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A NOTE ON RADICALS AND POLYNOMIAL RINGS

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1. Introduction.

All rings considered in this note are associative. A non-void class $\mathscr{R}$ of rings is called a *radical class* if $\mathscr{R}$ is homomorphically closed, every ring R has a largest ideal $\mathscr{R}(R)$ belonging to $\mathscr{R}$ and $\mathscr{R}(R/\mathscr{R}(R))=0$ for each R. With the exception that the disucssion is phrased in terms of classes rather than properties, we follow the terminology of Divinsky's book [3], to which the reader is referred for further details. Two remarks on notation: $R[x]$ or $R[x_1,\ldots,x_n]$ denotes the ring of polynomials over R in one or several commuting indeterminates; $I \lhd R$ means "I is an ideal of R".

We shall show that for every radical class $\mathscr{R}$, the rings R whose polynomial rings $R[x]$ belong to $\mathscr{R}$ form a radical class $\mathscr{R}^{(1)} \subseteqq \mathscr{R}$. By induction, a chain

$$\mathscr{R} = \mathscr{R}^{(0)} \supseteqq \mathscr{R}^{(1)} \supseteqq \ldots$$

of radical classes is obtained which can have any finite length. When $\mathscr{R}$ is hereditary, $\mathscr{R}^{(1)}(R)=R\cap\mathscr{R}(R[x])$ for every ring R. The equation $\mathscr{R}^{(1)}=\{0\}$ holds for hereditary $\mathscr{R}$ if and only if $\mathscr{R}$ is subidempotent. An application is given to rings R for which $\mathscr{R}(R)=R$ or 0 for every hereditary radical class $\mathscr{R}$.

2. The construction.

For a radical class $\mathscr{R}$, let $\mathscr{R}^{(1)}$ denote the class $\{R \mid R[x]\in\mathscr{R}\}$.

THEOREM 1. *$\mathscr{R}^{(1)}$ is a radical class, for any radical class $\mathscr{R}$.*

PROOF. If S is a homomorphic image of $R\in\mathscr{R}^{(1)}$, then $S[x]$ is a homomorphic image of $R[x]\in\mathscr{R}$, so $S[x]\in\mathscr{R}$, that is, $S\in\mathscr{R}^{(1)}$. For an arbitrary ring R, let $\{I_\lambda \mid \lambda\in\Lambda\}$ be the set of ideals belonging to $\mathscr{R}^{(1)}$. Each $I_\lambda[x]$ belongs to $\mathscr{R}$, so

$$I_\lambda[x] \subseteqq \mathscr{R}(\textstyle\sum_\Lambda I_\lambda[x])$$

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for each λ, that is,

$$(\sum_{\Lambda} I_\lambda)[x] \cong \sum_{\Lambda} I_\lambda[x] \in \mathscr{R} .$$

Thus $I=\sum_{\Lambda} I_\lambda \in \mathscr{R}^{(1)}$. If $J/I \lhd R/I$ and $J/I \in \mathscr{R}^{(1)}$, then

$$J[x]/I[x] \cong (J/I)[x] \in \mathscr{R} .$$

But $I[x] \subseteq \mathscr{R}(J[x])$, so $J[x]/\mathscr{R}(J[x]) \in \mathscr{R}$, that is,

$$J[x] = \mathscr{R}(J[x]) \in \mathscr{R} ,$$

whence $J=I$. This proves that $\mathscr{R}^{(1)}$ is a radical class and $\mathscr{R}^{(1)}(R)=I$.

Since every ring R is a homomorphic image of $R[x]$, we have $\mathscr{R}^{(1)} \subseteq \mathscr{R}$ for every radical class $\mathscr{R}$, and by inductively defining $\mathscr{R}^{(n+1)}=\mathscr{R}^{(n)(1)}$, we obtain a chain

$$\mathscr{R} = \mathscr{R}^{(0)} \supseteq \mathscr{R}^{(1)} \supseteq \dots \supseteq \mathscr{R}^{(n)} \supseteq \dots$$

of radical classes, where

$$\mathscr{R}^{(n)} = \{R \mid R[x_1,\dots,x_n] \in \mathscr{R}\} .$$

A radical class $\mathscr{R}$ is *hereditary* if all ideals of rings in $\mathscr{R}$ belong to $\mathscr{R}$; if in addition $\mathscr{R}$ contains all nilpotent rings (or, equivalently, all zero-rings) it is said to be *supernilpotent.*

PROPOSITION 2. *If $\mathscr{R}$ is a hereditary (resp. supernilpotent) radical class, then so is $\mathscr{R}^{(1)}$.*

PROOF. If R is a zeroring, so is $R[x]$, so if $\mathscr{R}$ contains all zerorings, $\mathscr{R}^{(1)}$ does also. The rest is obvious.

