THE CLASSIFICATION OF SIMPLY CONNECTED $H$-SPACES WITH THREE CELLS I

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0. Introduction.
Recently the list of known $H$-spaces with few cells was considerably enriched see [2], [4], [7] and [8]. Most of the non-classical $H$-spaces newly discovered were principal $G$-bundles over spheres where $G$ is a classical Lie group. In order to regain control of this new flow of $H$-spaces it seems desirable to obtain some necessary or sufficient conditions for $G$-bundles over spheres to be $H$-spaces. The following seems highly probable:

0.1. Conjecture. Let $(G_n,d)=(SU(n),2)$ or $(Sp(n),4)$. Then it is well-known that
\[ \pi_{dn-2}(G_{n-1}) = \mathbb{Z}_m, \quad m = 2^r(2k+1), \quad r > 0. \]
Let $M(n,\lambda)$ be the total space of the following induced principal fibration:

\[
\begin{array}{ccc}
G_{n-1} & \rightarrow & G_n \\
\downarrow & & \downarrow \\
M(n,\lambda) & \rightarrow & \quad \\
\downarrow & & \downarrow \\
S^{dn-1} & \xrightarrow{f_{\lambda}} & S^{dn-1},
\end{array}
\]
\[
\text{deg} f_{\lambda} = \lambda. \quad \text{(In [7] and in the sequel, $M(2,\lambda)$ is denoted by $M_{\lambda}^{10}$ for $d=4$.)}
\]

0.1.1. If $(\lambda,m)=1$, then $M(n,\lambda)$ is a loop space. $M(n,\lambda) \approx M(n,\lambda')$ if and only if $\lambda \equiv \pm \lambda' \pmod{m}$. For $d=4$ and $n=2$ one has $m=12$, $M(2,1)=Sp(2)$, and $M_{8}^{10}=M(2,5) \approx M(2,7)$ is the Hilton-Roitberg manifold proved to be a loop space by Stasheff [7].

0.1.2. If $\lambda$ is odd and satisfies: $p|\lambda$ and $p^r|m$ implies $p^r|\lambda$ for every prime $p$, then $M(n,\lambda)$ is an $H$-space. If $dn-1=7$, the restriction that $\lambda$ is odd is not necessary.

0.1.3. If $dn-1 > 7$ and $M(n,\lambda)$ admits an $H$-structure, then $\lambda$ is odd.
This paper is first in a sequence of two referred to as CSCH3 I and CSCH3 II. In CSCH3 II, conjectures 0.1.1 and 0.1.2 are given some consideration and an outline of a proof is given. In the case \( d n - 1 < 7 \), conjecture 0.1.3 is very simple. As a whole 0.1 is verified in [2] for the case \( d = 2, \ d n - 1 = 7 \). The main object of CSCH3 I is to settle the case \( d = 4, \ d n - 1 = 7 \). It turns out that the validity of 0.1.3 for \( d = 4, \ n = 2 \) was the last obstruction to the classification of simply connected \( H \)-spaces with three cells. This was also realized independently by Hilton–Roiter–berg [5] and by M. Curtis–Mislin–E. Thomas (Private communications). Thus, the ultimate goal of the present papers is to prove

0.2. The Classification Theorem. Let \( X \) be a simply connected CW complex with three cells. If \( X \) admits an \( H \)-structure, then \( X \) is homotopy equivalent to one of the following eight complexes:

\[ S^3 \times S^3, \quad SU(3), \quad M_k^{10}, \ k = 0, 1, 3, 4, 5, \quad S^7 \times S^7 \, . \]

This theorem is proved in CSCH3 II. The main theorem in CSCH3 I is the following:

0.3. Main Theorem I. Let \( X \) be a simply connected CW complex satisfying:

\begin{align*}
(1) & \quad H^*(X, \mathbb{Z}_2) = \Lambda(x_3, x_7) \text{ in dim } \leq 13, \ x_i \in H^i(X, \mathbb{Z}_2) , \\
(2) & \quad \pi_6(X) = \mathbb{Z}_2 .
\end{align*}

If \( X \) admits an \( H \)-structure, then \( H^{14}(X, \mathbb{Z}_2) \neq 0 \). In particular, the complexes \( M_k^{10}, \ k = 2 \text{ mod 4, do not admit } H \text{-structures.} \]

The proof of 0.3 is based on calculations involving high order cohomology operations carried out in chapter 2. The proof is quite complex as the obstruction for \( M_k^{10} \) to be an \( H \)-space can be essentially detected by a cohomology operation of order 5. Fortunately this operation can be decomposed in such a way that the calculations only involve operations of order at most three.

