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A REPRESENTATION THEOREM FOR A CONVEX CONE OF QUASI CONVEX FUNCTIONS

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0. Introduction and summary.

It was conjectured by Brøns [4] that if a family of quasi convex functions defined on R is closed under addition, then it could be transformed into a family of convex functions by a monotone transformation of the domaine of definition.

In this paper we give the proof of this conjecture and apply it to the problem of estimating a parameter from a one parameter statistical problem with unimodal likelihood function, see van Eeden [6].

A quasi convex function defined on R is just a function which is first decreasing and then increasing. The elementary properties are listed in section 1.

It should be noted that a sum of two quasi convex functions is in general not quasi convex. The condition that a family of quasi convex functions is in fact closed under addition is thus a strong condition.

The family of convex functions on R is clearly closed under addition and one could ask whether a family of quasi convex functions closed under addition is infact a family of convex functions. This can not be true since quasi convexity is invariant under homeomorphic transformations of R and convexity is not.

We can however, turn the problem around and ask if we can choose a transformation of R, depending on the family in question, such that in the new scale, the functions become convex. This is done in Theorem 2.8. Under differentiability conditions we can give an explicit form for the transformation, Theorem 2.4.

Since convex functions have derivatives whereas quasi convex functions need not, the condition on the family in fact implies that the functions considered as measures on R are equivalent.

In Section 3 we prove that for a family of quasi convex functions which is invariant under translation we can transform the domain of definition by a logarithm in order to get convexity.

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Section 4 gives a representation for a convex cone of functions which are increasing at zero, that is, functions which are negative, zero and then positive. These are essentially derivatives of quasi convex functions. Section 5 applies the result of Sections 2 and 3 to the problem of estimating in a statistical problem with a unimodal likelihood function.

The content of the results applied to this situation is that instead of assuming unimodality plus upper semi-continuity we can as well by reparametrizing assume that the likelihood function is logarithmically concave.

In a paper on mean values [1], Brøns has considered mean values defined by families of quasi convex functions and discussed their properties. Brøns, Brunk, Frank and Hanson [3] and Brøns [2] discuss generalized means as mean values of families of convex functions, see also Huber [9]. In this context our result states that up to a monotone transformation one gets the same mean values from convex functions as from quasi convex functions.

As is well known asymptotic properties of the maximum likelihood estimator depend on the smoothness of the likelihood function. So the parametrization used here is one in which the best asymptotic results can be proved.

1. Quasi convex functions.

1.1. Definition. A function $f\colon R \to R$ is called quasi convex if

$$z \in [x,y] \;\Rightarrow\; f(z) \leqq f(x) \vee f(y)\,.$$

There are several equivalent definitions of a quasi convex function. We list some of these in Lemma 1.2.

1.2. Lemma. *A function f is quasi convex if and only if one of the following conditions is satisfied*:

$$[f \leqq a] \text{ convex for all } a \in R\,. \tag{1.1}$$

$$f(x) < f(y) \;\Rightarrow\; f(u) \leqq f(v), \quad \text{for } [x,y] < [u,v]\,, \tag{1.2}$$

that is, $x \leqq u$, $y \leqq v$.

$$f(u) > f(v) \;\Rightarrow\; f(x) \geqq f(y), \quad \text{for } [x,y] < [u,v]\,. \tag{1.3}$$

Proof. Omitted.

1.3. Definition. For a quasi convex function f we define the extended real numbers

$$\underline{m}(f) = \inf\{x \mid y>x \Rightarrow f(y) \geqq f(x)\}$$
$$= \sup\{x \mid y<x \Rightarrow f(y) > f(x)\}$$

and

$$\underline{m}(f) = \sup\{x \mid y<x \Rightarrow f(y) \geqq f(x)\}$$
$$= \inf\{x \mid y>x \Rightarrow f(y) > f(x)\} .$$

It is easily seen that $\underline{m}(f)$ and $\overline{m}(f)$ are well defined and that $\underline{m}(f) \leqq \overline{m}(f)$. It follows that f is non-increasing on the interval $]-\infty, \overline{m}(f)]$ and non-decreasing on $]\underline{m}(f), \infty[$ and therefore constant on $]\underline{m}(f), \overline{m}(f)[$.