For hereditary $\mathscr{R}$, we have the following "local" description of $\mathscr{R}^{(1)}$:

THEOREM 3. *If $\mathscr{R}$ is a hereditary radical class, then*

$$\mathscr{R}^{(1)}(R) = R \cap \mathscr{R}(R[x])$$

for any ring R.

PROOF. Let $\varrho(R)=R\cap\mathscr{R}(R[x])$ and let R^* denote the Dorroh extension of R. Since

$$\varrho(R^*) \subseteq \mathscr{R}(R^*[x]) \lhd R^*[x]$$

and $x \in R^*[x]$, we have

$$\varrho(R^*)[x] \subseteq \mathscr{R}(R^*[x]) .$$

But $\varrho(R^*) \lhd R^*$, so $\varrho(R^*)[x] \lhd \mathscr{R}(R^*[x])$ and thus $\varrho(R^*)[x] \in \mathscr{R}$, that is, $\varrho(R^*) \in \mathscr{R}^{(1)}$, whence $\varrho(R^*) \subseteq \mathscr{R}^{(1)}(R^*)$. But

$$\mathscr{R}^{(1)}(R^*)[x] \lhd R^*[x] \quad \text{and} \quad \mathscr{R}^{(1)}(R^*)[x] \in \mathscr{R} ,$$

so

$$\mathscr{R}^{(1)}(R^*) = R^* \cap \mathscr{R}^{(1)}(R^*)[x] \subseteq R^* \cap \mathscr{R}(R^*[x]) = \varrho(R^*) .$$

In addition, $\mathscr{R}^{(1)}$ is hereditary, so (cf. [3, Theorem 48 p. 125])

$$\begin{aligned} \mathscr{R}^{(1)}(R) &= R \cap \mathscr{R}^{(1)}(R^*) = R \cap \varrho(R^*) = R \cap R^* \cap \mathscr{R}(R^*[x]) \\ &= R \cap \mathscr{R}(R^*[x]) = R \cap R[x] \cap \mathscr{R}(R^*[x]) = R \cap \mathscr{R}(R[x]) \\ &= \varrho(R) . \end{aligned}$$

Ortiz [7] has shown that for any radical class $\mathscr{R}$,

$$\mathscr{R}_A = \{R \mid \varrho(R) = R\}$$

is also a radical class (notation as in the proof of Theorem 3) and that $\mathscr{R}_A(R) = \varrho(R)$ whenever $\varrho(\varrho(R)) = \varrho(R)$. Thus Theorem 3 says that $\mathscr{R}_A = \mathscr{R}^{(1)}$ whenever $\mathscr{R}$ is hereditary, and that in such cases $\mathscr{R}_A(R) = \varrho(R)$ for every ring R.

Every non-void class $\mathscr{C}$ of rings is contained in a smallest radical class, the lower radical class $L(\mathscr{C})$ determined by $\mathscr{C}$. We now show that $\mathscr{C}$ can always be embedded in a smallest radical class $\mathscr{R}$ with the property that $\mathscr{R} = \mathscr{R}^{(1)}$.

PROPOSITION 4. *Let $\mathscr{C}$ be a non-void class of rings satisfying the condition*

$$R \in \mathscr{C} \Rightarrow R[x] \in \mathscr{C} .$$

Then $L(\mathscr{C})^{(1)} = L(\mathscr{C})$.

PROOF. If $R \in \mathscr{C}$, then $R[x] \in \mathscr{C} \subseteq L(\mathscr{C})$, so $R \in L(\mathscr{C})^{(1)}$. Thus $\mathscr{C} \subseteq L(\mathscr{C})^{(1)}$, so $L(\mathscr{C}) \subseteq L(\mathscr{C})^{(1)}$.

COROLLARY 5. *For a non-void class $\mathscr{C}$ of rings let*

$$L^*(\mathscr{C}) = L(\{R[x_1, \ldots, x_n] \mid R \in \mathscr{C},\ n = 1, 2, \ldots\}) .$$

Then $L^*(\mathscr{C})^{(1)} = L^*(\mathscr{C})$ and $L^*(\mathscr{C})$ is the smallest radical class containing $\mathscr{C}$ which has this property.

