ON HÖRMANDER’S
THEOREM ABOUT SURJECTIONS OF $\mathcal{D}'$

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Summary.

Consider an operator $P : \mathcal{D} \to \mathcal{D}$. It is proved here that for $P' \mathcal{D}' = \mathcal{D}'$ it is necessary and sufficient that $P^{-1}$ is sequentially continuous in a certain sense and that it “holds singularities”. Roughly speaking, the latter means that for a certain extension $\bar{P}$ of $P$, acting no more over infinitely smooth functions, to every compact $K_1$ it is possible to assign a compact $K_2$ such that $\text{singsupp} \bar{P} u \subset K_1$ implies $\text{singsupp} u \subset K_2$. Similar results are actually established not only for $\mathcal{D}$ but for a certain wide class of ($\mathcal{LF}$)-spaces. The paper is made selfcontained and includes some results announced in [5] and [6]. It provides an answer to some questions raised by Trèves in the introduction to [8].

For topological spaces $(W, \mu)$ and $(V, \nu)$ we write $(W, \mu) \leq (V, \nu)$ if $V \subset W$ and if the identical injection of $V$ into $W$ is continuous.

Denote by $K$ the compact subsets of the $N$-dimensional Euclidean space equal to the closure of their interior. We write

$$\mathcal{D}(K) = \{ f \in \mathcal{D} : \text{suppf} \subset K \},$$

and by $\tau_{\mathcal{D}}$ we denote the usual topology of $\mathcal{D}$. In what follows, $(\mathcal{E}', \tau_{\mathcal{E}'})$ shall denote the space of distributions with compact supports with the usual topology of $\mathcal{E}'$.

We shall say that a Banach space $(L, \| \cdot \|)$, briefly $(L)$, carries singularities over $K$ if

(*) \quad (\mathcal{D}(K), \tau_{\mathcal{D}}) \supseteq (L) \supseteq (\mathcal{E}', \tau_{\mathcal{E}'})

and if $\mathcal{D}(K)$ is dense in $(L)$.

Consider an open set $\Omega$ in the $N$-dimensional Euclidean space. A family $\xi$ of Banach spaces is said to be a projective component of $\mathcal{D}(\Omega)$ if every space from $\xi$ carries singularities over some compact $K \subset \Omega$ and if the following conditions hold.

1) To every compact $K \subset \Omega$ there corresponds an $(L) \in \xi$ which carries singularity over $K$.

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2) The family $\xi$ contains a sequence which is decreasing and cofinal with respect to the relation $\succeq$.

With every projective component $\xi$ we associate the space $L_\xi$ which is the union of all $L$ for $(L) \in \xi$. A sequence is said to be convergent in $L_\xi$ if it converges in an $(L) \in \xi$.

Consider open subsets $\Omega_1$ and $\Omega_2$ of the $N_1$- and $N_2$-dimensional Euclidean spaces respectively and a linear mapping $P$ from $\mathcal{D}(\Omega_1)$ to $\mathcal{D}(\Omega_2)$ continuous with respect to the usual topologies. Take components $\xi_1$ and $\xi_2$ of $\mathcal{D}(\Omega_1)$ and $\mathcal{D}(\Omega_2)$ respectively. Given $(L_\xi) \in \xi_2$, call $\{(f_n, g_n)\} \in L_{\xi_1} \times (L_2 + \mathcal{D}(\Omega_2))$ convergent if $\{f_n\}$ converges in $L_{\xi_1}$, $\{g_n\}$ converges in $L_{\xi_2}$ and in addition $\{g_n\}$ converges uniformly with all derivatives off a compact for which $(L_2)$ carries singularities. Let $\overline{P}_{L_2}$ denote the sequential closure of $P$ in $L_{\xi_1} \times (L_2 + \mathcal{D}(\Omega_2))$. We say that $P^{-1}$ holds singularities from $\xi_2$ to $\xi_1$ if the following condition holds

$$(H^*)_{\xi_2, \xi_1} \colon \text{To every } (L_\xi) \in \xi_2 \text{ there corresponds an } (L_\xi) \in \xi_1 \text{ such that the domain of } \overline{P}_{L_2} \text{ is contained in } L_1 + \mathcal{D}(\Omega_1).$$

**Theorem 1.** The adjoint mapping $P'$ is a surjection, that is, $P'\mathcal{D}'(\Omega_2) = \mathcal{D}'(\Omega_1)$ iff $P^{-1}$ is sequentially continuous from each $L_{\xi_2}$ to some $L_{\xi_1}$ and there exist projective components $\xi_1$ and $\xi_2$ such that $P^{-1}$ holds singularities from $\xi_2$ to $\xi_1$.

Theorem 1 can easily be expressed also for $\Omega_1$ and $\Omega_2$ being differentiable manifolds. However, the most important fact is that in the Theorem the condition for $P$ is invariant with respect to automorphisms of $\mathcal{D}$. This makes it possible to formulate an analogue to Theorem 1 for $P$ acting within a certain pretty large class of $(\mathcal{L}\mathcal{F})$-spaces and then provide a proof using purely functional analytic tools.

The case where $P$ is the convolution operator was investigated by Hörmander in [2]. One of the results of Section 4 of [2] can be expressed as follows.

**Theorem.** If $P$ is a convolution operator transforming $\mathcal{D}(\Omega_1)$ into $\mathcal{D}(\Omega_2)$, then for $P'$ to map $\mathcal{D}'(\Omega_2)$ onto $\mathcal{D}'(\Omega_1)$ it is necessary and sufficient that $P^{-1}$ is sequentially continuous from $\mathcal{D}(\Omega_2)$ to $\mathcal{D}(\Omega_1)$ and that to every compact $K_2 \subset \Omega_2$ there corresponds a compact $K_1 \subset \Omega_1$ such that for $u \in \mathcal{E}'(\Omega_1)$

\[ \text{singsupp } \overline{P}u \subset K_2 \implies \text{singsupp } u \subset K_1, \]

where $\overline{P}$ is the natural extension of $P$.

It is easy to see that (*) implies $(H^*)_{\xi_2, \xi_1}$ for $L_{\xi_2} = \mathcal{E}'(\Omega_2)$ and $L_{\xi_1} = \mathcal{E}'(\Omega_1)$. 
Because of the particular choice of the projective components, this version is not invariant under automorphisms of \( \mathcal{D} \). It would be interesting to find out if for a \( P \) which admits an extension to a continuous mapping from \( \mathcal{E}'(\Omega_1) \) to \( \mathcal{E}'(\Omega_2) \) and has \( P^{-1} \) sequentially continuous from \( \mathcal{D}(\Omega_2) \) to \( \mathcal{D}(\Omega_1) \), fulfilment of the condition \((H^*)_{\xi_2, \xi_1} \) for any pair of components implies its fulfilment for the special components which decompose \( \mathcal{E}'(\Omega_1) \) and \( \mathcal{E}'(\Omega_2) \) respectively.