1.4. Definition. A function f: $R \to R$ is called strictly quasi convex if for all $[x,y] < [u,v]$ we have

$$f(x) < f(y) \Rightarrow f(u) < f(v) ,$$

and

$$f(u) > f(v) \Rightarrow f(x) > f(y) .$$

It follows that a strictly quasi convex function is quasi convex, and that it is strictly decreasing on $]-\infty, \underline{m}(f)]$, constant on $]\underline{m}(f), \overline{m}(f)[$ and strictly increasing on $[\overline{m}(f), \infty[$.

1.5. Lemma. *The family of quasi convex functions is closed under pointwise limits and pointwise supremum.*

Let us recall the definition of lower-semi-continuity:

1.6. Definition. A function f: $R \to R$ is called lower semi-continuous (l.s.c.) if $[f \leqq a]$ is closed for all $a \in R$.

1.7. Lemma. *A lower semi-continuous quasi convex function f is left continuous when it is increasing and right continuous when decreasing.*

An l.s.c. quasi convex function f determines a non-negative measure on the interval $[\underline{m}(f), \infty[$ by the relation

$$\gamma^+[a,b[\; = f(b) - f(a), \quad \underline{m}(f) \leqq a < b$$

and a non-negative measure on $]-\infty, \overline{m}(f)]$ by

$$\gamma^-]a,b] = -f(b) + f(a), \quad a < b \leqq \overline{m}(f) .$$

If $\gamma^+ > 0$ and $\gamma^- > 0$ then

$$\inf_x f(x) > -\infty$$

and

$$f(x)-\inf_x f(x) = \gamma^+]-\infty,x[- \gamma^-]x,\infty[.$$

An l.s.c. quasi convex function can thus be considered as a measure on R, whose positive part and negative part are concentrated on half lines. We shall finally give the relation between functions which are increasing at zero and quasi convex functions.

1.8. DEFINITION. The function $h\colon R \to R$ is increasing at zero if

$$x<y,\ h(x)>0 \ \Rightarrow\ h(y)>0 ,$$

and

$$x<y,\ h(y)<0 \ \Rightarrow\ h(x)<0 .$$

This is easily seen to be equivalent to the conditions

$$x<y,\ h(x)\geqq 0 \ \Rightarrow\ h(y)\geqq 0 ,$$

and

$$x<y,\ h(y)\leqq 0 \ \Rightarrow\ h(x)\leqq 0 .$$

This was the definition used in Brunk and Johansen [5].

1.9. DEFINITION. If h is decreasing at zero, then we can define the extended real numbers

$$\overline{m}^*(h) = \sup\{x \mid h(x)\leqq 0\} = \inf\{x \mid h(x)>0\}$$

and

$$\underline{m}^*(h) = \inf\{x \mid h(x)\geqq 0\} = \sup\{x \mid h(x)<0\} .$$

It is easily seen that $\overline{m}^*(h)$ and $\underline{m}^*(h)$ are well defined and that h is positive on the interval $]\overline{m}^*(h),\infty[$, negative on $]-\infty,\underline{m}^*(h)[$, and zero on $]\underline{m}^*(h),\overline{m}^*(h)[$.

1.10. LEMMA. *Let h be increasing at zero and locally integrable, then*

$$f(x) = \int_0^x h(u)\,du$$

is strictly quasi convex and continuous. Further $\overline{m}(f)=\overline{m}^*(h)$ and $\underline{m}(f)=\underline{m}^*(h)$. If h is right continuous, then $D^+f(x)=h(x)$.

It should be emphasized that a derivative of a strictly quasi convex function f need not be increasing at zero. In order that this be true we need to know that the derivative does not vanish outside the interval $[\underline{m}(f),\overline{m}(f)]$.

2. Convex cones of quasi convex functions.

In the following we shall work with a class Q of real valued functions. There are four conditions that will be needed:

A. The functions in Q are quasi convex.

B. The functions in Q separate points, in the sense that for any $x<y$ there exist $f \in Q$, $g \in Q$, such that

$$f(x) < f(y) \quad \text{and} \quad g(x) > g(y) .$$

C. The set Q is a convex cone, that is,

$$f \in Q,\ g \in Q,\ a \geqq 0,\ b \geqq 0 \ \Rightarrow\ af+bg \in Q .$$

D. The functions in Q are lower semi-continuous.

We shall immediately prove the main theorem on cones of quasi convex functions.

2.1. THEOREM. *Let Q be a convex cone of quasi convex functions that separates points, that is, Q satisfies conditions* A, B, *and* C.

