{D^p, i^p\}$ according to the criterion of Cartan and Eilenberg. If we have degree(j^p) $= 1$ and degree$(k^p) = 0$ then $E^p = \mathsf{Z}$ for all p, and for each p there is an integer r such that $E^p = E_2{}^p = \ldots = E_r{}^p = \mathsf{Z}$ and $E^p_{r+1} = E^p_{r+2} = \ldots = E^p_\infty = 0$. But

$$F^p = \operatorname{im}\{D^p \to \varinjlim\{D^p, i^p\}\} = \varinjlim\{D^p, i^p\}$$

so $\bigcap F^p \neq 0$. So this example shows us an exact couple with $\varprojlim\{D^p, i^p\} = 0$ which converges weakly to $\varinjlim\{D^p, i^p\}$, and such that the filtration on $\varinjlim\{D^p, i^p\}$ is finite, but $\bigcap F^p$ is not zero.

Example 3.3 shows that one has to put additional conditions on the exact couple in order to be able to say something on the filtration $(F^p)_{p\in Z}$. We will study one such condition in paragraph 4.

We now turn to the filtration $(G^p)_{p\in Z}$ on $\varprojlim\{D^p,i^p\}$. We have defined $G^p=\ker\{\varprojlim\{D^p,i^p\}\to D^p\}$ so it follows at once that $\bigcap G^p=0$. There are short exact sequences

$$0\to G^p\to\varprojlim\{D^p,i^p\}\to X^p\to 0 \qquad 0\to X^p\to D^p\to Y^p\to 0$$

so in the following diagram the row and the column are exact:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ 0 & \to & \bigcap G^p & \to \varprojlim\{D^p,i^p\} \to & \varprojlim\{X^p,i^p|X^p\} & \to & \varprojlim^{(1)}\{G^p\} & \to & 0 \\ & & & & \downarrow & & & & \\ & & & & \varprojlim\{D^p,i^p\} & & & & \end{array}$$

The homomorphism $\varprojlim\{D^p,i^p\}\to\varprojlim\{D^p,i^p\}$ is induced by the canonical projections $\varprojlim\{D^p,i^p\}\to D^p$ so it is the identity, and so $\varprojlim^{(1)}\{G^p\}=0$.

In general, of course, $\bigcup G^p$ need not be $\varprojlim\{D^p,i^p\}$, so we make the extra assumption that the homomorphism $\varprojlim\{D^p,i^p\}\to\varinjlim\{D^p,i^p\}$ is zero. Since the sequences

$$0\to G^p\to\varprojlim\{D^p,i^p\}\to D^p$$

are exact it follows that

$$0\to\bigcup G^p\to\varprojlim\{D^p,i^p\}\to\varinjlim\{D^p,i^p\}$$

is exact, so $\bigcup G^p=\varprojlim\{D^p,i^p\}$ and we have proved the following proposition:

PROPOSITION 3.4. *Given an exact couple (D^p,E^p,i^p,j^p,k^p) such that $\varprojlim\{D^p,i^p\}\to\varinjlim\{D^p,i^p\}$ is zero. Then, if the spectral sequence converges weakly to $\varprojlim\{D^p,i^p\}$, it converges strongly to $\varprojlim\{D^p,i^p\}$.*

4. When the differentials are zero.

We will now study the following condition on the spectral sequence: For each p there is an integer N such that if $r\geqq N$ then $d_r{}^p=0$. If that condition is satisfied we say that the spectral sequence has *nice* differentials.

LEMMA 4.1. *If the spectral sequence has nice differentials then for each p there is an integer N such that if $r \geqq N$ then*

$$\operatorname{im}\{D^{p+N} \to D^p\} \cap \ker\{D^p \to D^{p-1}\} \subset \operatorname{im}\{D^{p+r} \to D^p\}.$$

PROOF. If $d_n{}^p = 0$ then $E_n{}^p \to D_n{}^{p+t} \to E_n{}^{p+s+t+n-1}$ is zero, so

$$\begin{aligned}\ker\{D_n{}^{p+t} \to D_n{}^{p+t-1}\} &= \operatorname{im}\{E_n{}^p \to D_n{}^{p+t}\} \subset \ker\{D_n{}^{p+t} \to E_n{}^{p+s+t+n-1}\} \\ &= \operatorname{im}\{D_n{}^{p+t+1} \to D_n{}^{p+t}\},\end{aligned}$$

and so

$$\operatorname{im}\{D^{p+t+n-1} \to D^{p+t}\} \cap \ker\{D^{p+t} \to D^{p+t-1}\} \subset \operatorname{im}\{D^{p+t+n} \to D^{p+t}\}$$

and the lemma follows.