Consider the following property concerning families of sets. Given any two families \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) of subsets of a fixed set, we say that \( \mathcal{F}_1 \) does not overrun \( \mathcal{F}_2 \) if to every \( U_1 \in \mathcal{F}_1 \) there corresponds an \( U_2 \in \mathcal{F}_2 \) such that \( U_1 \cap \bigcup \mathcal{F}_2 \subset U_2 \), where \( \bigcup \mathcal{F}_2 \) denotes the union of the sets from \( \mathcal{F}_2 \).

In this paper we shall consider the \((\mathcal{L}, \mathcal{F})\)-spaces \((X, \tau)\) as in [1] fulfilling the following additional requirement.

There exists an \((\mathcal{L}, \mathcal{F})\)-space \((L, \downarrow) \leq (X, \tau)\) which is the strict inductive limit of Banach spaces such that the family of Banach subspaces of \((L, \downarrow)\) does not overrun the family of Fréchet subspaces of \((X, \tau)\), where by a Banach (Fréchet) subspace we understand any subspace which is Banach (Fréchet) in the induced topology. We shall denote by \( \mathcal{F}(X, \tau) \) the family of all Fréchet subspaces of an \((\mathcal{L}, \mathcal{F})\)-space \((X, \tau)\).

A locally convex space \((L_{\xi}, \downarrow_{\xi})\) is said to be a strict \( p \)-component of \((X, \tau)\) if \((X, \tau) \supseteq (L_{\xi}, \downarrow_{\xi})\), \((L_{\xi}, \downarrow_{\xi})\) is the strict inductive limit of a sequence of Banach spaces, and if to every \( Z \in \mathcal{F}(L_{\xi}, \downarrow_{\xi}) \) there corresponds a \( U \in \mathcal{F}(X, \tau) \) such that \( Z \) is contained in the closure of \( U \) in \((L_{\xi}, \downarrow_{\xi})\). Clearly, all elements of \( \mathcal{F}(L_{\xi}, \downarrow_{\xi}) \) are Banach spaces.

A strict \( p \)-component is fully described by the family \( \mathcal{F}(L_{\xi}, \downarrow_{\xi}) \). We shall consider the family of all unit balls of spaces from \( \mathcal{F}(L_{\xi}, \downarrow_{\xi}) \). This family consists of all closed, absolutely convex bounded subsets \( C \) of \((L_{\xi}, \downarrow_{\xi})\). Then

\[
L_C = \bigcup_1^n nC \in \mathcal{F}(L_{\xi}, \downarrow_{\xi}).
\]

We shall denote by \( \xi \) the family of all \( C \) for which additionally \( X \cap L_C \) is dense in \((L_C, \downarrow_{\xi})\). Notice that the Minkowski functional \( \| \cdot \|_C \) of \( C \) induces on \( L_C \) the topology \( \downarrow_{\xi} \). We shall often write briefly \( \xi \) for \((L_{\xi}, \downarrow_{\xi})\).

The definition of \( \xi \) in [7] does not coincide with the one given here. It is, however, easily verified that every family, as defined here, can be uniquely extended to fulfil requirements of [7] with preservation of the space \((L_{\xi}, \downarrow_{\xi})\). Conversely, to every \( \xi \), as defined in [7], there corresponds a strict \( p \)-component \((L, \downarrow) \supseteq (L_{\xi}, \downarrow_{\xi})\). Notice that in [7], \( p \)-components are not assumed Hausdorff while here we deal only with Hausdorff components.
The family of all strict \( p \)-components \( \xi \) of \( (X, \tau) \) such that \( \mathcal{F}(L_{\xi}, \downarrow_{\xi}) \) does not overrun \( \mathcal{F}(X, \tau) \) we denote by \( \mathcal{P}(X, \tau) \). For \( \xi_{1}, \xi_{2} \in \mathcal{P}(X, \tau) \) we write \( \xi_{1} \leq \xi_{2} \) if \( (L_{\xi_{1}}, \downarrow_{\xi_{1}}) \leq (L_{\xi_{2}}, \downarrow_{\xi_{2}}) \).

For locally convex spaces \( (V_{i}, \nu_{i}), i = 1, 2 \), we write

\[
(V_{1}, \nu_{1}) \wedge (V_{2}, \nu_{2}) = (V_{1} + V_{2}, \nu_{1} \wedge \nu_{2})
\]

for the inductive limit of those spaces, that is for \( \nu_{1} \wedge \nu_{2} \) is set the finest locally convex topology such that \( (V_{1} + V_{2}, \nu_{1} \wedge \nu_{2}) \leq (V_{i}, \nu_{i}) \) for \( i = 1, 2 \).

Consider a pair \( (Y, \sigma) \) and \( (X, \tau) \) of \( (\mathcal{L}, \mathcal{F}) \)-spaces and a continuous linear mapping \( P \) from \( (Y, \sigma) \) to \( (X, \tau) \). Take \( \xi \in P(Y, \sigma) \) and \( \xi \in P(X, \tau) \).

We say that \( P^{-1} \) is \( \kappa \)-continuous from \( \xi \) to \( \xi \) or we write briefly \( P^{-1} \in (M)_{\xi, \xi} \) if the following condition holds.

\[ (M)_{\xi, \xi} : \text{To every } B \in \xi \text{ there corresponds a } C \in \xi \text{ such that to every } U \in \mathcal{F}(X, \tau) \text{ there corresponds a } V \in \mathcal{F}(Y, \sigma) \text{ in such a way that } P^{-1} \text{ maps } PY \cap (L_{B} + U) \text{ into } L_{C} + V \text{ and that it is continuous from } (L_{B}, \downarrow_{\xi}) \wedge (U, \tau) \text{ to } (L_{C}, \downarrow_{\xi}) \wedge (V, \sigma). \]

We say that \( P^{-1} \) is \( \kappa \)-continuous from \( (X, \tau) \) to \( (Y, \sigma) \) or we write briefly \( P^{-1} \in (M) \) if the following condition holds.

\[ (M) : \text{To every } \zeta \in \mathcal{P}(Y, \sigma) \text{ there corresponds a } \xi \in \mathcal{P}(X, \tau) \text{ such that } P^{-1} \in (M)_{\zeta, \xi}. \]