Then

$$0 < \frac{g(v)-g(u)}{g(y)-g(x)} \leqq \frac{f(v)-f(u)}{f(y)-f(x)} < \infty \tag{2.1}$$

for any intervals $[x,y]<[u,v]$ and for any g and f in Q, such that

$$f(x) < f(y) \quad \textit{and} \quad g(u) > g(v) .$$

PROOF. Let $[x,y]$ and $[u,v]$ be given and choose f and g as indicated. By the quasi convexity of f and g we know that

$$f(u) \leqq f(v) \quad \text{and} \quad g(x) \geqq g(y) .$$

Let us consider the function

$$k = af+bg .$$

By condition C we know that $k \in Q$ whenever $a \geqq 0$ and $b \geqq 0$. We now want to choose a and b such that $k(x)<k(y)$ since then $k(u) \leqq k(v)$ which turns out to be the desired inequality.

Let $c>0$, and let

$$\begin{aligned} a &= g(x)-g(y)+c > 0 , \\ b &= f(y)-f(x) \qquad > 0 . \end{aligned}$$

Then

$$k(y)-k(x) = a(f(y)-f(x))+b(g(y)-g(x)) = c(f(y)-f(x)) > 0$$

and therefore $k(u) \leqq k(v)$ which reduces to

$$0 < \frac{g(v)-g(u)}{g(y)-g(x)+c} \leqq \frac{f(v)-f(u)}{f(y)-f(x)} < \infty .$$

Now let $c \downarrow 0$ and we get the inequality (2.1) which proves the theorem.

2.2. Corollary. *Let Q satisfy* A, B, *and* C. *Then any $f \in Q$ is strictly quasi convex.*

Proof. The result follows from (2.1) and the definition of a strictly quasi convex function.

2.3. Corollary. *Let Q satisfy* A, B, C, *and* D. *Then all functions in Q are continuous.*

Proof. Let $[x,y] < [u,v]$, and let $f(x) < f(y)$ and $g(u) > g(v)$. By (2.1) we get

$$0 < \frac{g(v)-g(u)}{g(y)-g(x)} \leqq \frac{f(u)-f(v)}{f(y)-f(x)} < \infty .$$

Now f is left continuous at v. We therefore let $u \uparrow v$. Then $f(u) \to f(v)$ and therefore $g(u) \to g(v)$. Thus g is left continuous. However, g is right continuous at v if $v \leqq \overline{m}(g)$. Hence g is continuous in the interval $]-\infty, \overline{m}(g)]$. Thus all left branches are continuous, and using the same type of argument we get continuity of all right branches, and thereby continuity of all functions in Q.

The statement of Theorem 2.1 above is loosely speaking that a difference quotient of any increasing branch dominates the difference quotient of any decreasing branch. It is therefore tempting to try to separate the increasing and decreasing part of the function by a function r in such a way, that

$$\frac{r(v)-r(u)}{r(y)-r(x)}$$

lies between the two difference quotients of relation (2.1). Once the existence of such an r has been established we can prove our main result, namely that $h \circ r^{-1}$ is convex for any $h \in Q$, see Theorems 2.4 and 2.8.

In order that the construction be better understood we shall first give a proof using strong assumptions about the functions in Q, namely that they have continuous second derivatives. With this assumption the construction will be quite explicit.

2.4. THEOREM. *Let Q satisfy the conditions* A, B, *and* C *and assume further that all functions in Q have continuous second derivatives, and that the first derivatives are increasing at zero. Consequently we can define*

$$r_0(x) = \inf D \log Df(x) ,$$

where the infimum is taken over all $f \in Q$ for which $Df(x) > 0$, and

$$r(x) = \int_0^x \exp\left(\int_0^t r_0(u)\,du\right) dt .$$

Then the inequality

$$0 < \frac{g(v)-g(u)}{g(y)-g(x)} \leqq \frac{r(v)-r(u)}{r(y)-r(x)} \leqq \frac{f(v)-f(u)}{f(y)-f(x)} < \infty$$

holds. Further for any $f \in Q$, we have that $f \circ r^{-1}$ is convex.

