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A GENERALIZATION OF THE STRICT TOPOLOGY

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1. Introduction.

In [2] R. C. Buck introduced the strict topology on $C(X)$, where X is a locally compact Hausdorff space. In this paper we shall extend the definition of the strict topology to arbitrary Hausdorff spaces and by use of this topology prove some results on compactness in $C(X)$.

Let X be a Hausdorff space. Then $\mathscr{B}(X)$ denotes the Borel σ-algebra (=the σ-algebra generated by the closed subsets of X), and $C(X)$ denotes the set of all real, bounded, continuous functions on X.

Let m be a real valued finite measure on $(X,\mathscr{B}(X))$. Then $|m|$ denotes the *total variation* of m (see for example Definition 4 in Chapter III.1 of [4]), and m is called *regular* if

$$(1.1) \qquad |m|(A) = \sup\{|m|(K) \mid K \text{ compact}, K \subseteq A\} \quad \forall A \in \mathscr{B}(X) .$$

The set of all real valued, finite, regular measures on $(X,\mathscr{B}(X))$ is denoted by $M(X)$, and the positive part of $M(X)$ is denoted $M_+(X)$.

If $A \subseteq X$, we can define a seminorm, $\|\cdot\|_A$, on $C(X)$ by

$$\|f\|_A = \sup_{x\in A} |f(x)| \quad \forall f \in C(X) .$$

And we can define the closed balls:

$$B(A,a) = \{f \in C(X) \mid \|f\|_A \leqq a\} \quad \forall a>0 .$$

Using these seminorms we can define 3 topologies on $C(X)$:

The *pointwise topology*, p, which is generated by the seminorms

$$\{\|\cdot\|_A \mid A \text{ is a finite subset of } X\} .$$

The *compact topology*, $\mathscr{K}$, which is generated by the seminorms

$$\{\|\cdot\|_A \mid A \text{ is relatively compact in } X\} .$$

The *uniform topology*, u, which is generated by the norm $\|\cdot\|_X$.

It is obvious from the definitions that

$$(1.2) \qquad p \subseteq \mathscr{K} \subseteq u .$$

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If $f \in C(X)$ and $m \in M(X)$, then we may define the bilinear form $(\cdot, \cdot)$ by

$$(f, m) = \int_X f(x)\, m(dx) .$$

Under this bilinear form $\big(C(X), M(X)\big)$ becomes a dual pair. The weak topology on $C(X)$ arising from this duality is called w.

The *strict topology*, β, on $C(X)$ is the topology generated by the seminorms

$$(1.3) \qquad q(f) = \sup_{n \geqq 1}\{a_n \|f\|_{K_n}\} \quad \forall f \in C(X)$$

where $K_1, K_2, \ldots$ are compact subsets of X, and $\{a_n\}$ is a sequence of positive numbers with $\lim_{n\to\infty} a_n = 0$.

In [2] Buck has defined the strict topology as the topology generated by the seminorms

$$r(f) = \|gf\|_X ,$$

where g is a continuous function vanishing at infinity. This definition clearly works only in the case where X is locally compact, since otherwise $g \equiv 0$ is the only continuous function vanishing at infinity. It is easily seen that whenever X is locally compact then the definition of Buck coincides with the one given in (1.3). It is also rather obvious that

$$(1.4) \qquad w \subseteqq \beta \subseteqq u, \quad \mathscr{K} \subseteqq \beta .$$

We shall need some notions from general topology. Let X be a Hausdorff space. Then

(1.5) X is a *k-space*, if $A \subseteqq X$ is closed, whenever $A \cap K$ is compact for all compact $K \subseteqq X$;

(1.6) X is a *k^*-space*, if a bounded map, f, from X into the real line, R, is continuous, whenever $f|K$ is continuous on K for all compact $K \subseteqq X$;

(1.7) X is *almost σ-compact*, if there exist compact sets $K_1, \ldots, K_n, \ldots$ in X such that $\bigcup_1^\infty K_n$ is dense in X.

It is obvious that every k-space is a k^*-space. Furthermore every locally compact space, every metrizable space and every space satisfying the first countability axiom is a k-space.

If X is a σ-compact space or a separable space (that is X admits a countable dense subset), then X is almost σ-compact. Furthermore if there exists $m \in M(X)$, such that the support of m is equal to X, then X is almost σ-compact.