LEMMA 4.2. *If the spectral sequence has nice differentials then for each p and each n there is an integer N such that if $r \geqq N$ then*

$$\operatorname{im}\{D^{p+N} \to D^p\} \cap \ker\{D^p \to D^{p-n}\} \subset \operatorname{im}\{D^{p+r} \to D^p\}.$$

PROOF. Induction.

We will need the following well known result:

PROPOSITION 4.3. *Given an inverse system of modules $\{A^p, f^p\}_{p \in \mathbf{Z}}$ which satisfies Mittag-Leffler's condition:*
(ML) *For each p there is an integer N such that if $m, n \geqq N$ then*

$$\operatorname{im}\{A^m \to A^p\} = \operatorname{im}\{A^n \to A^p\}.$$

Then, $\varprojlim^{(1)}\{A^p\} = 0$.
Conversely, if $\varprojlim^{(1)}\{A^p\} = 0$ *and if A^p is countable for all p, then $\{A^p, f^p\}_{p \in \mathbf{Z}}$ satisfies* (ML).

PROOF. See Grothendieck [7] and Gray [6].

Now, put $K^{p,n} = \ker\{D^{p+n} \to D^p\}$ for $n \geqq 0$. Then $\{K^{p,n}, i^p | K^{p,n}\}_{n \in \mathbf{Z}}$ is an inverse system.

PROPOSITION 4.4. *If the spectral sequence has nice differentials then $\{K^{p,n}, i^p | K^{p,n}\}_{n \in \mathbf{Z}}$ satisfies* (ML) *for all p, hence* $\varprojlim_n{}^{(1)}\{K^{p,n}, i^p | K^{p,n}\} = 0$ *for all p.*

PROOF. For each p and for each n there is an integer N such that

$$\operatorname{im}\{D^{p+n+N} \to D^{p+n}\} \cap \ker\{D^{p+n} \to D^p\} \subset \operatorname{im}\{D^{p+n+r} \to D^{p+n}\}$$

for all $r \geqq N$. But

$$\operatorname{im}\{D^{p+n+N} \to D^{p+n}\} \cap \ker\{D^{p+n} \to D^p\} = \operatorname{im}\{K^{p,n+N} \to K^{p,n}\}$$

and

$$\operatorname{im}\{D^{p+n+r} \to D^{p+n}\} \cap \ker\{D^{p+n} \to D^p\} = \operatorname{im}\{K^{p,n+r} \to K^{p,n}\} .$$

In particular, if $D^p = 0$ for $p < 0$ then $K^{-1,n} = D^{n-1}$ and so we have the following well known result (see e.g. Bröcker and tom Dieck [1]): If the spectral sequence has nice differentials and if $D^p = 0$ for $p < 0$ then $\{D^p, i^p\}_{p \in \mathsf{Z}}$ satisfies (ML) and so $\varprojlim^{(1)}\{D^p, i^p\} = 0$.

One may also show that if $\{D^p, i^p\}_{p \in \mathsf{Z}}$ satisfies (ML) then the spectral sequence has nice differentials. This result does not require D^p to be zero for $p < 0$.

Proposition 4.5. *If the spectral sequence has nice differentials then* $\beta^p : G^{p-1}/G^p \to \ker\{I^p \to I^{p-1}\}$ *is an isomorphism for all* p.

Proof. There are short exact sequences

$$0 \to K^{p,n} \to D^{p+n} \to D_{n+1}{}^p \to 0 ,$$

so passing to the limit we get an exact sequence

$$0 \to \varprojlim_n \{K^{p,n}, i^p \mid K^{p,n}\} \to \varprojlim \{D^p, i^p\} \to I^p \to \varprojlim_n{}^{(1)} \{K^{p,n}, i^p \mid K^{p,n}\} \to \ldots ,$$

but $\varprojlim^{(1)}\{K^{p,n}, i^p \mid K^{p,n}\} = 0$ and so $\varprojlim\{D^p, i^p\} \to I^p$ is onto.

