GENERALIZED V-RINGS

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1. Introduction.

Kaplansky [11] noticed that a commutative ring is (von Neumann) regular if and only if it is a V-ring, that is, each simple module over the ring is injective. However in the general case, regularity of a ring is neither necessary nor sufficient to ensure that it is a left or a right V-ring, ([4], [9]). Here we consider right V-rings and also the generalized right V-rings which are written for short as GV-rings. These are the rings over which each simple right module is either projective or injective. We begin with the simple observation that every right V-ring possesses the property that to each element \( a \) in \( R \) there exists an element \( x \) in \( RaR \) such that \( a = ax \). Our investigation of GV-rings generalizes the work of [1], [3], [13]. For instance a ring \( R \) is a right GV-ring if and only if \( J(R) \cap Z(R) = 0 \), and every proper large right ideal of \( R \) is the intersection of all maximal right ideals containing it. In the presence of commutativity, von Neumann regularity is equivalent to the GV-ring condition. The right noetherian semiprime right GV-rings are precisely the rings \( R \) such that each semisimple right \( R \)-module is injective.

2. Preliminaries.

All the rings that we consider are associative rings with identity and all the modules, unitary right modules. A ring \( R \) will be called a right V-ring if each simple right \( R \)-module is injective and, a generalized right V-ring or for short, a GV-ring if each simple right \( R \)-module is either projective or injective. Observe that a GV-ring with zero socle is a V-ring. We shall require the following characterisation of right V-rings [5]: The following are equivalent for any ring \( R \):

1) \( R \) is a right V-ring,
2) each right ideal of \( R \) is an intersection of maximal right ideals,
3) each right \( R \)-module has zero radical.

A ring \( R \) is called (von Neumann) regular if for each element \( a \) in \( R \) there exists an element \( x \) in \( R \) such that \( a = axa \). A semiprime ring is

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one which has no nonzero nilpotent ideals and a Goldie ring is a ring \( R \) which satisfies the conditions

1) \( R \) has maximum condition on annihilator right ideals,
2) \( R \) contains no infinite direct sum of right ideals.

A right \( R \)-module \( M \) is called singular if each element of \( M \) is annihilated by a large right ideal of \( R \). It can be checked that a ring \( R \) is a right GV-ring if and only if each singular simple right \( R \)-module is injective. For any right \( R \)-module \( M \), \( Z(M), \text{Soc}M, J(M) \) will denote respectively the singular submodule of \( M \), the sum of all simple submodules of \( M \), and the intersection of all maximal submodules of \( M \). A submodule \( S \) will be called an absolute summand if for any submodule \( T \) of \( M \) such that \( T \) is maximal with respect to \( S \cap T = 0 \), we have \( S \oplus T = M \). A submodule \( S \) is fully invariant if \( S \) is invariant under every \( R \)-endomorphism of \( M \). The fully invariant submodules of injective modules are called quasi-injective. Observe that any simple \( R \)-module is quasi-injective. An \( R \)-module is \( \pi \)-quasi-injective if each direct product of copies of \( M \) is quasi-injective. An \( R \)-module is semisimple if it is a direct sum of simple modules.


**Proposition 3.1.** If \( R \) is a right \( V \)-ring, then for each element \( a \in R \) there is an element \( x \in RaR \) such that \( a = ax \).

**Proof.** If there is no element \( x \) in \( RaR \) such that \( a = ax \), then \( a \notin aRaR \). Hence by Villamayor (see [5, p. 130]), there is a maximal right ideal \( M \) of \( R \) which contains \( aRaR \) but not \( a \). Since \( R = aR + M \) we have \( 1 = as + m \) for some \( s \in R \) and \( m \in M \), so that \( a = asa + ma \) is an element of \( M \), a contradiction. Hence the assertion.

We call rings with the property of proposition 3.1 right weakly regular rings. We shall need the following characterisations of such rings (due to Ramamurthi [10] and Vanaja [12]).

**Proposition 3.2.** Any ring \( R \) has all or none of the following properties:

1) \( R \) is right weakly regular.
2) The quotient \( R/I \) is left \( R \)-flat for any two-sided ideal \( I \) of \( R \).
3) The equation \( I^2 = I \) holds for every right ideal \( I \) of \( R \).
We now give a characterisation of GV-rings, and this generalizes theorem 1.1 of [1].

**Theorem 3.3.** The following are equivalent for any ring $R$.

1) Every proper large right ideal of $R$ is an intersection of maximal right ideals of $R$, and $Z(R) \cap J(R) = 0$.

2) $R$ is a right GV-ring.

3) The module $J(M)$ vanishes for any right $R$-module $M$ with $Z(M)$ large in $M$.