2. Properties of the strict topology.

PROPOSITION 1. *A subset D of $C(X)$ is β-bounded if and only if D is u-bounded, and if D is β-bounded then β and $\mathscr{K}$ coincide on D.*

PROOF. Since $\beta \leqq u$ it follows that u-bounded sets are β-bounded. So let D be a u-unbounded subset of $C(X)$, we can then find $f_n \in D$ and $x_n \in X$, such that $|f_n(x_n)| \geqq n^2$. Let $U = \bigcap_1^\infty B(\{x_n\}, n)$, then U is a β-neighbourhood, which obviously cannot absorb D, hence D is β-unbounded.

The last statement is trivial.

THEOREM 1. *Let X be a completely regular Hausdorff space, then the following* 3 *conditions are equivalent*:
(1) $(C(X), \beta)$ *is complete,*
(2) $(C(X), \beta)$ *is quasi-complete,*
(3) X *is a k^*-space.*

PROOF. (1) $\Rightarrow$ (2): Trivial.

(2) $\Rightarrow$ (3). Let f be a bounded map from X into R, such that $f|K$ is continuous for all K a compact subset of X. Let $f_K = f|K$; since X is completely regular and K is compact, there exists $g_K \in C(X)$ such that

$$g_K(x) = f_K(x) \quad \forall x \in K \quad \forall K \text{ compact},$$
$$\|g_K\|_X \leqq \|f\|_X \quad \forall K \text{ compact}.$$

Hence $\{g_K \mid K \text{ compact}\}$ is a bounded subset of $(C(X), \beta)$ by Proposition 1. Let us order $\mathscr{K}(X) = \{K \mid K \text{ compact subset of } X\}$ by inclusion. If $\{K_n\} \subseteq \mathscr{K}(X)$ and $\{a_n\}$ is a sequence of positive numbers with $\lim_{n\to\infty} a_n = 0$, then for every $\varepsilon > 0$ we can find $N \geqq 1$, such that

$$a_n < \frac{\varepsilon}{2\|f\|_X + 1} \quad \forall n \geqq N .$$

If $L, K \supseteq \bigcup_{j=1}^N K_j$, then

$$\|g_K - g_L\|_{K_n} = 0 \qquad \forall n \leqq N$$

and

$$\|g_K - g_L\|_{K_n} \leqq 2\|f\|_X \quad \forall n \geqq N .$$

Hence

$$\sup_{n\geqq 1}\{a_n \|g_K - g_L\|_{K_n}\} < \varepsilon ,$$

for all $L, K \in \mathscr{K}(X)$ with $L, K \supseteq \bigcup_{j=1}^N K_j$.

This shows that $\{g_K \mid K \in \mathscr{K}(X)\}$ is a generalized bounded β-Cauchy sequence. Hence by assumption $\{g_K\}$ is β-convergent, and since f is the only possible limit of g_K, we see that $f \in C(X)$, which means that X is a k^*-space.

(3) $\Rightarrow$ (1). Let $\{f_\alpha\}$ be a β-Cauchy sequence. Then obviously we can find a real function f on X such that $f_\alpha \to_\alpha f$ uniformly on every compact subset of X. Hence $f|K$ is continuous for all K compact, and since X is a k^*-space, f is continuous. Suppose that f is unbounded, then we can find $x_n \in X$ with $|f(x_n)| \geqq 2n$ for all $n \geqq 1$. On the other hand there exists α_0 such that

$$|f_\alpha(x_n) - f_\beta(x_n)| \leqq n \quad \forall \alpha, \beta \geqq \alpha_0 \quad \forall n \geqq 1 .$$

Taking limit over β we find

$$|f_{\alpha_0}(x_n) - f(x_n)| \leqq n \quad \forall n \geqq 1$$

and so $|f_{\alpha_0}(x_n)| \geqq n$ for all $n \geqq 1$, which contradicts the fact that f_{α_0} is bounded.

Hence f is bounded and so $f \in C(X)$. But this implies that $\mathscr{K}$-$\lim_\alpha f_\alpha = f$, and since β has a base around 0 consisting of $\mathscr{K}$-closed sets, we find that β-$\lim_\alpha f_\alpha = f$, which shows that $(C(X), \beta)$ is complete.