PROOF. Let $z \in R$ and choose f and g such that

$$Dg(z) < 0 < Df(z) .$$

By the continuity of the derivatives it is possible to find x, y, u, v such that $x<y<z<u<v$ and

$$Df(x) > 0 \quad \text{and} \quad Dg(v) < 0 .$$

Clearly

$$f(x) < f(y) \quad \text{and} \quad g(u) > g(v)$$

so that

$$0 < \frac{g(v)-g(u)}{g(y)-g(x)} \leqq \frac{f(v)-f(u)}{f(y)-f(x)} < \infty .$$

We now multiply by $(y-x)/(v-u)$ and let $x \uparrow y$ and $v \downarrow u$. Then we obtain

$$0 < \frac{Dg(u)}{Dg(y)} \leqq \frac{Df(u)}{Df(y)} < \infty .$$

Taking logarithms and dividing by $u-y$ we can let $u \downarrow z$ and $y \uparrow z$, and we get

$$D \log(-Dg(z)) \leqq D \log Df(z) .$$

Now we define r_0 as indicated in the theorem. Then r_0 is upper semicontinuous and therefore measurable, and

$$D\log(-Dg(z)) \leqq r_0(z) \leqq D\log Df(z)$$

whenever $Dg(z) < 0 < Df(z)$.

Now integrating, exponentiating and integrating again yields the desired inequality.

In order that $f \circ r^{-1}$ be convex it is necessary and sufficient that

$$Dr(x)D^2f(x) - Df(x)D^2r(x) \geqq 0$$

which reduces to

$$D\log Df(x) = \frac{D^2f(x)}{Df(x)} \geqq \frac{D^2r(x)}{Dr(x)} = D\log Dr(x) = r_0(x)$$

whenever $Df(x) > 0$, and

$$D\log(-Df(x)) = \frac{D^2f(x)}{Df(x)} \leqq \frac{D^2r(x)}{Dr(x)} = r_0(x)$$

whenever $Df(x) < 0$. Thus the r_0 and r constructed immediately transforms f into a convex function.

Let us now turn to the general case where no extra regularity is assumed except lower semi-continuity. The problem then is to replace the two differentiations, the infinum and the two integrations by operations which do not involve differentiation.

The tool necessary for this is similar to the integration of a function of intervals employed in a different context but for the same purpose by Goodman and Johansen [7]. For a systematic account see Hildebrandt [8].

2.5. Lemma. *Let Q satisfy the conditions* A, B, *and* C. *Then the function*

$$m([x,y],[u,v]) = \inf \frac{h(v)-h(u)}{h(y)-h(x)}$$

where the infimum is taken over all $h \in Q$ with $h(x) < h(y)$, is submultiplicative, that is,

$$m([x,y],[u,v])\,m([u,v],[s,t]) \leqq m([x,y],[s,t]), \qquad [x,y] < [u,v] < [s,t]\,,$$

and subadditive, that is,

$$m([x,y],[u,v]) + m([x,y],[v,w]) \leqq m([x,y],[u,w]), \qquad [x,y] < [u,v] < [v,w]\,.$$

Further

$$0 < \frac{g(v)-g(u)}{g(y)-g(x)} \leqq m([x,y],[u,v]) \leqq \frac{f(v)-f(u)}{f(y)-f(x)} < \infty \tag{2.2}$$

whenever $g(u)>g(v)$ *and* $f(x)<f(y)$.

PROOF. The inequality (2.2) follows directly from (2.1) and the properties of m follow from those of the difference quotient of $h \in Q$. Notice that $0<m<\infty$.

The problem now is to replace m by a difference quotient of r.

2.6. LEMMA. *The function* m_1 *defined by*

$$m_1([x,y],[u,v]) = \inf \textstyle\prod_{i=1}^{n-1} m([x_i,y_i],[x_{i+1},y_{i+1}]) \,,$$

where the infimum is taken over all $x_1,y_1,\ldots,x_n,y_n$ *with*

$$[x,y] = [x_1,y_1] < \ldots < [x_n,y_n] = [u,v] \,,$$

is multiplicative and satisfies the inequality

$$0 < \frac{g(v)-g(u)}{g(y)-g(x)} \leqq m_1([x,y],[u,v]) \leqq \frac{f(v)-f(u)}{f(y)-f(x)} < \infty \,,$$

for $g(u)>g(v)$ *and* $f(x)<f(y)$.