PROOF. $\mathscr{C} \subseteq L^*(\mathscr{C})$, as any ring R is a homomorphic image of $R[x]$. If $\mathscr{I}$ is a radical class for which $\mathscr{C} \subseteq \mathscr{I} = \mathscr{I}^{(1)}$, then for each $R \in \mathscr{C}$, we have $R[x] \in \mathscr{I}$, and by induction $R[x_1, \ldots, x_n] \in \mathscr{I}$ for each n. Thus $L^*(\mathscr{C}) \subseteq \mathscr{I}$.

On the other hand, every radical class $\mathscr{R}$ has a largest radical subclass $\mathscr{M}$ for which $\mathscr{M}^{(1)} = \mathscr{M}$.

PROPOSITION 6. *Let $\mathscr{R}$ be a radical class, $\mathscr{R}_* = \bigcap_{n=1}^{\infty} \mathscr{R}^{(n)}$. Then $\mathscr{R}_*$ is a radical class, $(\mathscr{R}_*)^{(1)} = \mathscr{R}_*$ and if $\mathscr{M} \subseteq \mathscr{R}$ is a radical class which shares these properties, then $\mathscr{M} \subseteq \mathscr{R}_*$.*

PROOF. The intersection of any collection of radical classes is itself a radical class (see [6]). If $R \in \mathscr{R}_*$, then $R \in \mathscr{R}^{(n+1)} = (\mathscr{R}^{(n)})^{(1)}$, that is $R[x] \in \mathscr{R}^{(n)}$ for $n = 1, 2, \ldots$, so $R[x] \in \mathscr{R}_*$, that is, $(\mathscr{R}_*)^{(1)} = \mathscr{R}_*$. If $\mathscr{M}$ is as described, then

$$\mathscr{M} = \mathscr{M}^{(1)} = \ldots = \mathscr{M}^{(n)} \subseteq \mathscr{R}^{(n)}$$

for each n, so $\mathscr{M} \subseteq \mathscr{R}_*$.

In this context, an obvious question is: Does the chain

$$\mathscr{R} = \mathscr{R}^{(0)} \supseteq \mathscr{R}^{(1)} \supseteq \ldots \supseteq \mathscr{R}^{(n)} \supseteq \ldots$$

terminate for every radical class $\mathscr{R}$? We are unable to answer this question, but we shall show, by an example, that chains of arbitrary finite length can occur. The next couple of results prepare the way for this example.

LEMMA 7. *Let R be a ring with identity, S a ring with identity and no other non-zero idempotents. S belongs to $L(\{R\})$ if and only if it is a homomorphic image of R.*

PROOF. If $S \in L(\{R\})$ there exists a finite chain

$$I_1 \lhd I_2 \lhd \ldots \lhd I_n \lhd S$$

where $I_1 (\neq 0)$ is a homomorphic image of R (see [8]). But then I_1 has an identity, which must be the identity of S, so $I_1 = \ldots = I_n = S$. The converse is obvious.

PROPOSITION 8. *Let Q denote the field of rational numbers. Then $\mathsf{Q}[x] \notin L(\{\mathsf{Q}\})$ and*

$$\mathsf{Q}[x_1, \ldots, x_{n+1}] \notin L(\{\mathsf{Q}[x_1, \ldots, x_n]\}), \quad n = 1, 2, \ldots .$$

PROOF. Each of the rings referred to has an unique non-zero idempotent—the identity. By Lemma 7, it suffices to show that $\mathsf{Q}[x]$ is not a homomorphic image of Q and $\mathsf{Q}[x_1, \ldots, x_{n+1}]$ is never a homomorphic image of $\mathrm{Q}[x_1,\ldots,x_n]$. The first assertion is obvious and both follow from Theorem 28, p. 101 of [9].

COROLLARY 9. *Let* $\mathscr{R}=L(\{\mathrm{Q}[x_1,\ldots,x_n]\})$. *Then*

$$\mathscr{R} = \mathscr{R}^{(0)} \supsetneqq \mathscr{R}^{(1)} \supsetneqq \ldots \supsetneqq \mathscr{R}^{(n)} \supsetneqq \mathscr{R}^{(n+1)} .$$

PROOF. If $\mathscr{R}^{(k)}=\mathscr{R}^{(m)}$, where $k<m$, then

$$\mathscr{R}^{(k)} \supseteqq \mathscr{R}^{(k+1)} \supseteqq \mathscr{R}^{(m)} = \mathscr{R}^{(k)}, \quad \text{so} \quad \mathscr{R}^{(j)} = \mathscr{R}^{(k)} \quad \text{for all } j>k .$$

We need only show, therefore, that $\mathscr{R}^{(n+1)} \neq \mathscr{R}^{(n)}$. Now Q belongs to $\mathscr{R}^{(n)}$, but not to $\mathscr{R}^{(n+1)}$, since then $\mathscr{R}=L(\{\mathrm{Q}[x_1,\ldots,x_n]\})$ would contain $\mathrm{Q}[x_1,\ldots,x_{n+1}]$, contradicting Proposition 8.