Theorem 1. *Given an exact couple* $(D^p, E^p, i^p, j^p, k^p)$ *such that the spectral sequence has nice differentials. Then the spectral sequence converges weakly to* $\varprojlim\{D^p, i^p\}$ *if and only if* $D^p \to \varinjlim\{D^p, i^p\}$ *is onto for all* p, *and it converges strongly to* $\varprojlim\{D^p, i^p\}$ *if and only if* $D^p \to \varinjlim\{D^p, i^p\}$ *is onto for all* p *and* $\varprojlim\{D^p, i^p\} \to \varinjlim\{D^p, i^p\}$ *is zero.*

Proof. $\beta^p : G^{p-1}/G^p \to \ker\{I^p \to I^{p-1}\}$ is an isomorphism for all p, so the spectral sequence is weakly convergent if and only if $F^p/F^{p+1} = 0$ for all p which is equivalent to $D^p \to \varinjlim\{D^p, i^p\}$ being onto for all p. The rest of the theorem follows from proposition 3.4.

Corollary. *Given an exact couple* $(D^p, E^p, i^p, j^p, k^p)$ *such that the spectral sequence has nice differentials and* $\varinjlim\{D^p, i^p\} = 0$. *Then the spectral sequence converges strongly to* $\varprojlim\{D^p, i^p\}$.

Corollary. *Given an exact couple* $(D^p, E^p, i^p, j^p, k^p)$ *such that the spectral* *sequence has nice differentials and* $\varinjlim\{D^p,i^p\}=0$. *Suppose that for some* p *there is an integer* N *such that* $\ker\{D^p\to D^{p-N}\}=\ker\{D^p\to D^{p-r}\}$ *for all* $r\geqq N$. *Then the spectral sequence converges strongly to* $\varprojlim\{D^p,i^p\}$ *and* $\varprojlim^{(1)}\{D^p,i^p\}=0$.

PROOF. It is clear that

$$\ker\{D^p\to D^{p-N}\} = \ker\{D^p\to \varinjlim\{D^p,i^p\}\} = D^p ,$$

so $K^{p-N,n}=D^{p-N+n}$ for $n\geqq N$, and so $\varprojlim^{(1)}\{D^p,i^p\}=0$.

THEOREM 2. *Given an exact couple* (D^p,E^p,i^p,j^p,k^p) *such that the spectral sequence has nice differentials. Then the spectral sequence converges weakly to* $\varinjlim\{D^p,i^p\}$ *if and only if* $\varprojlim\{D^p,i^p\}\to D^p$ *is mono for all* p.

PROOF. $\beta^p:G^{p-1}/G^p\to\ker\{I^p\to I^{p-1}\}$ is an isomorphism for all p, so the spectral sequence is weakly convergent to $\varinjlim\{D^p,i^p\}$ if and only if $G^{p-1}/G^p=0$ for all p, which is equivalent to G^p being zero for all p.

COROLLARY. *Given an exact couple* (D^p,E^p,i^p,j^p,k^p) *such that the spectral sequence has nice differentials and* $\varprojlim\{D^p,i^p\}=0$. *Then the spectral sequence converges weakly to* $\varinjlim\{D^p,i^p\}$.

THEOREM 3. *Given an exact couple* (D^p,E^p,i^p,j^p,k^p) *such that the spectral sequence has nice differentials,* $\varprojlim\{D^p,i^p\}\to D^p$ *is mono for all* p *and such that for some* p *there is an integer* N *such that* $\ker\{D^p\to D^{p-N}\}=\ker\{D^p\to D^{p-r}\}$ *for all* $r\geqq N$. *Then the homomorphism* $\varprojlim\{D^p,i^p\}\to\bigcap F^p$ *is an isomorphism.*