More precisely: Two operations \( \varphi \) and \( \bar{\varphi} \) are studied. Both are operations defined on \( H^*(\cdot, \mathbb{Z}_4) \) classes with values being cosets of \( H^*(\cdot, \mathbb{Z}_2) \) (referred to as \( \mathbb{Z}_4/\mathbb{Z}_2 \) operations). The operation \( \varphi \) is a third-order operation of degree 8, while \( \bar{\varphi} \) is a secondary operation of degree 4. The following type of relation between them is established (Proposition 2.3):

\[ \varphi 2 \Rightarrow S_{g^4} \bar{\varphi} + S_{g^8} q_2 , \]

where \( q_2 \) is the reduction \( H^*(\cdot, \mathbb{Z}_4) \rightarrow H^*(\cdot, \mathbb{Z}_2) \). The evaluations of \( \varphi \) and \( \bar{\varphi} \) on the projective plane \( B_2(\hat{X}) \) of \( \hat{X} \) (where \( \hat{X} \) is essentially \( X \) made 4-connected) imply the condition \( H^{14}(X, \mathbb{Z}_2) \neq 0 \).
1. Some definitions and notations.

Let $I = (n_1, n_2, \ldots, n_k)$ be a finite sequence of natural numbers. Put $k = l(I)$ and write $K(Z_p, I)$ for the product $\prod_{j=1}^{k} K(Z_p, n_j)$. The vector in $H^\ast(K(Z_p, I), Z_p)$ consisting of the images of the fundamental classes of $H^n(q, K(Z_p, n_j), Z_p)$ will be referred to as the fundamental vector.

If $I_1$ and $I_2$ are two sequences, an $H$-mapping $h$ (and hence $\infty$-loop map) between $K(Z_p, I_1)$ and $K(Z_p, I_2)$ can be given by an $l(I_2) \times l(I_1)$ matrix $B$ with entries in the Steenrod algebra $\mathcal{A}(p); h \ast \tau_2 = B \ast \tau_1$, where the $t_k$ are the fundamental vectors.

A (stable) generalized $k$-stage Postnikov system (mod $p$) is an $\infty$-deloopable diagram (\$\mathcal{G}\$) of the form

\[
\begin{array}{c}
\Omega K_{k-1} \xrightarrow{\delta_{k-1}} E_k \\
\downarrow r_{k-1} \\
\Omega K_{k-2} \xrightarrow{\delta_{k-2}} E_{k-1} \xrightarrow{h_{k-1}} K_{k-1} = K(Z_p, I_{k-1}) \\
\downarrow r_{k-2} \\
\quad \vdots \\
\downarrow r_2 \\
\Omega K_1 \xrightarrow{\delta_1} E_2 \xrightarrow{h_2} K_2 = K(Z_p, I_2) \\
\downarrow r_1 \\
K(Z_p, I_0) = E_1 \xrightarrow{h_1} K_1 = K(Z_p, I_1),
\end{array}
\]

where

\[
\Omega K_{i-1} \xrightarrow{\delta_{i-1}} E_i \xrightarrow{r_{i-1}} E_{i-1}
\]

is the principal $\Omega K_{i-1}$ fibration induced by $h_{i-1}$. Let $B_i$ be the $l(I_i) \times l(I_{i-1})$ matrix with entries in $\mathcal{A}(p)$ corresponding to $h_i$ if $i=1$ and to $h_i \circ \delta_{i-1}$ if $k > i > 1$. We refer to (\$\mathcal{G}\$) as to the geometric realization of $B_1, \ldots, B_{k-1}$. Conversely, given $B_1, \ldots, B_{k-1}$, $B_i$ an $m_i \times m_{i-1}$ matrix with entries in $\mathcal{A}(p)$, we write $0 \in \langle B_1, \ldots, B_{k-1} \rangle$ if $B_1, \ldots, B_{k-1}$ admit a geometric realization. A necessary condition for $0 \in \langle B_1, \ldots, B_{k-1} \rangle$ is that $B_{i+1} B_i = 0$. If $k = 3$ this condition is sufficient.