THEOREM 2. *Let X be a completely regular space, then the dual of $(C(X), \beta)$ equals $M(X)$.*

PROOF. Let $m \in M(X)$, and put

$$F(f) = \int_X f\, dm \quad \forall f \in C(X) .$$

We shall then prove that F is β-continuous. By regularity of m we can find compact sets $K_1 \subseteqq K_2 \subseteqq \ldots$, such that

$$|m|(X \setminus K_n) < 2^{-2n} \quad \forall n \geqq 1 .$$

If $\|f\|_{K_n} \leqq 2^{n-2}$ for all $n \geqq 1$, then

$$|F(f)| \leqq 1 + |m|(X) ,$$

and so F is β-continuous.

Let F be a β-continuous linear functional on $C(X)$. Then F is u-continuous since $\beta \subseteqq u$. So, if $\hat{X}$ is the Stone-Čech compactification of X, we find that there exists $\hat{m} \in M(\hat{X})$, such that

$$F(f) = \int_X \hat{f}\, d\hat{m} \quad \forall f \in C(X) ,$$

where $\hat{f}$ is the unique extension of f to $\hat{X}$.

We shall now prove that X is $\hat{m}$-measurable and $\hat{m}(\hat{X}\setminus X)=0$. Let $K_1 \subseteqq K_2 \subseteqq \ldots$ be compact sets and $a_1 \leqq a_2 \leqq \ldots$ positive numbers with $\lim_{n\to\infty} a_n = \infty$, such that

$$|F(f)| \leqq 1 \quad \forall f \in \textstyle\bigcap_{n=1}^{\infty} B(K_n, a_n) .$$

Let $f \in \bigcap_{n=1}^{\infty} B(K_n, a_n)$ and $\hat{f}$ its extension to $\hat{X}$. Then

$$\begin{aligned}\int_{\hat{X}} |\hat{f}|\, d|\hat{m}| &= \sup\left\{\left|\int_{\hat{X}} \hat{g}\hat{f}\, d\hat{m}\right| \;\middle|\; \|\hat{g}\|_{\hat{X}} \leqq 1,\ \hat{g} \in C(\hat{X})\right\} \\ &= \sup\{|F(gf)| \mid \|g\|_X \leqq 1,\ g \in C(X)\} \leqq 1 ,\end{aligned}$$

since $gf \in \bigcap_1^{\infty} B(K_n, a_n)$ for all $g \in C(X)$ with $\|g\|_X \leqq 1$. Hence we have shown that

$$\int_{\hat{X}} |\hat{f}|\, d|\hat{m}| \leqq 1 \quad \forall f \in \textstyle\bigcap_{n=1}^{\infty} B(K_n, a_n) . \tag{2.1}$$

By regularity of $\hat{m}$ there exist open sets $\hat{U}_n$ in $\hat{X}$ such that $\hat{U}_n \supseteqq K_n$ and

$$|\hat{m}|(\hat{U}_n \setminus K_n) \leqq a_n^{-1} \quad \forall n \geqq 1 .$$

By complete regularity of $\hat{X}$ there exists a continuous function, $\hat{f}_n$, from $\hat{X}$ into $[0, a_n]$ such that

$$\begin{aligned}\hat{f}_n(x) &= 0 \qquad \text{for } x \in K_n \\ &= a_n \qquad \text{for } x \notin \hat{U}_n .\end{aligned}$$

Then

$$\|f_n\|_{K_j} = 0 \leqq a_j \qquad \forall\, 1 \leqq j \leqq n ,$$

and

$$\|f_n\|_{K_j} \leqq a_n \leqq a_j \qquad \forall j \geqq n ,$$

where $f_n = \hat{f}_n | X$, and so f_n belongs to $\bigcap_1^{\infty} B(K_j, a_j)$ for all $n \geqq 1$. So by (2.1) we find

$$\begin{aligned}a_n |\hat{m}|(\hat{X} \setminus K_n) &= a_n |\hat{m}|(\hat{U}_n \setminus K_n) + a_n |\hat{m}|(X \setminus \hat{U}_n) \\ &\leqq 1 + \int_X \hat{f}_n\, d|\hat{m}| \leqq 2 .\end{aligned}$$

Since $\lim_{n\to\infty} a_n = \infty$ we have $\lim_{n\to\infty} |\hat{m}|(\hat{X} \setminus K_n) = 0$, hence

$$|\hat{m}|(\hat{X} \setminus \textstyle\bigcup_1^{\infty} K_n) = 0 ,$$

and since $\hat{X} \setminus X \subseteqq \hat{X} \setminus \bigcup_1^{\infty} K_n$ we find that $\hat{X} \setminus X$ is an $\hat{m}$-null set. So, if we put

$$m(B) = \hat{m}(B) \quad \forall B \in \mathscr{B}(X),$$

then $m \in M(X)$ and

$$F(f) = \int_X f\,dm \quad \forall f \in C(X),$$

which proves the theorem.