We say that \( P^{-1} \) holds singularities from \( \xi \) to \( \zeta \) or we write briefly \( P^{-1} \in (H)_{\xi, \zeta} \) if the following condition holds.

\[ (H)_{\xi, \zeta} : \text{To every } B \in \xi \text{ there corresponds a } C \in \zeta \text{ such that for every } \{y_{n}\} \subseteq Y \text{ tending to } y \text{ in } (L_{\xi}, \downarrow_{\xi}) \text{ with } \{Py_{n}\} \text{ converging in } (L_{B}, \downarrow_{\xi}) \wedge (X, \tau) \text{ we have } y \text{ belonging to } L_{C} + Y. \]

We say that \( P^{-1} \) totally holds singularities or we write briefly \( P^{-1} \in (H) \) if the following condition holds.

\[ (H) : \text{To every } \zeta \in \mathcal{P}(Y, \sigma) \text{ there corresponds a } \xi \in \mathcal{P}(X, \tau) \text{ such that } P^{-1} \in (H)_{\xi, \zeta}. \]

We say that \( P^{-1} \) is sequentially continuous on components if to every \( \zeta \in \mathcal{P}(Y, \sigma) \) there corresponds \( \xi \in \mathcal{P}(X, \tau) \) such that \( P^{-1} \) is sequentially continuous from \( (L_{\xi}, \downarrow_{\xi}) \) to \( (L_{\xi}, \downarrow_{\xi}) \).

A \( p \)-component is said to be reflexive if \( \mathcal{F}(L_{\xi}, \downarrow_{\xi}) \) consists of reflexive Banach spaces. From now on we shall assume that to every \( \xi_{1} \in \mathcal{P}(X, \tau) \) there corresponds a reflexive \( \xi_{2} \in \mathcal{P}(X, \tau) \) with \( \xi_{1} \leq \xi_{2} \).

**Theorem 2.** The adjoint \( P' \) of \( P \) maps the dual \( X' \) of \( (X, \tau) \) onto the dual \( Y' \) of \( (Y, \sigma) \) iff \( P^{-1} \) is \( \kappa \)-continuous from \( (X, \tau) \) to \( (Y, \sigma) \).

**Theorem 3.** The mapping \( P^{-1} \) is \( \kappa \)-continuous from \( (X, \tau) \) to \( (Y, \sigma) \) iff it is sequentially continuous on components and totally holds singularities.
Theorem 4. Take ζ, λ ∈ ℙ(Y, σ). If ζ ≥ λ and λ does not overrun ζ, then if \( P^{-1} \) holds singularities from some \( \xi \in ℙ(X, τ) \) to λ, it holds singularities from \( \eta \) to \( \xi \) for any \( \xi \leq \eta \in ℙ(X, σ) \).

Remark. Denoting by \( \overline{P_B} \) the sequential closure of \( P \) in \( (L_\xi, \downarrow_\xi) \times [(L_B, \downarrow_\xi) \wedge (X, τ)] \) the condition \((H)_{\xi, \xi, \tau} \) is equivalent to the following condition.

\((H^*)_{\xi, \tau} \): To every \( B \in \xi \) there corresponds a \( C \in \xi \) such that the domain of \( \overline{P_B} \) is contained in \( L_C + Y \).

The rest of this paper shall be devoted to verifying Theorems 2, 3 and 4, and here we notice that Theorem 1 is an easy consequence of these theorems. Indeed, the not overrunning conditions are certainly fulfilled for projective components of \( ℙ(Ω) \) as they were defined here. This is because neither the family of Fréchet subspaces of \( ℙ(Ω) \) nor any component of \( ℙ(Ω) \) can be overrun by the “ultimate” component which is the decomposition of \( ℋ(Ω) \) into Banach spaces (of course not a strict decomposition). Joining Theorems 2, 3 and 4 with the Remark, we obtain Theorem 1 as a trivial corollary.

Proposition 1. Let \( \xi \in ℙ(X, τ) \) and \( C \in \xi \). Then \( (L_C, \downarrow_\xi) \wedge (X, τ) \) is again an \( (L\mathcal{F}) \)-space and it is the inductive limit of \( (L_C, \downarrow_\xi) \wedge (X_n, τ) \) for every decomposition \( \{X_n\} \subset \mathcal{F}(X, τ) \) of \( X \).

Proof. Since \( \{(X_n, τ)\} \) is strict, \( \{(L_C, \downarrow_\xi) \wedge (X_n, τ)\} \) is strict as well and thus limind \( (L_C, \downarrow_\xi) \wedge (X_n, τ) \) is an \( (L\mathcal{F}) \)-space. A seminorm on \( L_C + X \) is continuous in \( (L_C, \downarrow_\xi) \wedge (X, τ) \) iff it is continuous in every \( (L_C, \downarrow_\xi) \wedge (X_n, τ) \) and the Proposition follows.

We shall write \( P_\xi \geq \eta \) if for every \( B \in \xi \) there is a \( C \in \eta \) such that \( P(Y \cap B) \subset C \). This simply means that \( P \) is continuous from \( (L_\xi, \downarrow_\xi) \) to \( (L_\eta, \downarrow_\eta) \). We also put

\[ I_{(U \cap C)^o} = \bigcup_{i=1}^{\infty} n(U \cap C)^o, \]

where \(^o\) denotes the polar in \( U' \).

Lemma 1. If \( P^{-1} \) is \( \lambda \)-continuous from \( \xi \) to \( \xi \), and if \( \eta \in ℙ(X, τ) \) is such that \( \eta \leq \xi \) and \( P_\xi \geq \xi \), then setting \( U = PY \) we have \( U \in (ACC)_{\eta, \xi} \), that is, \( U \) fulfills the following condition.

\((ACC)_{\eta, \xi} \): To every \( B \in \xi \) there corresponds a \( D \in \eta \) such that for every \( Z \in ℙ(X, τ) \) we have
\[ L_B \cap (U \cap Z) \subset (U \cap L_D)^-, \]

where the closures — are taken subsequently in \((L_B, \downarrow_{\xi}) \land (X, \tau)\)
and in \((L_D, \downarrow_{\eta})\).