4) If $M$ is any right $R$-module, then every proper large submodule of $M$ is an intersection of maximal submodules of $M$ and $Z(M) \cap J(M) = 0$

**Proof.** 1 $\Rightarrow$ 2: Let $X$ be a simple singular right $R$-module and $I$ a large right ideal of $R$ with a nonzero $R$-homomorphism $f: I \to X$. If $K (\neq I)$ is the kernel of $f$ and $K$ is not a large $R$-submodule of $I$, then $K$ is a summand of $I$, and so $I = K \oplus S$. Here $S \cong X$ is simple so that $S^2 = 0$ or $S = eR$ where $e^2 = e$. Since $S$ is singular, $S \subset Z(R)$ and hence $S \neq eR$. Thus $S \subset J(R) \cap Z(R) = 0$, a contradiction. Thus $K$ is large in $I$ and hence in $R$. Let $M$ be a maximal right ideal of $R$ containing $K$ but not $I$. Then $R/K = M/K \oplus I/K$. Let $g$ be the natural map $R \to R/K$, let $h$ be the projection $R/K \to I/K$ and $k$ be the isomorphism $I/K \to X$. Then the composite map $khg$ extends $f$ from $I$ to $R$. Thus $X$ is injective.

2 $\Rightarrow$ 3: Let $0 \neq m \in M$. Since $Z(M)$ is large in $M$, we get $0 \neq mr = x \in Z(M)$ for some $r \in R$. Let $L$ be a submodule of $M$ maximal with respect to $x \notin L$. Then the module $(xR + L)/L$ is simple singular and hence is an injective submodule of the subdirectly irreducible module $M/L$, so that $M/L = (xR + L)/L$. This implies that $L$ is a maximal submodule of $M$. Clearly $m \notin L$. Thus $J(M) = 0$.

3 $\Rightarrow$ 4: If $M$ is any right $R$-module and $S \neq M$ is large in $M$, then, by hypothesis, $J(M/S) = 0$ so that $S$ is the intersection of all the maximal submodules of $M$ containing $S$. Suppose

$$0 \neq m \in Z(M) \cap J(M).$$

Let $L$ be a submodule of $M$ maximal with respect to $m \notin L$. Clearly $M/L$ is subdirectly irreducible and is not simple since $L$ is not a maximal submodule of $M$ so that $J(M/L) \neq 0$. But $(mR + L)/L$ is singular so that $Z(M/L)$ is large in $M/L$ and hence, by hypothesis, $J(M/L) = 0$, a contradiction.

4 $\Rightarrow$ 1: Trivial.
REMARK 1. The condition \( Z(R) \cap J(R) = 0 \) in (1) cannot be dropped. In fact, if \( R \) is the ring of integers modulo \( p^2 \) where \( p \) is a prime, then trivially each proper large right ideal of \( R \) is an intersection of maximal right ideals, but \( R \) is not a GV-ring, since \( R/pR \) is neither injective nor projective.

REMARK 2. The condition (4) implies that if \( R \) is a GV-ring, then for any right \( R \)-module \( M \), we get \( J(M) \subseteq \text{Soc } M \). This is because \( \text{Soc } M \) is the intersection of all large submodule of \( M \) and \( J(M/S) = 0 \) for every large submodule of \( M \). In particular, \( J(R) \subseteq \text{Soc } R \) and hence \( (J(R))^2 = 0 \).

PROPOSITION 3.4. If \( R \) is a right GV-ring in which every primitive idempotent is central, then \( R \) is a right V-ring.

PROOF. Let \( S \) be a simple projective right \( R \)-module so that \( S \cong eR \) where \( e \) is an idempotent. Let \( I \) be an arbitrary right ideal of \( R \) and \( f: I \rightarrow eR \) any nonzero epimorphism with kernel \( K \). Then \( I = K \oplus T \) where \( T \cong eR \). Since \( e \) is central, \( eR \) is a fully invariant summand of the right \( R \)-module \( R \) and hence \( eR = T \). Clearly \( K \subseteq (1-e)R \) and hence the map given by \( g = 0 \) on \( (1-e)R \) and \( g = f \) on \( eR \) is an extension of \( f \). Hence \( S \) is injective.

COROLLARY 3.5. A right GV-ring with no nonzero nilpotent elements is a V-ring.

Proposition 3.4 at once yields a characterisation of commutative GV-rings.

THEOREM 3.6. For any commutative ring \( R \) the following are equivalent

1. \( R \) is a GV-ring,
2. \( R \) is a V-ring,
3. \( R \) is a von Neumann regular ring.

PROOF. \( 1 \Rightarrow 2 \) by proposition 3.4 and \( 2 \Rightarrow 3 \) by proposition 3.1. Since for any regular ring \( R \), \( J(R) = 0 \) and furthermore every homomorphic image of a regular ring is regular, we have \( 3 \Rightarrow 1 \) by theorem 3.3.