COROLLARY 1. *If X is a completely regular k^*-space then the Mackey topology on $C(X)$ arising from the dual pair $(C(X), M(X))$ is complete.*

THEOREM 3. *Let X be a completely regular space, then the following 4 statements are equivalent:*
(1) *X is compact,*
(2) $\beta = u$,
(3) *$(C(X), \beta)$ is bornological,*
(4) *$(C(X), \beta)$ is barrelled.*

PROOF. (1) $\Rightarrow$ (2): Trivial.

(2) $\Rightarrow$ (1). If $\beta = u$ then there exist compact sets $K_1 \subseteqq K_2 \subseteqq \dots$ and positive numbers $a_1 \geqq a_2 \geqq \dots$ with $\lim_{n\to\infty} a_n = 0$ such that

$$\|f\|_X \leqq \sup_{n\geqq 1}\{a_n \|f\|_{K_n}\}.$$

Let N be chosen such that $a_n \leqq \frac{1}{2}$ for all $n \geqq N$. Then I claim that

$$K_N = X.$$

Suppose this is not the case, then there exists a point $x_0 \in X \setminus K_N$. By complete regularity of X there exists a continuous map, f, from X into $[0,1]$ with

$$\begin{aligned} f(x) &= 1 \quad \text{for } x = x_0, \\ &= 0 \quad \text{for } x \in K_N. \end{aligned}$$

Then

$$\|f\|_X = 1 > \tfrac{1}{2} \geqq \sup_{n\geqq 1}\{a_n \|f\|_{K_n}\}$$

which contradicts the choice of $\{K_n\}$ and $\{a_n\}$.

(2) $\Rightarrow$ (3): Trivial.

(2) $\Rightarrow$ (4): Trivial.

(3) $\Rightarrow$ (2): Let T be the identity map from $(C(X), \beta)$ into $(C(X), u)$, then T is a bounded linear operator by Proposition 1, and so by assumption T is continuous. That is $\beta \geqq u$ and since the converse inequality is trivially satisfied, (2) follows.

(4) $\Rightarrow$ (2): Let S be the unit ball in $C(X)$, that is,

$$S = \{f \in C(X) \mid \|f\|_X \leqq 1\}.$$

Then S is a weakly closed barrel in $C(X)$, and so by assumption S is a β-neighbourhood of zero. That is, $\beta \supseteqq u$, and so (2) follows.

3. w-compact subsets of $C(X)$.

THEOREM 4. *Let X be a completely regular k^*-space and A a relatively countably compact subset of $(C(X), w)$. Then the closed balanced convex hull of A is w-compact.*

PROOF. This is an immediate consequence of Corollary 1 and 24.1 (1) p. 316 and 24.5 (4) p. 328 in [5].

THEOREM 5. *Let X be an almost σ-compact completely regular space and A a relatively countably compact subset of $(C(X), w)$. Then A is relatively sequentially compact in $(C(X), w)$.*

PROOF. Let $K_1, K_2, \ldots$ be compact sets in X such that $\bigcup_1^\infty K_n$ is dense in X. Let

$$d(f,g) = \sum_{n=1}^{\infty} 2^{-n} \operatorname{Arctan} \|f-g\|_{K_n}.$$

Then d is a locally convex metric on $C(X)$, which generates a topology weaker than β. So by Theorem 2 and 24.1 (3) on p. 314 of [5] the theorem follows.

THEOREM 6. *Let X be an almost σ-compact Hausdorff space and A a relatively countably compact subset of $(C(X), p)$. Then A is relatively sequentially compact in $(C(X), p)$.*

PROOF. Let $K_1, K_2, \ldots$ be compact subsets of X such that $\bigcup_1^\infty K_n$ is dense in X and let $\{f_n\}$ be an ordinary sequence in A.