**Proof.** Take \(B \in \xi\) and adjust \(C \in \zeta\) according to \((M)_{\xi, \zeta}\). Subsequently, take \(D \in \eta, D \supset B\), in such a way that it is \(P(Y \cap C) \subset D\).
Fix \(Z\) and take a sequence \(\{Py_n\} \subset U \cap Z\) convergent to some \(x\) in \((L_B, \downarrow_{\xi}) \land (Z, \tau)\). From \((M)_{\xi, \zeta}\) it follows that \(\{y_n\}\) converges to some \(y\) in \((L_C, \downarrow_{\zeta}) \land (V, \sigma)\) for some \((\mathcal{F})\)-subspace \(V\) of \((Y, \sigma)\). Hence, \(y_n = c_n + v_n\), where \(\{c_n\}\) tends to some \(c\) in \((L_C, \downarrow_{\zeta})\) and \(\{v_n\}\) tends to some \(v\) in \((V, \sigma)\).
Since \(P(Y \cap C) \subset D\), the sequence \(\{Pc_n\}\) has a limit \(d\) in \((L_D, \downarrow_{\eta})\) and we have \(Py_n = Pc_n + Pv_n\) converging to \(x = d + Pv \in L_B \subset L_D\). Thus \(Pv = x - d \in U \cap L_D\) and \(P(c_n + v)\) tends to \(x\) in \((L_D, \downarrow_{\eta})\) which concludes the proof.

Put \((L_{A'}, \|\cdot\|_{A'}) = \text{the adjoint of } (L_A, \|\cdot\|_A)\). Write \(U \in (A_0)_{\eta, \xi}\) if the following condition holds.

\((A_0)_{\eta, \xi}\): To every \(B \in \xi\) there corresponds a \(D \in \eta, D \supset B\), such that to every \(\varepsilon > 0\), every \(Z \in \mathcal{F}(X, \tau)\) and every \(z' \in L_D\) vanishing on \(U \cap D\) there corresponds an \(x' \in X'\) bounded on \(X \cap B\) and vanishing on \(U \cap Z\) such that \(\|\bar{x}' - \bar{z}'\|_{B'} < \varepsilon\), where \(\bar{x}' \in L_{B'}\) denotes the extension of the restriction of \(x'\) to \(X \cap L_B\) and \(\bar{z}'\), denotes the restriction of \(z'\) to \(L_B\).

We then prove the following lemma, cf. [5, Proposition 2]. (The property \((A_0)\) seems to be related to the notion of orthogonality introduced in [3] by Pták.)

**Lemma 2.** If \(\xi\) is reflexive, then \((ACC)_{\eta, \xi} \subset (A_0)_{\eta, \xi}\).

**Proof.** Notice that \((A_0)_{\eta, \xi}\) amounts to the following statement. Given \(B \in \xi\), we can find \(D \in \eta, D \supset B\), such that for every Fréchet subspace \(Z\) of \((X, \tau)\) the closure of the subspace

\[ V_1 = \{\bar{x}' \in L_{B'} : x' \in L_{(X \cap B')}, x'(U \cap Z) = \{0\}\} \]

with respect to the norm \(\|\cdot\|_{B'}\) contains the subspace

\[ V_2 = \{\bar{z}' \in L_{B'} : z' \in L_{D'}, z'(U \cap L_D) = \{0\}\} \]

Due to reflexivity of \(\xi\), it is sufficient to show weak* density of \(V_1\), that is, if for \(z \in L_B\) all functionals from \(V_1\) vanish on \(z\), then all functionals from \(V_2\) vanish on \(z\) as well.
The space \( V_1 \) consists of the restrictions to \( L_B \) of functionals from
\[
V_1 = \{ x' \in M' : x'(U \cap Z) = \{0\} \},
\]
where \((M, \mu) = (L_B, \downarrow \xi) \land (X, \tau)\). Hence, to prove the Lemma, we have to show that from \((ACC)_{\eta, \xi}\) it follows that if for \( z \in L_B \) all functionals from \( V_1 \) vanish on \( z \), then all the functionals from \( V_2 \) vanish on \( z \) as well. The first part of this means that \( z \in (U \cap Z)^e \), where the closure \( - \) is taken in \((M, \mu)\), and the second part amounts to \( z \in (U \cap L_D)^e \), where the closure \( - \) is taken in \((L_D, \downarrow \eta)\), so that the above-stated implications amounts to the inclusion from \((ACC)_{\eta, \xi}\) and the Lemma follows.

**Lemma 3.** (Cf. [7, Theorem 5.1].) Consider \( \xi, \eta \in \mathcal{P}(X, \tau) \), \( \xi \sqsupseteq \eta \), and \( \lambda \in \mathcal{P}(Y, \sigma) \). If \( P^{-1} \in (M)_{\eta, \lambda} \) and \( PY \in (A_\eta)_{\eta, \xi} \), then \( P' \in (NO^*)_{\xi, \lambda} \), that is, \( P' \) satisfies the following condition.

\((NO^*)_{\xi, \lambda}\): To every \( B \in \xi \) there corresponds a \( C \in \lambda \) such that to every \( y' \in (Y \cap C)^c \) and every \( Z \in \mathcal{F}(Y, \sigma) \) there corresponds an \( x' \in (X \cap B)^c \) such that \( y'y = (P'x')y \) for \( y \in Z \).

**Proof.** Take \( B \in \xi \) and adjust \( D \in \eta, D \supset B \), to fulfil \((A_\eta)_{\eta, \xi}\). Subsequently, adjust to \( D \) a \( C \in \lambda \) to fulfil \((M)_{\eta, \lambda}\). The condition \((M)_{\eta, \lambda}\) allows us to make it so that
\[(*) \quad P \in D \implies x \in \frac{1}{2} C .\]

Fix a \( Z \in \mathcal{F}(Y, \sigma), Z \supset Y \cap L_C \). Since \( P \) is continuous, we can find a \( V \in \mathcal{F}(X, \tau) \) such that \( PZ \subset V \). Additionally, we shall require that \( V \supset X \cap L_D \). Take \( y' \in (Y \cap C)^c \). Since \( y'P^{-1} \) is continuous in
\[(L_D, \downarrow \eta) \land (V, \tau) ,\]
we can extend it over \( V \) to a \( \downarrow \eta \land \tau \)-continuous functional and then we can still extend the obtained functional over to \( u' \in X' \). Denoting by \( ||u'|| \) the sup norm of \( u' \) in \((PY \cap L_D, ||.||_D)\), we get from \((*)\)
\[||u'|| \leq \frac{1}{2} ||P'u'||_{(Y \cap C)^c} .\]

Thus, for the norm-preserving extension \( z' \in L_D' \) of the restriction of \( u' \) to \((PY) \cap L_D \) we obtain
\[||z'||_{L_D'} \leq \frac{1}{2} ||P'u'||_{(Y \cap C)^c} .\]