The following two propositions give sufficient conditions for a right GV-ring and a right weakly regular ring to be a V-ring.
Proposition 3.7. A ring $R$ is a V-ring if and only if $R$ is a right GV-ring and every simple right ideal of $R$ is an absolute summand of $R$.

Proof. Let $R$ be a V-ring and $S$ be a simple right ideal of $R$. Let $T$ be a right ideal of $R$ maximal with respect to $S \cap T = 0$. If $X = S \oplus T$ and $X \to S$ is the obvious projection, then, by the injectivity of $S$, this extends to an epimorphism $f: R \to S$. Clearly $\ker f \cap S = 0$ and $\ker f \supseteq T$. Hence $\ker f = T$. Thus $S \oplus T = R$ as $\ker f$ is a maximal right ideal in $R$. Conversely, let $R$ be a right GV-ring and let every simple right ideal of $R$ be an absolute summand. If $X$ is a projective simple right $R$-module and $f: I \to X$ is a nonzero morphism with kernel $K$, then $K$ is a summand of $I$, that is $I = K \oplus L$ for a right ideal $L$ of $R$. As $L$ is isomorphic to $X$, the ideal $L$ is simple and hence, if $T$ is a right ideal of $R$ containing $K$ and maximal with respect to $T \cap L = 0$, then $L \oplus T = R$. The projection $R \to L$, followed by the isomorphism $L \to X$, extends $f$ from $I$ to $R$. Thus $X$ is injective.

Proposition 3.8. A ring $R$ is a right V-ring if and only if it is right weakly regular and each simple right $R$-module is \( \pi \)-quasi injective.

Proof. Let $X$ be a simple right $R$-module that is \( \pi \)-quasi-injective. Then, by Fuller [6], $X$ is an injective right $R/\ell(X)$-module where $\ell(X)$ is the annihilator of $X$ in $R$. If $R$ is right weakly regular then, by proposition 3.2, the left $R$-module $R/\ell(X)$ is flat, so that

$$\text{inj dim}_RX \leq \text{inj dim}_{R/\ell(X)}X$$

by Cartan and Eilenberg [2, exercise 9, chapter VI]. Thus $X$ is an injective right $R$-module. The converse is trivial.

We conclude this section by noting the following.

Proposition 3.9. A ring $R$ is a V-ring if and only if $R_n$ (the ring of all $n \times n$ matrices over $R$) is a V-ring.

Proof. This follows from the Morita equivalence of the categories of $R$-modules and $R_n$-modules.

4. Semiprime GV-rings.

In general a GV-ring need not be semiprime. We give below, conditions under which a right GV-ring is semiprime. Our main result in this sec-
tion asserts that, noetherian semiprime GV-rings are precisely the rings over which each semisimple module is injective.

**Lemma 4.1.** In any right GV-ring each large right ideal is idempotent.

**Proof.** Let $I$ be a large right ideal of the right GV-ring $R$ such that $I^2 + I$. Then there is an element $a \in R$, such that $a \in I$ but $a \notin I^2$. By Zorn’s lemma, choose a right ideal $T$, maximal among those containing $I^2$ but not $a$. Then $(aR + T)/T$ is a simple singular right $R$-module and hence injective. Since $R/T$ is an essential extension of $(aR + T)/T$, it follows that $T$ is a maximal right ideal of $R$. Since $a \notin T$, there are elements $r \in R, t \in T$ such that $ar + t = 1$. Thus $ara + ta = a$ is an element of $T$, a contradiction.

**Proposition 4.2.** A right GV-ring is semiprime if and only if each two-sided ideal of $R$ is idempotent.

**Proof.** Let $A$ be a two-sided ideal of the semiprime right GV-ring $R$. If $B$ is the right annihilator of $A$ in $R$, then $B$ is a two-sided ideal and since $R$ is semiprime, $A \cap B = 0$ and $A + B$ is a large right ideal. Hence, by lemma 4.1,

$$A \oplus B = (A \oplus B)^2 = A^2 + AB + BA + B^2 = A^2 \oplus B^2.$$ 

Thus $A = A^2$. The converse is obvious.

**Proposition 4.3.** A ring $R$, in which each large right ideal is two-sided, is semiprime right GV, if and only if it is right weakly regular.