PROOF. $\varprojlim\{D^p,i^p\}\to D^p$ is a monomorphism for each p, so

$$\varprojlim\{D^p,i^p\}\to\varinjlim\{D^p,i^p\}$$

is a monomorphism. If $x\in\bigcap F^p$ there are elements $x^p\in D^p$ which map to x in $\varinjlim\{D^p,i^p\}$. Without loss of generality we may assume that there is an integer N such that $\ker\{D^0\to D^{-N}\}=\ker\{D^0\to D^{-r}\}$ for all $r\geqq N$. Then, for $p\geqq 0$ let x^p map to y^p in D^0. Since the elements y^p map to the same element x in $\varinjlim\{D^p,i^p\}$, they map to the same element y in D^{-N}. So $y\in I^{-N}$ and since $\varprojlim\{D^p,i^p\}\to I^{-N}$ is onto,

$$x\in\operatorname{im}\{\varprojlim\{D^p,i^p\}\to\varinjlim\{D^p,i^p\}\} .$$

COROLLARY. *Given an exact couple* (D^p,E^p,i^p,j^p,k^p) *such that the spectral sequence has nice differentials,* $\varprojlim\{D^p,i^p\}=0$ and $i^p:D^p\to D^{p-1}$ *is a* *monomorphism for* $p<0$. *Then the spectral sequence converges weakly to* $\varinjlim\{D^p, i^p\}$ *and* $\bigcap F^p = 0$.

COROLLARY. *Given an exact couple* $(D^p, E^p, i^p, j^p, k^p)$ *in the category of graded R-modules, where R is a noetherian ring, such that* D^p *is finitely generated for some p. Then, if the spectral sequence has nice differentials and if* $\varprojlim\{D^p, i^p\}=0$, *it converges weakly to* $\varinjlim\{D^p, i^p\}$ *and* $\bigcap F^p=0$.

COROLLARY. *Given an exact couple* $(D^p, E^p, i^p, j^p, k^p)$ *such that the spectral sequence has nice differentials,* $\varprojlim\{D^p, i^p\}=0$, $\varprojlim^{(1)}\{D^p, i^p\}=0$ *and such that for some p there is an integer N such that*

$$\ker\{D^p \to D^{p-N}\} = \ker\{D^p \to D^{p-r}\} \quad \textit{for all } r \geqq N\,.$$

Then the spectral sequence converges strongly to $\varinjlim\{D^p, i^p\}$.

COROLLARY. *Given an exact couple* $(D^p, E^p, i^p, j^p, k^p)$ *such that the spectral sequence has nice differentials,* $\varprojlim\{D^p, i^p\}=0$, $E_\infty^p=0$ *for all but finitely many p and such that for each p there is an integer N such that*

$$\ker\{D^p \to D^{p-N}\} = \ker\{D^p \to D^{p-r}\} \quad \textit{for all } r \geqq N\,.$$

Then the spectral sequence converges strongly to $\varinjlim\{D^p, i^p\}$.

The following example is due to Eilenberg and Moore [5].

EXAMPLE 4.6. Let A be a differential filtered group, and suppose that the filtration $(F^pA)_{p\in \mathbf{Z}}$ is complete. Put $D^p = H(F^pA)$ and $E^p = H(F^pA/F^{p+1}A)$. Then we get an exact couple and $\varinjlim\{D^p, i^p\} = H(A)$ and the filtration on $H(A)$: $F^p = \operatorname{im}\{H(F^pA) \to H(A)\}$ is complete. Hence, if the spectral sequence has nice differentials it converges strongly to $\varinjlim\{D^p, i^p\} = H(A)$.

5. When $E_{r+1}^{\,p} \subset E_r^{\,p}$ for r sufficiently big.

We will now study the special case when for each p there is an integer N such that

$$d_r^{\,p-s-t-r+1} : E_r^{\,p-s-t-r+1} \to E_r^{\,p}$$

is zero for all $r \geqq N$, that is $E_{r+1}^{\,p} \subset E_r^{\,p}$ for $r \geqq N$. If the spectral sequence has nice differentials, this means that for each p there is an integer N such that $E_N^{\,p} = E_{N+1}^{\,p} = \ldots = E_\infty^{\,p}$. Now, assume that for each p, $\varprojlim_r^{(1)}\{E_r^{\,p}\} = 0$. Put

$$C_r^p = E_r^p/\operatorname{im}\{D_r^{p-s-r+1} \to E_r^p\}.$$

Then $\varprojlim_r^{(1)}\{C_r^p\} = 0$ for all p, and there are short exact sequences

$$0 \to C_r^p \to D_r^{p+t} \to D_{r+1}^{p+t-1} \to 0,$$

and so there is an exact sequence

$$0 \to \varprojlim_r\{C_r^p\} \to I^{p+t} \to I^{p+t-1} \to \varprojlim_r^{(1)}\{C_r^p\} \to \ldots$$