Denoting by \( \tilde{z}' \) the restriction of \( z' \) to \( L_B \), we obtain
\[||\tilde{z}'||_{L_B'} \leq ||z'||_{L_D'} \leq \frac{1}{2} ||P'u'||_{(Y \cap C)^c} ,\]
\[\leq \frac{1}{2} (||y' - P'u'||_{(Y \cap C)^c} + ||y'||_{(Y \cap C)^c}) \leq \frac{1}{2} .\]
Writing \( u^* \in L_D' \) for the extension of the restriction of \( u' \) to \( X \cap L_D \), we notice that \( z' - u^* \in L_D' \) vanishes on \((PY) \cap L_D\). Using \((A_0)_{n,\xi}\) we find \( v' \in X' \) bounded on \( X \cap B \) and vanishing on \((PY) \cap V\) such that
\[
\|z' - u^*\|_{B'} < \frac{1}{2},
\]
where \( z', u', v' \in L_B' \) denote the extensions of the restrictions to \( X \cap L_B \) of \( z', u', v' \) respectively. Since \( PZ \subset V \), we have \((P'v')y = 0 \) for \( y \in Z \). Since \((y' - P'u')y = 0 \) for \( y \in Z \), setting \( x' = u' + v' \), we obtain \( y'y = (P'x')y \) for \( y \in Z \). Moreover,
\[
\|x'\|_{(X \cap B)^o} \leq \|v' - (z' + u')\|_{B'} + \|z'\|_{B'} \leq 1,
\]
and the Lemma follows.

**Lemma 4.** We have \((NO^*)_{\xi,\lambda} \subset (O)_{\xi,\lambda}\), where \( P' \in (O)_{\xi,\lambda} \) if the following condition holds.

\((O)_{\xi,\lambda}: \) To every \( B \in \xi \) there corresponds a \( C \in \lambda \) such that to every \( y' \in (Y \cap C)^o \) there corresponds an \( x' \in (X \cap B)^o \) such that \( y' = P'x' \).

**Proof.** Take an ascending sequence \( \{B_n\} \in \xi \) cofinal with \( \xi \) such that \( B_1 = B \). Subsequently, to every \( B_n \) assign \( C_n \in \lambda \) according to the requirements of \((NO^*)_{\xi,\lambda}\). Finally, let \( \{Z_n\}, \ Y = \bigcup_1^\infty Z_n \), be an ascending sequence of \( (\mathcal{F}) \)-subspaces of \( (Y, \sigma) \) such that \( Z_n \supset Y \cap C_n \) for \( n = 1, 2, \ldots \).

Take \( y' \in (Y \cap C)^o, C = C_1 \), and assign to it \( x_1' \in 2^{-1}(X \cap B_1)^o \) such that \( y'y = (P'x_1')y \) for \( y \in Z_2 \). Hence \( y_2' = y' - P'x_1' = (Y \cap C_2)^o \) and we can find \( x_2' \in 2^{-2}(X \cap B_2)^o \) such that \( y_2'y = (P'x_2')y \) for \( y \in Z_3 \). Continuing this way, we produce a sequence \( x_n' \in 2^{-n}(X \cap B_n)^o \) such that \( y'y = (P'x_n')y \) for \( y \in Z_{n+1} \). It is easy to see that \( x' = \sum_1^\infty x_n' \) is a well defined functional belonging to \( (X \cap B)^o \) such that \( y' = P'x' \), and this concludes the proof of Lemma 4.

**Proposition 2.** If \( P \) is such that to every \( \xi \in \mathcal{P}(Y, \sigma) \) there corresponds \( \xi \in \mathcal{P}(X, \tau) \) such that \( P^{-1} \) is \( \lambda \)-continuous from \( \xi \) to \( \xi \), then to every \( \lambda \in \mathcal{P}(Y, \sigma) \) there corresponds a \( \xi \in \mathcal{P}(X, \tau) \) such that \( P' \) fulfills \((O)_{\xi,\lambda}\).

**Proof.** To a given \( \lambda \in \mathcal{P}(Y, \sigma) \) we first assign \( \eta \in \mathcal{P}(X, \tau) \) such that \( P^{-1} \in (M)_{\eta,\lambda} \). Then we find \( \xi \in \mathcal{P}(Y, \sigma), \xi \geq \lambda, \) such that \( P_\xi \geq \eta \) and finally to \( \zeta \) we assign a reflexive \( \xi \in \mathcal{P}(X, \tau), \xi \geq \eta, \) such that \( P^{-1} \in (M)_{\xi,\tau} \). The following commutative diagram describes the situation,
From Lemma 1 we obtain $PY \in (ACC)_{\eta, \xi} \subset (A_0)_{\eta, \xi}$ and applying subsequently Lemmas 3 and 4, we obtain $P' \in (NO^*)_{\xi, \lambda} \subset (O)_{\xi, \lambda}$ which concludes the proof.

**Corollary 1.** If $P$ fulfils the requirements of Proposition 2, then $P' X' = Y'$.

**Proof.** It is sufficient to notice that to every $y' \in Y'$ there corresponds a $\lambda \in \mathcal{P}(Y, \sigma)$ such that $y'$ is bounded on $Y \cap B$ for every $B \in \lambda$ and then apply Proposition 2.

Take an $(L \mathcal{F})$-space $(Z, \delta)$ and $\lambda \in \mathcal{P}(Z, \delta)$. Let $Z'$ be the adjoint of $(Z, \delta)$. We shall define a metric topological group $(Z', \xi_\lambda^o)$ as follows. First we choose an ascending cofinal sequence $\{C_n\} \subset \lambda$. Then we put for $x' \in Z'$

$$q_n(x') = t_n/(1 + t_n) \quad \text{for} \quad t_n = ||x'||_{(Z \cap C)'} < \infty$$

$$= 1 \quad \text{otherwise}$$

and then

$$q_\lambda^o(x') = \sum_{n=1}^{\infty} 2^{-n} q_n(x') .$$

It is easy to see that the topology induced by the metric $q_\lambda^o(x' - y')$ does not depend on the choice of the sequence $\{C_n\}$. The convergence of $\{x'_n\} \subset Z'$ to zero in $(Z', q_\lambda^o)$ means that given $C \in \lambda$, there exists an $n_C$ such that $x'_n \in C^o$ for $n > n_C$. Therefore, the set of polars $C^o$ of $C$ from $\lambda$ constitutes a basis of neighbourhoods of zero in $(Z', q_\lambda^o)$.