Let $0 \in \langle B_1, \ldots, B_k \rangle$. A stable $k$-order cohomology operation $\varphi$ associated with $B_1, \ldots, B_k$ is an operation on $m_0 = l(I_0)$ variables with $m_k = l(I_k)$ values given by the universal example (in the sense of [1]) $\langle x, E_k, y \rangle$, where $E_k$ is obtained from a $k+1$ stage Postnikov system (\$\mathcal{G}\$) realizing $B_1, \ldots, B_k$ geometrically, and where
x = r_{k-1}^*r_{k-2}^* \cdots r_1^*t_0, \quad y = h_k^*t_k.

If \( X \) is a \( CW \) complex, the domain \( D(\varphi) \) of \( \varphi \) consists of all vectors \( z \) of length \( l(I_0) \) of cohomology classes in \( H^*(X, \mathbb{Z}_p) \) with the property that

\[
 f_z^{(1)}: X \to E_1, \quad f_z^{(1)*}t_0 = z,
\]

can be lifted to \( f_z^{(k)}: X \to E_k \). The operation \( \varphi(z) \) is then the set \( \{f_z^{(k)*}(y)\} \), \( f_z^{(k)} \) running over all such liftings.

A similar situation occurs when \( E_1 \) is not \( K(\mathbb{Z}_p, I) \) but \( K(\mathbb{Z}_p, I) \). The operation \( \varphi \) is then referred to as being a \( \mathbb{Z}_p^*\mathbb{Z}_p \) operation as \( D(\varphi) \subset H^*(\cdot, \mathbb{Z}_p) \) while \( \varphi \subset H^*(\cdot, \mathbb{Z}_p) \). In this study, we restrict ourselves to \( \mathbb{Z}_4^*\mathbb{Z}_2 \) operations defined on a single class (that is, \( l(I_0) = 1 \)) with a single value \( (l(I_k) = 1) \) and \( k \leq 3 \). One only has to note that in this case, the condition \( B_2B_1 = 0 \) should read \( B_2B_1 \equiv 0 \mod(\#(2)S^1) \).

Throughout this paper we shall consider only \( \mathbb{Z}_2 \) Postnikov approximations, that is: we consider only the \( \mathbb{Z}_2m - k \) invariants or the integral \( k \)-invariants of order \( 2^m \).

2. Some relations among high-order operations.

We consider here two high-order \( \mathbb{Z}_4^*\mathbb{Z}_2 \) operations: A third-order operation \( \varphi \) and a secondary operation \( \tilde{\varphi} \).

Let

\[
 B_1 = \begin{pmatrix} Sq^2 \\ Sq^6 \end{pmatrix},
\]

\[
 B_2 = \begin{pmatrix} Sq^2 & 0 \\ 0 & Sq^2 \\ Sq^7 + Sq^{4,2,1} & Sq^{2,1} \end{pmatrix},
\]

\[
 B_3 = (Sq^{4,2} \quad Sq^2 \quad Sq^1).
\]

2.1 Lemma. \( 0 \in \langle B_1, B_2, B_3 \rangle \).

Proof. In order to prove this lemma one should construct a three-stage Postnikov system:

\[
 \Omega K_2 \xrightarrow{j_2} E_3 \xrightarrow{h_3} K(\mathbb{Z}_2, n + 8) = K_3
\]

\[
 \Omega K_1 \xrightarrow{j_1} E_2 \xrightarrow{h_2} K(\mathbb{Z}_2; n + 3, n + 7, n + 8) = K_2
\]

\[
 E_1 = K(\mathbb{Z}_4, n) \xrightarrow{h_1} K(\mathbb{Z}_2, n + 2, n + 6) = K_1
\]
\[ h_1^* = B_1 e_2 , \]

(where \( e_2 : H^*(\cdot, Z_4) \to H^*(\cdot, Z_2) \) is the reduction),

\[ j_i^{*-1} h_i^* = B_i , \quad i = 2, 3 . \]