By Krein's theorem (see for example 24.5 (1) on p. 327 of [5]), and the usual diagonalization method we can find $n_1 < n_2 < \ldots$ such that

$$f^*(x) = \lim_{j \to \infty} f_{n_j}(x)$$

exists for all $x \in \bigcup_1^\infty K_n$. If f is a limit point of $\{f_{n_j}\}$, then obviously $f(x) = f^*(x)$ for all $x \in \bigcup_1^\infty K_n$, and since $\bigcup_1^\infty K_n$ is dense in X we find that $\{f_{n_j}\}$ has at most one limit point in $(C(X), p)$. But $\{f_{n_j}\}$ is relatively countably compact in $(C(X), p)$, and so $\{f_{n_j}\}$ is convergent in $(C(X), p)$.

That is, A is sequentially compact in $(C(X), p)$.

PROPOSITION 2. *If X is a Hausdorff space and A a u-bounded relatively sequentially compact subset of $(C(X),p)$, then A is relatively sequentially compact in $(C(X),w)$.*

PROOF. Immediate consequence of Lebesgue's dominated convergence theorem.

Combining the Theorems 4, 5, 6 and Proposition 2, we find,

THEOREM 7. *Let X be a completely regular, almost σ-compact k^*-space. Then the following 7 statements are equivalent for a u-bounded subset A of $C(X)$.*
(1) *A is relatively countably compact in $(C(X),p)$.*
(2) *A is relatively sequentially compact in $(C(X),p)$.*
(3) *A is relatively compact in $(C(X),p)$.*
(4) *A is relatively countably compact in $(C(X),w)$.*
(5) *A is relatively sequentially compact in $(C(X),w)$.*
(6) *A is relatively compact in $(C(X),w)$.*
(7) *The closed balanced convex hull of A is w-compact.*

4. The Stone-Weierstrass theorem.

THEOREM 8. *Let X be a Hausdorff space and F a subset of $C(X)$ satisfying:*
(a) *If $g,f \in F$, then $\max(f,g)$ and $\min(f,g)$ belong to F.*
(b) *$\exists a \geqq 1$ such that: $\forall K$ compact, $\forall b > 0$ there exist functions $f,g \in F$ with*

$$f(x) \geqq b \quad \forall x \in K \quad \text{and} \quad \|f\|_X \leqq ab\,,$$
$$g(x) \leqq -b \quad \forall x \in K \quad \text{and} \quad \|g\|_X \leqq ab\,.$$

Then a function $h \in C(X)$ belongs to the β-closure of F if and only if it satisfies:
(c) *For all $x,y \in X$ and $\varepsilon > 0$, there exist an $f \in F$ with*

$$|f(x)-h(x)| < \varepsilon, \quad |f(y)-h(y)| < \varepsilon\,.$$

PROOF. The "only if" follows trivially from the fact that $\beta \geqq p$. Now let h be a function satisfying (c) and let $K_1,K_2,\ldots$ be compact sets, $a_1,a_2,\ldots$ positive numbers with $\lim_{n\to\infty} a_n = 0$ and $\varepsilon > 0$. Let us put $a_0 = 1$ and

$$b = \|h\|_X + \varepsilon, \quad c = \sup_{n\geqq 0}\{a_n\}\,.$$

Then we choose N such that

$$a_n < \varepsilon(ab + \|h\|_X)^{-1} \quad \forall n \geqq N\,. \tag{4.1}$$

Let $K=\bigcup_{j=1}^N K_j$. Then K is compact. Let $y \in K$ be fixed for a moment. Then to each $x \in K$ we can find $f_x \in F$ with

$$|f_x(y)-h(y)| < \varepsilon c^{-1}, \quad |f_x(x)-h(x)| < \varepsilon c^{-1}.$$

If $U_x=\{z \mid f_x(z)<h(z)+\varepsilon c^{-1}\}$, then U_x is a neighbourhood of x for all $x \in K$. By compactness of K there exist $x_1,\ldots,x_m \in K$ with

$$K \subseteqq \bigcup_{j=1}^m U_{x_j}.$$

Let $g_y=\max(f_{x_1},\ldots,f_{x_m})$. Then $g_y \in F$ for all $y \in K$ and clearly

$$\begin{aligned} g_y(x) &< h(x)+\varepsilon c^{-1} \quad \forall x \in K \quad \forall y \in K, \\ g_y(y) &> h(y)-\varepsilon c^{-1} \quad \forall y \in K. \end{aligned}$$