**Proof.** Suppose that $R$ is a right weakly regular ring in which every large right ideal is two-sided. Then by Ramamurthi [10], we have $J(R) = 0$ and $J(R/I) = 0$ for each large right ideal $I$ of $R$, so that, by theorem 3.3, $R$ is a right GV-ring. Conversely, let $a$ be any element of the semiprime right GV-ring $R$ and let $H$ be a right ideal of $R$, maximal with respect to $aR \cap H = 0$. Then, $aR + H$ is a large right ideal. As this is a two-sided ideal, $RaR \subseteq (aR + H)$ so that

$$aR + (RaR \cap H) = RaR = (RaR)^2$$

$$= aRaR + aR(RaR \cap H) + (RaR \cap H)aR + (RaR \cap H)(RaR \cap H).$$

Thus

$$aR = aRaR + aR(RaR \cap H) = aRaR,$$

which implies that $R$ is a right weakly regular ring, by proposition 3.2.
Remark 3. It is known that a self-injective regular ring need not be a V-ring [9]. It is easy to deduce from the above proposition, that, any self-injective regular ring in which each large right ideal is two-sided (these are precisely the semiprime q-rings of [7]) is a V-ring.

Next we consider noetherian GV-rings. The following lemma is proved in [10].

Lemma 4.4. Every ring $R$ contains a largest two-sided ideal $W$, which is right weakly regular, such that 0 is the only right weakly regular ideal in $R/W$. $W$ is called the RWR-radical of $R$.

Proposition 4.5. Any prime right GV-ring $R$ is right weakly regular.

Proof. If $R$ has zero socle then it is a V-ring and hence right weakly regular by proposition 3.1. Let $R$ have a nonzero socle $S$. Then $S$ is a regular ideal of $R$ (see the proof of theorem 1.8 of [1]) so that the RWR-radical of $R$ is nonzero. Since $R$ is a prime ring and $W$ is a two-sided ideal of $R$, it is a large right ideal of $R$. Hence by lemma 4.1, the ring $R/W$ is right weakly regular, which implies that $R/W = 0$ or $R = W$. Hence the proposition.

Corollary 4.6. Any prime Goldie right GV-ring is simple.

Proof. Follows from proposition 4.5 and the fact that any prime right weakly regular Goldie ring is simple (see [10]).

We need the following result, due to Ornstein [8], for our next theorem.

Lemma 4.7. Let $R$ be a non-prime, semiprime ring. Then $R$ is a right Artin ring if and only if $R/A$ is right noetherian for every nonzero two-sided ideal $A$ of $R$, and $R/P$ is a simple ring for every nonzero prime ideal $P$ of $R$.

Theorem 4.8. The following are equivalent for any ring $R$.

1) $R$ is a semiprime right noetherian right GV-ring.
2) $R$ is either a simple right noetherian V-ring with zero socle or is a semisimple Artin ring.
3) Every semisimple right $R$-module is injective.

Proof. 1 $\Rightarrow$ 2: If $R$ is prime, then the implication follows by corollary 4.6. Let $R$ be not prime. If $A$ is any nonzero two-sided ideal of $R$,
then \( R/A \) is right noetherian because \( R \) is so. If \( P \) is any nonzero prime ideal of \( R \) then \( R/P \) is a prime right noetherian right GV-ring so that by corollary 4.6, the ring \( R/P \) is simple. Thus by lemma 4.7, \( R \) is a semi-simple Artin ring.

\[ 3 \Rightarrow 1: \text{ It is sufficient to prove that } R \text{ is right noetherian. Let } \]

\[ I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq \ldots \]

be an ascending sequence of distinct right ideals of \( R \). As \( R \) is a V-ring, there are maximal right ideals \( M_k \) (\( k = 1, 2, \ldots \)) such that \( I_k \subseteq M_k \) but \( I_{k+1} \not\subseteq M_k \). Let \( p_k \) denote the natural projection \( R \to R/M_k \). If \( I = \bigcup_{k=1}^{\infty} I_k \), define a morphism

\[ f: I \to \sum_{k=1}^{\infty} R/M_k, \]

by \( f(x) = \sum_{k=1}^{\infty} p_k(x) \) (the summation on the right being meaningful since every \( x \in I \) belongs to all but a finite number of the \( M_k \)'s). Then \( f \) extends to a morphism

\[ g: R \to \sum_{k=1}^{\infty} R/M_k, \]

since

\[ \sum_{k=1}^{\infty} R/M_k \]

is a semi-simple \( R \)-module and hence injective. Since \( R \) has an identity, \( g(R) \) and hence \( g(I) \) is contained in

\[ \sum_{k=1}^{t} R/M_k \]

for some positive integer \( t \). This implies that the assumed chain of right ideals is finite.

\[ 2 \Rightarrow 3: \text{ Since } R \text{ is right noetherian, each direct sum of injective right } \]

\( R \)-modules is injective. Because \( R \) is furthermore a V-ring, 3 follows.

**Corollary 4.9. ([3])** A commutative ring \( R \) is Artin semisimple if and only if every semisimple \( R \)-module is injective.

**References**


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