PROOF. The multiplicativity follows easily from the definition, and that it still satisfies the inequality follows from the fact that the difference quotients of f and g are multiplicative for the intervals considered. Now let us choose intervals

$$\ldots < [x_n,y_n] < \ldots < [x_1,y_1] \,,$$

such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = -\infty \,.$$

Then for any $[x,y]$, there exists an n, such that

$$[x_n,y_n] < [x,y] \,.$$

Let us now define

$$m_2(x,y) = \frac{m_1([x_n,y_n],[x,y])}{m_1([x_n,y_n],[x_1,y_1])}$$

for n sufficiently large. The multiplicativity ensures that the definition of m_2 does not depend on n.

2.7. LEMMA. *The function m_2 is subadditive, $0<m_2<\infty$, and*

$$m_1([x,y],[u,v]) = m_2(u,v)/m_2(x,y) .$$

PROOF. The subadditivity follows from that of m and the relation to m_1 from the multiplicativity of m_1.

2.8. LEMMA. *The function m_3 defined by*

$$m_3(u,v) = \inf \textstyle\sum_{i=1}^{n-1} m_2(u_i,u_{i+1}) ,$$

where the infimum is taken over all $u_1,u_2,\ldots,u_n$ with $u=u_1<\ldots<u_n=v$, is additive and satisfies the inequality

$$0 < \frac{g(v)-g(u)}{g(y)-g(x)} \leqq \frac{m_3(u,v)}{m_2(x,y)} \leqq \frac{f(v)-f(u)}{f(y)-f(x)} < \infty .$$

PROOF. The additivity follows from the construction, and the inequality is satisfied since the difference quotients are additive.

Now take a sequence

$$\ldots < u_n < \ldots < u_1$$

such that $u_n \to -\infty$ and define

$$r(u) = m_3(u_n,u)-m_3(u_n,u_1) .$$

We can now state and prove the main theorem.

2.8. THEOREM. *Let Q satisfy the conditions* A, B, *and* C. *Then there exists a strictly increasing function r, such that*

$$0 < \frac{g(v)-g(u)}{g(y)-g(x)} \leqq \frac{r(v)-r(u)}{r(y)-r(x)} \leqq \frac{f(v)-f(u)}{f(y)-f(x)} < \infty \tag{2.2}$$

for all f and g in Q, such that $f(x)<f(y)$ and $g(u)>g(v)$.

If further Q satisfies condition D, *then r is continuous and $h\circ r^{-1}$ is convex for all $h \in Q$.*

PROOF. Let us define r as above. Then

$$m_3(u,v) = r(v)-r(u)$$

and from Lemma 2.8 it follows that

$$0 < \frac{g(v)-g(u)}{g(y)-g(x)} \leqq \frac{r(v)-r(u)}{m_2(x,y)} \leqq \frac{f(v)-f(u)}{f(y)-f(x)} < \infty . \tag{2.3}$$

Taking reciprocals throughout we can repeat the argument of Lemma 2.8 in order to replace $m_2(x,y)$ by $r(y)-r(x)$, and this completes the proof of the inequality.

Let finally the functions in Q be lower semi-continuous, that is, Q satisfies D. By Corollary 2.3 all functions are continuous and it easily follows that r is continuous and strictly increasing.

Convexity of $h \circ r^{-1}$ is proved if

$$\frac{h(y)-h(x)}{r(y)-r(x)} \leqq \frac{h(v)-h(u)}{r(v)-r(u)}$$

for $[x,y] < [u,v]$.

If $h(x) < h(y)$, then this follows immediately from (2.3) for $f=h$. If $h(x) \geqq h(y)$ and $h(u) > h(v)$, then we can use the left hand side of (2.3) for $g=h$.

Finally, if $h(x) \geqq h(y)$ and $h(u) \leqq h(v)$, the convexity relation is trivially satisfied.

3. Translation invariant families of quasi convex cones.

Let us consider some special cases of the above results. Let there be given a quasi convex function f which is not monotone and define

$$Q = \{h \mid h(x) = \textstyle\sum_{i=1}^{n} a_i f(x-x_i),\ x_i \in R,\ a_i \geqq 0,\ i=1,\ldots,n,\ n=1,2,\ldots,\ x \in R\}.$$

Then Q satisfies conditions B and C.

3.1. Lemma. *Let Q satisfy condition* A, *that is, assume that all functions in Q are quasi convex. Then f is continuous.*

Proof. We have seen in Corollary 2.3 that if an increasing branch is continuous at a point, then all decreasing branches are continuous at the same point. But by translating the decreasing branch of f we get that it is continuous of all points. A similar argument gives the continuity of the increasing branch and thereby continuity of the function.