Instead, one seeks a \( Z_2 \times Z_2 \) geometric realization for \( \tilde{B}_1, \tilde{B}_2, \) and \( \tilde{B}_3, \) where

\[
\tilde{B}_1 = \begin{pmatrix} Sq^1 \\ Sq^2 \\ Sq^6 \end{pmatrix},
\]

\[
\tilde{B}_2 = \begin{pmatrix} Sq^3 \\ Sq^7 + Sq^{4,2,1} \\ Sq^{6,2} \\ Sq^7 + Sq^{4,2,1} \\ Sq^{2,1} \end{pmatrix},
\]

\[
\tilde{B}_3 = B_3 = (Sq^{4,2} \ Sq^2 \ Sq^1) .
\]

That is, one seeks a three-stage Postnikov system:

\[
\begin{array}{c}
\Omega \bar{K}_2 \to j_2 \to \tilde{E}_3 \to K(Z_2, n+8) = \bar{K}_3 = K_3 \\
\downarrow \bar{r}_2 \\
\Omega \bar{K}_1 \to j_1 \to \tilde{E}_2 \to K(Z_2, n+3, n+7, n+8) = \bar{K}_2 = K_2 \\
\downarrow \bar{r}_1 \\
\tilde{E}_1 = K(Z_2, n) \to K(Z_2, n+1, n+2, n+6) = \bar{K}_1 \\
\end{array}
\]

\[
\bar{h}_1^* = \tilde{B}_1 \quad \text{and} \quad \bar{j}_i^{*-1} \bar{h}_i^* = \tilde{B}_i, \quad i = 2, 3 .
\]

Once such a realization is established one gets the following comparison:

\[
\Omega K_2 = \Omega \bar{K}_2 \to j_2 = j_2 \to E_3 = \tilde{E}_3 \to \bar{h}_3 = \bar{h}_3 \to K_3 = \bar{K}_3
\]

\[
\Omega K_1 \xrightarrow{j_1} \tilde{E}_2 \xrightarrow{\bar{h}_2} K_2 = \bar{K}_2
\]

\[
\Omega \bar{K}_1 \xrightarrow{\tilde{E}_1 \xrightarrow{\bar{h}_1} \tilde{E}_1} K_1 = K(Z_2, n + 1, n + 2, n + 6) \xrightarrow{\bar{r}_2} K(Z_2, n+1)
\]
where \( p_1 \) is the projection and \( E_i, K_i, r_i, h_i, j_i \) are the desired realization as

\[
\text{inj}^* = \begin{pmatrix}
0 & 0 \\
1 & 0 \\
0 & 1
\end{pmatrix}.
\]

To show \( 0 \in \langle \tilde{\mathcal{B}}_1, \tilde{\mathcal{B}}_2, \tilde{\mathcal{B}}_3 \rangle \) (or equivalently the existence of \( (\mathcal{F}) \)) one realizes that \( \tilde{\mathcal{B}}_2 \tilde{\mathcal{B}}_1 = 0 \). Hence \( 0 \in \langle \tilde{\mathcal{B}}_1, \tilde{\mathcal{B}}_2 \rangle \) and there exists a partial realization

\[
\begin{array}{ccc}
\Omega \tilde{\mathcal{K}}_2 & \xrightarrow{j_1} & \tilde{\mathcal{E}}_3 \\
\downarrow & & \downarrow \tilde{r}_2 \\
\Omega \tilde{\mathcal{K}}_1 & \xrightarrow{j_1} & \tilde{\mathcal{E}}_2 \xrightarrow{\tilde{h}_2} \tilde{\mathcal{K}}_2 \\
\downarrow \tilde{r}_1 & & \downarrow \tilde{h}_1 \\
\tilde{E}_1 & \xrightarrow{\tilde{h}_1} & \tilde{K}_1.
\end{array}
\]

(\( \mathcal{F} \)-partial)

As \( \tilde{\mathcal{B}}_3 \tilde{\mathcal{B}}_2 = 0 \) if \( \sigma^* \tilde{r}_1 \) and \( \tilde{r}_2 \) are the fundamental vectors of \( H^*(\Omega \tilde{\mathcal{K}}_1, \mathbb{Z}_2) \) and \( H^*(\tilde{\mathcal{K}}_2, \mathbb{Z}_2) \) respectively,

\[
\tilde{j}_1^* \tilde{h}_2^* \tilde{\mathcal{B}}_3 \tilde{r}_2 = \tilde{\mathcal{B}}_3 \tilde{\mathcal{B}}_2 \sigma^* \tilde{r}_1 = 0.
\]