It is left to the reader to verify that $(Z', q_\lambda^o)$ is always complete. The object $(Z', q_\lambda^o)$ was introduced already in [7] and was called the $(\mathcal{F})$-class polar to $\lambda$.

Now, let us return to our original setup with two $(L \mathcal{F})$-spaces $(Y, \sigma)$, $(X, \tau)$ and a continuous mapping $P$ from $Y$ to $X$. We have the following

**Lemma 5.** Given $\lambda \in \mathcal{P}(Y, \sigma)$ and $\xi \in \mathcal{P}(X, \tau)$, $P' \in (O)_{\xi, \lambda}$ from Lemma 4 means that $P'$ is an open mapping of $(X', q_\xi^o)$ onto $(Y', q_\lambda^o)$ and if $P' \in (O)_{\xi, \lambda}$, then $P^{-1}$ is $\lambda$-continuous from $\xi$ to $\lambda$. 
PROOF. Suppose that $P'$ admits $(O)_{\xi, \lambda}$ and $P^{-1}$ is not $\lambda$-continuous from $\xi$ to $\lambda$. Then there exists a $B \in \xi$ such that for no $C \in \lambda$ the condition $(M)_{\xi, \lambda}$ is fulfilled. Choose $C \in \lambda$ which corresponds to $B$ according to $(O)_{\xi, \lambda}$. Since, in particular, $(M)_{\xi, \lambda}$ does not hold for this choice of $C$, either there exists a $U \in \mathcal{F}(X, \tau)$ such that for no $V \in \mathcal{F}(Y, \sigma)$ the image by $P^{-1}$ of $PY \cap (L_B + U)$ is contained in $V + L_C$, or, if for some $V$ this image is contained in $V + L_C$, the mapping $P^{-1}$ is not continuous from $(L_B, \upward)(U, \tau)$ to $(L_C, \upward)(V, \sigma)$. In the first case we take a sequence $\{y_n\} \subset Y$ such that $\{Py_n\} \subset (L_B + U)$ and that for a cofinal ascending $\{V_n\} \subset \mathcal{F}(Y, \sigma)$ we have $y_n \in (L_C + V_{n+1}) - (L_C + V_n)$. Multiplying if necessary by $t_n > 0$, we can always make $\{Py_n\}$ bounded in the space $(L_B, \upward)(U, \tau)$, while $\{y_n\}$ cannot be bounded in the space $(L_C, \upward)(Y, \sigma)$. Hence there exists a $y'$ in the adjoint of $(L_C, \upward)(Y, \sigma)$ on which $\{y_n\}$ is not bounded, and from $(O)_{\xi, \lambda}$ it follows that there must exist an $x'$ in the adjoint of $(L_B, \upward)(X, \tau)$ such that $y'y = x'Py$ for $y \in Y$ and this contradicts boundedness of $\{Py_n\}$. In the alternative case, if $P^{-1}$ maps $PY \cap (L_B + U)$ into $L_C + V$ for some $V \in \mathcal{F}(Y, \sigma)$ but $P^{-1}$ is not continuous from $(L_B, \upward)(U, \tau)$ to $(L_C, \upward)(V, \sigma)$, we take a bounded $\{Py_n\} \subset L_B + U$ such that $\{y_n\}$ is not bounded in $(L_C, \upward)(V, \sigma)$ and choosing again $y'$ in the adjoint of $(L_C, \upward)(Y, \sigma)$ on which $\{y_n\}$ is not bounded, we arrive at a contradiction to the existence of $x'$ in the adjoint of $(L_B, \upward)(X, \tau)$ such that $y'y = x'Py$ for $y \in Y$. This concludes the proof of Lemma 5.

LEMMA 4'. For given $\lambda \in \mathcal{P}(Y, \sigma)$ and $\xi \in \mathcal{P}(X, \tau)$ we have the mapping $P'$ open from $(X', \varrho_{\xi}^o)$ to $(Y', \varrho_{\lambda}^o)$ if $P' \in (NO)_{\xi, \lambda}$, that is, if $P'$ fulfills the following condition.

$(NO)_{\xi, \lambda}$: To every $B \in \xi$ there corresponds a $C \in \lambda$ such that for every $y' \in (Y \cap C)^o$, every $D \in \lambda$ and every $\varepsilon > 0$ there corresponds an $x' \in (X \cap B)^o$ with $\|y' - P'x'\|_{(Y \cap D)^o} < \varepsilon$.

PROOF. Though it is a consequence of Proposition 12 of [4], we shall prove it independently. (This is actually repetition of Banach's proof given for $(\mathcal{F})$-spaces.) Notice at first that $(NO)_{\xi, \lambda}$ amounts to the following statement. To every $\varepsilon > 0$ there corresponds a $\delta > 0$ such that the closure in $(Y', \varrho_{\lambda}^o)$ of the image by $P'$ of the ball $\{x' \in X': \varrho_{\xi}^o(x') < \varepsilon\}$ contains the ball $\{y' \in Y': \varrho_{\lambda}^o(y') < \delta\}$. Fix any $\varepsilon > 0$ and adjust a sequence $0 < t_n \to 0$ in such a way that the closure of $P'\{x' \in X': \varrho_{\xi}^o(x') < 2^{-n}\varepsilon\}$ contains $\{y' \in Y': \varrho_{\lambda}^o(y') < t_n\}$. Then for $y' \in Y'$ with $\varrho^o(y') < t_1$ we can find $x'_1 \in X'$ with $\varrho^o(x'_1) < 2^{-1}\varepsilon$ such that $\varrho^o(y' - P'x'_1) < t_2$. Continuing this procedure, we define a sequence $\{x'_n\} \subset X'$ such that
\( \varrho_\varepsilon^\circ(x_n') \leq 2^{-n\varepsilon} \) and \( \varrho_\lambda^\circ(y' - P'(x_1' + \ldots + x_n')) < t_{n+1} \)

for every \( n \). Setting \( x' = \sum_1^\infty x_n' \), we obtain \( \varrho_\varepsilon^\circ(x') < \varepsilon \) and \( y' = P'x' \).

Hence to every \( \varepsilon > 0 \) we assigned \( \delta = t_1 > 0 \) such that

\[
P'(x' \in X' : \varrho_\varepsilon^\circ(x') < \varepsilon) \supset \{ y' \in Y' : \varrho_\lambda^\circ(y') < \delta \}
\]

and this amounts to \((O)_{\varepsilon, \lambda} \), that is, to the openness of \( P' \) from \((X', \varrho_\varepsilon^\circ)\) to \((Y', \varrho_\lambda^\circ)\).