If $V_y=\{z \in X \mid g_y(z)>h(z)-\varepsilon c^{-1}\}$, then V_y is a neighbourhood of y for all $y \in K$. By compactness of K there exist $y_1,\ldots,y_k \in K$, with

$$K \subseteqq \bigcup_{j=1}^k V_{y_j}.$$

Let $g=\min\{g_{y_1},\ldots,g_{y_k}\}$. Then $g \in F$ and obviously

$$|g(x)-h(x)| < \varepsilon c^{-1} \quad \forall x \in K.$$

Let g_1 and g_2 belong to F such that

$$\begin{aligned} g_1(x) &\geqq b = \|h\|_X+\varepsilon \quad \forall x \in K, \\ g_2(x) &\leqq -b \quad \forall x \in K, \end{aligned}$$

and $\|g_i\|_X \leqq ab$, $i=1,2$. Then

$$h_0 = \max\{g_2, \min\{g_1,g\}\}$$

belongs to F, and it is easily seen that

$$\begin{aligned} h_0(x) &= g(x) \quad \forall x \in K, \\ \|h_0\|_X &\leqq ba. \end{aligned}$$

Hence we find by (4.1) that

$$\begin{aligned} a_n\|h-h_0\|_{K_n} &= a_n\|h-g\|_{K_n} \leqq \varepsilon \quad \forall\, 1 \leqq n \leqq N, \\ a_n\|h-h_0\|_{K_n} &\leqq a_n(\|h\|_X+ba) \leqq \varepsilon \quad \forall\, n \geqq N \end{aligned}$$

which shows that h belongs to the β-closure of F.

Corollary 2. *Let X and Y be completely regular spaces, and T an algebra homomorphism from $C(X)$ into $C(Y)$ such that $T(1_X)=1_Y$ and T is (β,β)-continuous. Then there exists a continuous function t from Y into X such that T is given by*

$$Tf = f \circ t \quad \forall f \in C(X).$$

If T is an algebra isomorphism from $C(X)$ onto $C(Y)$ and T is a (β,β)-homeomorphism, then t becomes a homeomorphism of Y onto X.

PROOF. Let $\hat{X}$ and $\hat{Y}$ be the Stone–Čech compactifications of X and Y. Then by Theorem 26 in Chapter IV.6 of [4] there exists a continuous map $\hat{t}$ from $\hat{Y}$ into $\hat{X}$ such that

$$(Tf)(y) = \hat{f}(\hat{t}(y)) \quad \forall f \in C(X) \quad \forall y \in Y ,$$

where $\hat{f}$ is the unique extension of f to $\hat{X}$.

I claim that $\hat{t}$ maps Y into X, so let $y_0 \in Y$ and suppose that $\hat{t}(y_0) = \hat{x} \notin X$. Now let

$$F = \{f \in C(X) \mid \hat{f}(\hat{x}) = 0\} .$$

Then obviously F satisfies (a) of Theorem 8. Let K be a compact subset of X and $b > 0$. Then, since $\hat{x} \in \hat{X} \setminus X \subseteq \hat{X} \setminus K$, there exists a continuous map, $\hat{f}$, from $\hat{X}$ into $[0,b]$, with

$$\begin{aligned} \hat{f}(\hat{y}) &= 0 \quad \text{if } \hat{y} = \hat{x} , \\ &= b \quad \text{if } \hat{y} \in K . \end{aligned}$$

Hence (b) in Theorem 8 is satisfied. Since

$$F = T^{-1}\{g \in C(Y) \mid g(y_0) = 0\} ,$$

F is β-closed. Let $h \in C(X)$ and $\varepsilon > 0$ and $x,y \in X$. Since $\hat{x} \notin X$ we can find $f \in F$ with $f(x) = h(x)$ and $f(y) = h(y)$. So by Theorem 8, F equals $C(X)$, which is obviously impossible. That is, $\hat{t}(Y) \subseteq X$, so if $t = \hat{t} \mid Y$, then t is a continuous map from Y into X with $Tf = f \circ t$ for all $f \in C(X)$.

If T is an algebra isomorphism and a homeomorphism, then we can find a continuous map, s, from X into Y such that

$$T^{-1}g = g \circ s \quad \forall g \in C(Y) .$$

Now it is easily seen that $s = t^{-1}$ and so t is a homeomorphism.

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