3.2. Theorem. *Assume that Q satisfies A, and further that f has a continuous second derivative and that the first derivatives are increasing at zero, then there exists a constant c, such that*

$$f(c^{-1} \log(cx+1))$$

is convex.

PROOF. We can apply Theorem 2.4 and define $r_0(x)$ as before, that is,

$$r_0(x) = \inf D \log \sum_{i=1}^n a_i Df(x-x_i) ,$$

where the infimum is taken over all $x_1,\ldots,x_n$, $a_1,\ldots,a_n$ for which

$$\sum_{i=1}^n a_i Df(x-x_i) > 0 .$$

It is seen here that $r_0(x)$ is a constant independent of x. Call this c. Then we can choose

$$r(x) = \int_0^x \exp\left(\int_0^t c\,du\right) dt = c^{-1}(e^{cx}-1) ,$$

where this is interpreted as x if $c=0$, in which case the function f is convex from the beginning.

Another example where simple results can be obtained is if f is symmetric around 0. But here we can also give a direct proof.

3.3. THEOREM. *Let f be a quasi convex function, such that $f(x)=f(-x)$, and such that*

$$\sum_{i=1}^n a_i f(x-x_i)$$

is quasi convex for all $x_1,\ldots,x_n$, $a_1 \geqq 0,\ldots,a_n \geqq 0$ and $n=1,2,\ldots$. Then f is convex.

PROOF. We know from Lemma 3.1 that f is continuous. Choose $x_1 < x_2$ and consider the function h defined by

$$h(x) = f(x)+f(x-x_1-x_2) .$$

Then h is quasi convex and we therefore have

$$h\left(\tfrac{1}{2}(x_1+x_2)\right) \leqq h(x_1) \vee h(x_2)$$

or

$$f\left(\tfrac{1}{2}(x_1+x_2)\right)+f\left(-\tfrac{1}{2}(x_1+x_2)\right) \leqq \left(f(x_1)+f(-x_2)\right) \vee \left(f(x_2)+f(-x_1)\right)$$

or

$$2f\left(\tfrac{1}{2}(x_1+x_2)\right) \leqq f(x_1)+f(x_2) .$$

Together with continuity this inequality gives convexity.

4. Convex cones of functions which are increasing at zero.

We shall apply the results of Section 2 to convex cones of functions which are increasing at zero.

4.1. THEOREM. *Let K be a convex cone of functions which are locally integrable, right continuous, and increasing at zero, and assume that K separates points, that is, for any x there exist h and g in K, such that $h(x) < 0 < g(x)$.*

Then there exists a function $k > 0$, such that hk is non decreasing for any $h \in K$.

PROOF. The cone Q defined by

$$Q = \{f \mid f(x) = \textstyle\int_0^x h(u)\,du,\ h \in K\}$$

satisfies the conditions A, B, C, and D of Theorem 2.8.

According to Theorem 2.8 there exists a function r satisfying the relation (2.2). Now the measure determined by r is dominated by the measure determined by f, but this again is absolutely continuous with respect to Lebesgue measure. Thus r has a derivative and $D^+r(x)$ exists except for x in some null set.

Let us now choose $x<y<u<v$ and choose x and u, such that D^+r exists. We further choose g and $f \in Q$, such that $D^+g(u) < 0 < D^+f(x)$. Now multiply (2.2) by $(y-x)/(v-u)$ and let $v \downarrow u$ and $y \downarrow x$. We get

$$0 < \frac{D^+g(u)}{D^+g(x)} \leqq \frac{D^+r(u)}{D^+r(x)} \leqq \frac{D^+f(u)}{D^+f(x)} < \infty\,. \tag{4.1}$$

Right continuity of D^+f and D^+g implies that D^+r is right continuous when defined. We therefore define

$$k_0(x) = D^+r(x+0)\,.$$

By right continuity (4.1) extends to

$$0 < \frac{D^+g(u)}{D^+g(x)} \leqq \frac{k_0(u)}{k_0(x)} \leqq \frac{D^+f(u)}{D^+f(x)} < \infty \tag{4.2}$$

which implies that $k_0 > 0$. We can therefore define k by

$$k(x) = (k_0(x))^{-1}\,.$$

Then k has the desired property.

We have tried to reduce the theorem for functions which are increasing at zero to the corresponding theorem for quasi convex functions. It is however possible to prove the theorem directly using the methods of Section 2 and in this way to avoid the condition on right continuity of h.