**Lemma 6.** (Cf. [7, Theorem 7.1].) Suppose that \( P'X' = Y' \). Then to every \( \lambda \in \mathcal{P}(Y, \sigma) \) there corresponds a \( \xi \in \mathcal{P}(X, \tau) \) such that \( P' \in (NO)_{\xi, \lambda} \).

**Proof.** Let \( \{ Z_n \} \subset \mathcal{F}(X, \tau) \) be an ascending sequence of \((F)\)-subspaces cofinal with \( \mathcal{F}(X, \tau) \). For every \( n \) we provide a point-wise non-decreasing sequence of norms \( \{ \| ||_{n, m} : m = 1, 2, \ldots \} \) inducing the topology \( \tau \) on \( Z_n \). Denote by \( Z^*_n \) the subspace of \( X' \) consisting of all functionals from \( X \) which on \( Z_n \) are continuous with respect to the norm \( || ||_{n, m} \).

Clearly, for every \( n \) we have \( X' = \bigcup_{m=1}^\infty Z^*_n \).

Hence, there exists an \( m_1 \) such that \( P'Z^*_{1, m_1} \) is a second category subset of \((Y', \varrho_\lambda^\circ)\). Suppose that we have selected \( m_1, \ldots, m_n \) such that \( P'(Z^*_{1, m_1} \cap \ldots \cap Z^*_{n, m_n}) \) is a second category subset of \((Y', \varrho_\lambda^\circ)\).

Then, since

\[
P'(Z^*_{1, m_1} \cap \ldots \cap Z^*_{n, m_n}) = \bigcup_{m=1}^\infty P'(Z^*_{1, m_1} \cap \ldots \cap Z^*_{n, m_n} \cap Z^*_{n+1, m_1}),
\]

we can still find \( m_{n+1} \) such that \( P'(Z^*_{1, m_1} \cap \ldots \cap Z^*_{n+1, m_{n+1}}) \) is a second category subset of \((Y', \varrho_\lambda^\circ)\). Continuing this procedure, we produce a sequence \( \{ Z^*_{n, m_n} \} \) of sets with the above property. Notice that we can then produce a norm \( || \cdot || \) such that \( ||x||_{n, m_n} \leq a_n ||x|| \) for \( x \in Z_n, n = 1, 2, \ldots \), and that \( || \cdot || \) is continuous in every \((Z_n, \tau)\). Let \( \xi \in \mathcal{P}(X, \tau) \) be such that \((Z_n, \downarrow \xi) = (Z_n, || \cdot ||)\). Then to every \( B \in \xi \) there corresponds \( n \) such that

\[
L_{(X \cap B)^r} \supset Z^*_{1, m_1} \cap \ldots \cap Z^*_{n, m_n},
\]

that is, for every \( B, P'L_{(X \cap B)^r} \) is a second category subset of \((Y', \varrho_\lambda^\circ)\).

Here we could directly apply Proposition 10 of [4], but to make the paper self-contained, we repeat the argument essentially due to Banach.

Fix \( \varepsilon > 0 \) and define \( H_m = P'\{ x' \in X' : \varrho_\varepsilon^\circ(x'/m) < \frac{1}{2} \varepsilon \} \).

For this \( \varepsilon \) we can find a \( B \in \xi \) such that

\[
P'L_{(X \cap B)^r} \subset \bigcup_1^\infty H_m,
\]

and since the set on the left is a second category subset, there exists \( m_0 \) such that \( H_{m_0} \) is not nowhere dense in \((Y', \varrho_\lambda^\circ)\). Hence, there are \( y_0' \in Y' \) and \( r > 0 \) such that

\[
H_{m_0} \supset \{ y' \in Y' : \varrho_\lambda^\circ(y' - y_0') < r \}.
\]
where \(-\) denotes the closure in \((Y', \mathfrak{e}_\lambda^\circ)\). Since \(\mathfrak{e}_\lambda^\circ(y' - y_0'/m_0) < r/m_0\) implies
\[
\mathfrak{e}_\lambda^\circ(m_0y' - y_0') = \mathfrak{e}_\lambda^\circ(m_0(y' - y'/m_0)) \leq m_0\mathfrak{e}_\lambda^\circ(y' - y_0'/m_0) < r,
\]
we have \(m_0y' \in H_{m_0}^-\) and thus \(y' \in H_1^-\). Taking \(y_1' \in H_1\) with
\[
\mathfrak{e}_\lambda^\circ(y_1' - y_0'/m_0) < \eta = r/2m_0,
\]
we obtain \(H_1^- \ni \{y' \in Y' : \mathfrak{e}_\lambda^\circ(y_1' - y') < \eta\}\). For \(\mathfrak{e}_\lambda^\circ(y') < \eta\), we have \(\mathfrak{e}_\lambda^\circ((y_1' - y') - y_1')\). Hence \(y_1' - y' \in H_1^-\), and thus also \(y' - y_1' \in H_1^-\) so that finally
\[
(*) \quad \{y' \in Y' : \mathfrak{e}_\lambda^\circ(y') < \eta\} \subseteq (y_1' - H_1^-).
\]
Moreover, if \(y' = y_1' - u' \in y_1' - H_1\), then there are \(x_1', v' \in X'\) with \(\mathfrak{e}_\xi^\circ(x_1'), \mathfrak{e}_\xi^\circ(v') < 1/2\varepsilon\) such that \(y_1' = P'x_1'\) and \(u' = P'v'\). Setting \(x' = x_1' - v'\), we have \(y' = P'x'\) and \(\mathfrak{e}_\xi^\circ(x') < \varepsilon\) so that
\[
(**) \quad y_1' - H_1 \subseteq P'\{x' \in x' : \mathfrak{e}_\xi^\circ(x') < \varepsilon\}.
\]
Joining \((*)\) and \((**)\), we obtain
\[
\{y' \in Y' : \mathfrak{e}_\lambda^\circ(y') < \eta\} \subseteq (P'\{x' \in x' : \mathfrak{e}_\xi^\circ(x') < \varepsilon\})^-,
\]
and due to the arbitrariness of the choice of \(\varepsilon > 0\) this amounts exactly to \((NO)_{\xi,\lambda}\). Hence, the Lemma 6 is proved.

**Proposition 3.** If \(P'X' = Y'\), then to every \(\lambda \in \mathcal{P}(Y, \sigma)\) there corresponds a \(\xi \in \mathcal{P}(X, \tau)\) such that \(P^{-1}\) is \(\lambda\)-continuous from \(\xi\) to \(\lambda\).