A hereditary radical class is *subidempotent* if it consists of rings in which all ideals are idempotent (see [2]).

THEOREM 10. *Let* $\mathscr{R}$ *be a hereditary radical class. Then* $\mathscr{R}^{(1)}=\{0\}$ *if and only if* $\mathscr{R}$ *is subidempotent.*

PROOF. If $\mathscr{R}$ is subidempotent and R belongs to $\mathscr{R}^{(1)}$, then for any $a \in R$, we have $ax \in I^2$, where I is the principal ideal of $R[x]$ generated by ax. This can happen only when $a=0$, so $\mathscr{R}^{(1)}=\{0\}$. Conversely, if $\mathscr{R}^{(1)}=\{0\}$ and $I \lhd R \in \mathscr{R}$, then $\mathscr{R}$ contains the zeroring I/I^2, and hence also $(I/I^2)[x]$, as the latter is isomorphic to a direct sum of copies of I/I^2. But then $I/I^2 \in \mathscr{R}^{(1)}$, so $I=I^2$.

3. An application.

A radical theoretic problem which has not so far received much attention is to determine, for a ring R, which ideals can have the form $\mathscr{R}(R)$ for some radical class $\mathscr{R}$. (See [5] for some remarks on the question for hereditary radicals). A special case is the characterization of those rings R for which $\mathscr{R}(R)=R$ or 0 for all radical classes $\mathscr{R}$ or for those with some additional condition imposed. We shall call a ring R *h-unequivocal* if $\mathscr{R}(R)=R$ or 0 for every hereditary radical class $\mathscr{R}$.

If $\mathscr{A}$ is a radical class of abelian groups, then the rings whose additive groups belong to $\mathscr{A}$ form a radical class $\bar{\mathscr{A}}$ (see [4] for further details) and if $\mathscr{A}$ is hereditary, so is $\bar{\mathscr{A}}$. A consequence of this is that if a ring R is to be h-unequivocal, its additive group must be torsion-free or primary. The radical behaviour of polynomial rings over h-unequivocal rings is partially described by the next result.

PROPOSITION 11. *If an h-unequivocal ring R has a torsion-free additive group or admits an algebra structure over an infinite field of finite characteristic, then* $R[x_1, \ldots, x_n]$ *is h-unequivocal for* $n = 1, 2, \ldots$.

PROOF. Amitsur [1] has shown that for any radical class $\mathscr{R}$ and any ring R which has a torsion-free additive group or is an algebra over an infinite field, $\mathscr{R}(R[x]) \neq 0$ implies $R \cap \mathscr{R}(R[x]) \neq 0$. (Amitsur's results are stated for a restricted class of radicals, but the restrictions do not influence the proof of the result cited). As we have seen, for hereditary $\mathscr{R}$, we have $R \cap \mathscr{R}(R[x]) = \mathscr{R}^{(1)}(R)$. Thus if R is as described and $\mathscr{R}$ is a hereditary radical class with $\mathscr{R}(R[x]) \neq 0$, we have $\mathscr{R}^{(1)}(R) \neq 0$, so, since $\mathscr{R}^{(1)}$ is hereditary, $\mathscr{R}^{(1)}(R) = R$. But then $R[x] \in \mathscr{R}$, that is $\mathscr{R}(R[x]) = R[x]$. Hence $R[x]$ is h-unequivocal. In addition, it inherits the other property from R, so the result follows by induction.

If R is as in Proposition 11 and $\mathscr{R}(R) = 0$ for some hereditary radical class $\mathscr{R}$, then $R[x]$ cannot belong to $\mathscr{R}$, for then its homomorphic image R would also. Thus we obtain

COROLLARY 12. *If R is an h-unequivocal ring which has a torsion-free additive group or admits an algebra structure over an infinite field of finite characteristic, and if* $\mathscr{R}(R) = 0$ *for a hereditary radical class* $\mathscr{R}$, *then*

$$\mathscr{R}(R[x_1, \ldots, x_n]) = 0 \quad \textit{for} \quad n = 1, 2, \ldots .$$

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