Hence, $I^p \to I^{p-1}$ is onto for all p, so $\varprojlim\{D^p, i^p\} \to I^p$ is onto for all p, and so

$$\beta^p : G^{p-1}/G^p \to \ker\{I^p \to I^{p-1}\}$$

is an isomorphism for all p. Hence we have proved most of the following theorem:

THEOREM 4. *Given an exact couple $(D^p, E^p, i^p, j^p, k^p)$ such that in the spectral sequence for each p there is an integer N such that $E_{r+1}^p \subset E_r^p$ for all $r \geqq N$. Then, in theorem 1, 2 and 3 and their corollaries, the condition that the spectral sequence should have nice differentials can be replaced by the following condition: For each p, $\varprojlim_r^{(1)}\{E_r^p\} = 0$.*

The proof of the second corollary of theorem 1 requires a new argument: $\varprojlim_r^{(1)}\{E_r^p\} = 0$ for all p, so

$$\varprojlim_r^{(1)}\{D_r^p\} \to \varprojlim_r^{(1)}\{D_r^{p-1}\}$$

is an isomorphism for all p. But for some p there is an integer N such that $D^p \to D^{p-N}$ is zero, hence $\varprojlim_r^{(1)}\{D_r^{p-N}\} = 0$, hence $\varprojlim_r^{(1)}\{D_r^p\} = 0$ for all p. The homomorphism $I^p \to I^{p-1}$ is onto for each p, and so it follows that $\varprojlim_p^{(1)}\{D^p, i^p\} = 0$. (This can be shown by picking convergent sequences in complete modules. In the work of Boardman there is a stronger result: There is a short exact sequence:

$$0 \to \varprojlim^{(1)}\{I^p\} \to \varprojlim^{(1)}\{D^p, i^p\} \to \varprojlim_p\{\varprojlim_r^{(1)}\{D_r^p\}\} \to 0.)$$

One may ask if the condition that $\varprojlim_r^{(1)}\{E_r^p\} = 0$ for all p is necessary for convergence. This is in general not the case, which the following example shows:

EXAMPLE 5.1. Let $(D^p, E^p, i^p, j^p, k^p)$ be an exact couple such that $D^p = \mathbf{Z}$ for $p \geqq 0$, $D^p = 0$ for $p < 0$, $i^p : D^p \to D^{p-1}$ is zero if $p \leqq 0$ and is multiplication by 2 if $p > 0$. Assume that degree$(j^p) = 1$ and that degree$(k^p) = 0$. Then $E_\infty^p = 0$ for all p, and since $\varprojlim\{D^p, i^p\} = 0$ and

$\varinjlim\{D^p, i^p\} = 0$ the spectral sequence converges strongly both to $\varprojlim\{D^p, i^p\}$ and to $\varinjlim\{D^p, i^p\}$. But $E_r{}^0 = 2^{r-1}\mathbf{Z}$ and so $\varprojlim_r{}^{(1)}\{E_r{}^0\} \neq 0$.

The following proposition applies to the Atiyah–Hirzebruch spectral sequence:

Proposition 5.2. *Let* $(D^p, E^p, i^p, j^p, k^p)$ *be an exact couple such that* $D^p = 0$ *for* $p < 0$. *Then* $\varprojlim_r{}^{(1)}\{E_r{}^p\} = 0$ *for all* p *if and only if the spectral sequence converges strongly to* $\varprojlim\{D^p, i^p\}$ *and* $\varprojlim^{(1)}\{D^p, i^p\} = 0$.

Proof. If the spectral sequence is strongly convergent and if $\varprojlim^{(1)}\{D^p, i^p\} = 0$, then $\varprojlim_r{}^{(1)}\{D_r{}^p\} = 0$ for all p. We show by induction that the homomorphism $\varprojlim\{D^p, i^p\} \to I^p$ is onto for all p. For $r > p - s + 1$ there are short exact sequences

$$0 \to E_r{}^p \to D_r{}^{p+t} \to D_{r+1}^{p+t-1} \to 0,$$

and so there is an exact sequence

$$0 \to E_\infty{}^p \to I^{p+t} \to I^{p+t-1} \to \varprojlim_r{}^{(1)}\{E_r{}^p\} \to \varprojlim_r{}^{(1)}\{D_r{}^{p+t}\} = 0,$$

and $I^{p+t} \to I^{p+t-1}$ is onto, so $\varprojlim_r{}^{(1)}\{E_r{}^p\} = 0$.