**Proof.** Given \(\lambda \in \mathcal{P}(Y, \sigma)\), we apply first Lemma 6 and find \(\xi \in \mathcal{P}(X, \tau)\) such that \((NO)_{\xi,\lambda}\) holds. Then from Lemma 4' we conclude that also \((O)_{\xi,\lambda}\) holds for this choice of \(\lambda\). Then it remains only to apply Lemma 5 to find that for the pair \(\lambda\) and \(\xi\) the condition \((M)_{\xi,\lambda}\) holds as well, and this concludes the proof of Proposition 3.

**Proof of Theorem 2.** This proof amounts to combining Corollary 1 and Proposition 3.

**Proposition 4.** Fix arbitrary \(\xi \in \mathcal{P}(Y, \sigma)\) and \(\xi \in \mathcal{P}(X, \tau)\). Then \(P^{-1}\) is \(\lambda\)-continuous from \(\xi\) to \(\xi\) if and only if \(P^{-1}\) is sequentially continuous from \(\xi\) to \(\zeta\) and holds singularities from \(\xi\) to \(\zeta\).

**Proof.** Take \(B \in \xi\) and assign to it \(C \in \zeta\) according to \((H)_{\xi,\zeta}\). Subsequently, fix \(U \in \mathcal{F}(X, \tau)\) and take \(D \in \xi\) such that \(D \supseteq B\) and \(L_D \supseteq U\).
There exists an $A \in \zeta$, $A \supset C$, such that $P^{-1}$ is sequentially continuous from $(L_D, \downarrow \zeta)$ to $(L_A, \downarrow \zeta)$ so that we can produce an extension $S$ of $P^{-1}$ to an operator closed in $(L_D, \downarrow \zeta) \times (L_A, \downarrow \zeta)$. We shall show that $S$ which is defined on the closure of $PY \cap (L_B + U)$ transforms this closure into $L_C + Y$ and is closed in $[(L_B, \downarrow \zeta) \wedge (U, \tau)] \times [(L_C, \downarrow \zeta) \wedge (Y, \sigma)]$. Indeed, if $\{Py_n\}$ tends to some $x$ in $(L_B, \downarrow \zeta) \wedge (U, \tau)$, then it tends to $x$ in $(L_D, \downarrow \zeta)$ as well. Hence $\{y_n\}$ tends to some $y$ in $(L_A, \downarrow \zeta)$ and by $(H)_{\xi, \zeta}$ we have $y \in L_C + Y$. Clearly, $Sx = y$. Applying the closed graph theorem, we obtain continuity of $S$ from $(L_B, \downarrow \zeta) \wedge (U, \tau)$ to $(L_C, \downarrow \zeta) \wedge (Y, \sigma)$. Hence $P^{-1} \in (M)_{\xi, \zeta}$ which concludes the proof of sufficiency. To verify necessity, notice first that $\lambda$-continuity of $P^{-1}$ from $\xi$ to $\zeta$ trivially implies its sequential continuity from $(L_{\xi}, \downarrow \zeta)$ to $(L_{\zeta}, \downarrow \zeta)$. Indeed, by $(M)_{\xi, \zeta}$ one can assign to $B \in \xi$ a $C \in \zeta$ in such a way that $P^{-1}$ is continuous from $(L_B, \downarrow \zeta)$ to $(L_C, \downarrow \zeta) \wedge (V, \sigma)$ for some $V \in \mathcal{F}(Y, \sigma)$, and thus from $(L_B, \downarrow \zeta)$ to some $(L_D, \downarrow \zeta)$ for a suitable $D \in \zeta$. Since $(H)_{\xi, \zeta}$ is also an immediate consequence of $(M)_{\xi, \zeta}$, Proposition 4 holds.

**Proof of Theorem 3.** Suppose $P^{-1}$ is sequentially continuous on components and totally holds singularities from $(X, \tau)$ to $(Y, \sigma)$. Take $\zeta \in \mathcal{P}(Y, \sigma)$. Since $P^{-1}$ is continuous on components there exists a $\xi \in \mathcal{P}(X, \tau)$ such that $P^{-1}$ is sequentially continuous from $(L_{\xi}, \downarrow \zeta)$ to $(L_{\zeta}, \downarrow \zeta)$. Since $P^{-1} \in (H), \xi$ can be chosen such that $P^{-1} \in (H)_{\xi, \zeta}$, and then applying Proposition 4, we get $P^{-1} \in (M)_{\xi, \zeta}$, and by arbitrariness of $\zeta \in \mathcal{P}(Y, \sigma)$ we conclude that $P^{-1} \in (M)$. Conversely, from Proposition 4 it follows that $P^{-1} \in (M)$ implies $P^{-1} \in (H)$, and implies its sequential continuity as well, and this concludes the proof of Theorem 3.

**Lemma 7.** Given $\lambda, \zeta \in \mathcal{P}(Y, \sigma)$ such that $\lambda \leq \zeta$ and that does not overrun $\zeta$, to every $C \in \lambda$ there corresponds a $D, D \in \zeta$, such that

$$L_\zeta \cap (L_C + Y) \subset L_D + Y. $$

**Proof.** Take $D \in \zeta$ such that $L_\zeta \cap L_C \subset L_D$. If $u \in L_\zeta \cap (L_C + Y)$, then $u = c + y$, where $c \in L_C$ and $y \in Y$. Hence $c = u - y \in L_\zeta \cap L_C \subset L_D$, and consequently, $u \in L_D + Y$ which finishes the proof.

**Proof of Theorem 4.** Let $P^{-1} \in (H)_{\xi, \lambda}$. Take $B \in \xi$ and assign to it a $C \in \lambda$ according to $(H)_{\xi, \lambda}$. By Lemma 7 we can find $D \in \zeta$ such that $L_\zeta \cap (L_C + Y) \subset L_D + Y$. To verify $(H)_{\xi, \zeta}$, take $\{y_n\} \subset Y$ tending to $y$ in $(L_\zeta, \downarrow \zeta)$ with $\{Py_n\}$ convergent in $(L_B, \downarrow \zeta) \wedge (X, \tau)$. Since $(L_\zeta, \downarrow \zeta) \supset (L_\lambda, \downarrow \lambda)$, $\{y_n\}$ converges in $(L_\lambda, \downarrow \lambda)$ as well, and thus by $(H)_{\xi, \lambda}$ we have $y \in L_C + Y$. However, $y \in L_\zeta$ and thus $y \in L_\zeta \cap (L_C + Y) \subset L_D + Y$, and the Theorem follows.
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