5. Application to estimation from a unimodal likelihood function.

Let us consider a one parameter statistical problem given by the positive densities

$$f(x,\theta), \quad \theta \in R .$$

For each set of independent observations $x_1,\ldots,x_n$ we define the likelihood function

$$L_{x_1,\ldots,x_n}(\theta) = \prod_{i=1}^n f(x_i,\theta) .$$

5.1. Definition. We call L unimodal if $-\log L$ is quasi convex for all $x_1,\ldots,x_n$ and strictly unimodal if $-\log L$ is strictly quasi convex.

5.2. Lemma. *If the functions*

$$-\log L_{x_1,\ldots,x_n}(\theta)$$

are quasi convex, then

$$-\sum_{i=1}^n a_i \log L_{x_i}(\theta) \tag{5.1}$$

is quasi convex for any $x_i \in R$, $a_i \geqq 0$, $i=1,\ldots,n$.

Proof. By repeating k_i times the observation x_i, we get that

$$-\sum_{i=1}^n k_i \log L_{x_i}(\theta)$$

is quasi convex for any integers k_i, $i=1,\ldots,n$. Now let $k_i(N)$, $i=1,\ldots,n$, $N=1,\ldots$ be chosen such that

$$\lim_{N\to\infty} k_i(N)/N = a_i, \quad i=1,\ldots,n .$$

It follows that also (5.1) is quasi convex.

5.3. Theorem. *Let us assume that in a one parameter problem with positive density, the likelihood is unimodal and upper semi continuous. Let us further assume that the likelihood functions separate points.*

Then there exists a function r, which is strictly increasing and continuous, and such that if we reparametrize by $\tau = r(\theta)$ then the likelihood function becomes logarithmically concave.

Proof. The proof follows from Theorem 2.8 by considering the cone generated by the likelihood functions.

5.4. Theorem. *In a translation parameter problem, where the second derivative of the likelihood function is continuous, and the derivative is increasing at zero, there exists a constant c, such that in the parameter*

$$\tau = c^{-1} \log(c\theta + 1)$$

the likelihood function is logarithmically concave.

PROOF. This follows from Theorem 3.2.

5.5. THEOREM. *Assume that the likelihood function has a continuous derivative, and that the likelihood equation*

$$D \log L_{x_1,\ldots,x_n}(\theta) = 0$$

has a unique solution for any $x_1, \ldots, x_n$ *and any* $n = 1, 2, \ldots$. *Then one can reparametrize the problem in such a way that*

$$D \log L_{x_1,\ldots,x_n}(\theta)$$

becomes non increasing.

PROOF. This follows from Theorem 4.1.

Notice that in the θ-parameter the likelihood functions must be continuous, whereas in the τ-parameter they have right and left derivatives. This seems to indicate that for results on asymptotic distribution of estimates, we have found the proper parametrization where, say, asymptotic normality can be proved. For results on consistency of estimates we can use the original parametrization since it only requires continuity of the likelihood function.

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REFERENCES

1. H. K. Brøns, *Mean-values*, Preprint, University of Copenhagen, 1969.
2. H. K. Brøns, *Forelæsningsnoter i Matematisk Statistik og Sandsynlighedsregning 9–15*, Institut for Matematisk Statistik, Københavns Universitet, 1967. (In Danish).
3. H. K. Brøns, H. D. Brunk, W. E. Franck, and D. L. Hanson, *Generalized means and associated families of distributions*, Ann. Math. Statist. 40 (1969), 339–355.
4. H. K. Brøns, Private Communication, 1968.
5. H. D. Brunk and S. Johansen, *A generalized Radon–Nikodym derivative*, Pacific. J. Math. 34 (1970), 585–617.
6. C. van Eeden, *Maximum likelihood estimation of partially or completely ordered parameters, I and II*, Indag. Math. 19 (1957), 128–136, 201–211.

7. G. S. Goodman, and S. Johansen, *Operational timescales and Kolmogorov's differential equations for some non-stationary Markov processes on a countable state space*, 1972 (To be published in Proc. Cambridge Philos. Soc.)
8. T. H. Hildebrandt, *Introduction to the theory of integration*, Academic Press, New York · London, 1963.
9. P. J. Huber, *Robust estimation of a location parameter*. Ann. Math. Statist. 35 (1964), 73–101.

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