Example 5.3. Let h be a cohomology theory and let X be a CW-complex. Then the bigraded exact couple defined by $D^{p,q} = h^{p+q}(X^p)$, $E^{p,q} = h^{p+q}(X^p, X^{p-1})$ etc., generates the Atiyah–Hirzebruch spectral sequence. Then it follows from proposition 5.2 that the spectral sequence converges to $\varprojlim\{h^n(X^p)\}$ and the natural map

$$h^{n+1}(X) \to \varprojlim\{h^{n+1}(X^p)\}$$

is an isomorphism if and only if $\varprojlim_r{}^{(1)}\{E_r{}^{p,n-p}\} = 0$ for all p. In particular, if $\varprojlim_r{}^{(1)}\{E_r{}^{p,n-p}\} = 0$ and $\varprojlim_r{}^{(1)}\{E_r{}^{n-1-p}\} = 0$ for all p then the spectral sequence converges strongly in degree n to $h^n(X)$.

Remark. If, in the example above, h has countable coefficients and X has finite sceletons then $D^{p,q}$ and $E^{p,q}$ are countable abelian groups, and so $\varprojlim_p{}^{(1)}\{D^{p,n-p}\} = 0$ if and only if it satisfies (ML) which is equivalent to the spectral sequence having nice differentials in degree n. Hence, the map

$$h^{n+1}(X) \to \varprojlim\{h^{n+1}(X^p)\}$$

is an isomorphism if and only if for each p there is an integer N such that $E_N{}^{p,n-p} = E_{N+1}^{p,n-p} = \ldots = E_\infty{}^{p,n-p}$. Compare Buchstaber and Miščenko [2] and Landweber [8].

EXAMPLE 5.4. Let h be a cohomology theory defined by an Ω-spectrum. Then, if X is a paracompact space, there is an isomorphism $\varinjlim_\alpha \{h^n(X_\alpha)\} \to h^n(X)$ for each n, where $\{X_\alpha\}$ is the family of nerves of coverings of X. We define an exact couple $(D^p, E^p, i^p, j^p, k^p)$ by taking the direct limit of the exact couples $(D_\alpha{}^p, E_\alpha{}^p, i_\alpha{}^p, j_\alpha{}^p, k_\alpha{}^p)$, where $D_\alpha{}^p = h(X_\alpha{}^p)$ etc. We get

$$\begin{aligned} E_2{}^{p,n-p} &= \varinjlim_\alpha \{E_{2,\alpha}^{p,n-p}\} = \varinjlim_\alpha \{H^p(X_\alpha;\, h^{n-p}(\text{point}))\} \\ &= \check{H}^p(X;\, h^{n-p}(\text{point}))\,. \end{aligned}$$

So if $\varprojlim_r{}^{(1)} \{E_r{}^{p,n-p}\} = 0$ for all p then the spectral sequence converges strongly in degree n to $\varprojlim_p \{D^{p,n-p}\}$, and $\varprojlim_p{}^{(1)} \{D^{p,n-p}\} = 0$. Byt using a variant of Milnor's lemma (cf. [10]) we see that there is a short exact sequence

$$0 \to \varprojlim_p{}^{(1)} \{\varinjlim_\alpha \{h^n(X_\alpha{}^p)\}\} \to \varinjlim_\alpha \{h^{n+1}(X_\alpha)\} \to \varprojlim_p \{\varinjlim_\alpha \{h^{n+1}(X_\alpha{}^p)\}\} \to 0\,,$$

so $\varprojlim^{(1)} \{D^{p,n-p}\} = 0$ if and only if the homomorphism

$$h^{n+1}(X) \to \varprojlim \{D^{p,n+1-p}\}$$

is an isomorphism. Hence, if $\varprojlim_r{}^{(1)} \{E_r{}^{p,n-p}\} = 0$ and $\varprojlim_r{}^{(1)} \{E_r{}^{p,n-1-p}\} = 0$ for all p, then the spectral sequence converges strongly to $h^n(X)$.

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