Hence \( \tilde{h}_2^* \tilde{\mathcal{B}}_3 \tilde{r}_2 \in \ker \tilde{j}_1^* \), therefore,

\[
\tilde{h}_2^* \tilde{\mathcal{B}}_3 \tilde{r}_2 \in \tilde{r}_1^* PH^{(n+9)}(\tilde{\mathcal{E}}_1, \mathbb{Z}_2) \quad \text{and} \quad \tilde{h}_2^* \tilde{\mathcal{B}}_3 \tilde{r}_3 = \tilde{r}_1^* \alpha \tilde{r}_1,
\]

\( \alpha \in \mathfrak{u}(2) \), \( \deg \alpha = 9 \). Now, as

\[
S^0 = (S^8 + S^6,2)S^1 + (S^7 + S^4,2,1)S^2, \quad S^8,1 = S^8 S^1,
\]

\[
S^7,2 = S^7 S^2, \quad S^6,3 = S^6,1 S^2, \quad S^6,2,1 = S^6,2 S^1
\]

form a basis for \( \mathfrak{u}(2) \) in \( \dim 9 \),

\[
\alpha = \alpha_1 S^1 + \alpha_2 S^2 \quad \text{and} \quad \alpha \tilde{r}_1 \in \ker \tilde{r}_1^*.
\]

It follows that

\[
\tilde{h}_2^* \tilde{\mathcal{B}}_3 \tilde{r}_2 = 0, \quad \tilde{\mathcal{B}}_3 \tilde{r}_2 \in \ker \tilde{h}_2^*,
\]

and consequently

\[
\tilde{\mathcal{B}}_3 \sigma^* \tilde{r}_2 \in \text{im} \tilde{j}_2^*.
\]

where \( \sigma^* \tilde{r}_2 \) is the fundamental vector in \( H^*(\Omega \tilde{\mathcal{K}}_2, \mathbb{Z}_2) \), and \( (\mathcal{F} \)-partial) can be completed to \( (\mathcal{F}) \) by adding \( \tilde{h}_3: \tilde{\mathcal{E}}_3 \rightarrow \tilde{\mathcal{K}}_3, \tilde{j}_2^* \tilde{h}_3^* \tilde{r}_3 = \tilde{\mathcal{B}}_3 \sigma^* \tilde{r}_2 \).

2.1.1. Remark. Note that during the proof of 2.1 the arbitrary original choice of \( \tilde{h}_2^* \tilde{r}_2 \) satisfying
\[ \tilde{j}_1^* \tilde{h}_2^* t_2 = \tilde{j}_1^* h_2^* t_2 = \tilde{B}_2 \sigma^* t_1 \]

was not altered and hence \( h_2^* t_2 \) can be freely changed by any element in \( \text{im} \tilde{\sigma}_1^* \).

Let
\[ \tilde{B}_1 = \begin{pmatrix} Sq^2 \\ Sq^4 \end{pmatrix}, \quad \tilde{B}_2 = (Sq^2,1,Sq^4). \]

As \( \tilde{B}_2 \tilde{B}_1 \equiv 0 \mod \mu(2) Sq^1 \), one gets a geometric realization of \( 0 \in \langle \tilde{B}_1, \tilde{B}_2 \rangle \) as follows:

\[ \Omega \tilde{K}_1 \xrightarrow{j_1} \tilde{E}_2 \xrightarrow{\kappa_2} \tilde{K}_2 = K(\mathbb{Z}_2, n+4) \]

\( (\mathcal{G}_1) \)

\[ \begin{pmatrix} \tilde{E}_1 = K(\mathbb{Z}_4, n) \xrightarrow{\tilde{h}_1} \tilde{K}_1 = K(\mathbb{Z}_2, n+2, n+4) \end{pmatrix}. \]

\[ \tilde{h}_1^* = \tilde{B}_1, \quad \tilde{j}_1^* \tilde{h}_2^* = \tilde{B}_2. \]

Our main concern in this section is to choose \( h_2 \) and \( h_3 \) in \( (\mathcal{G}) \) in such a way that the third-order operation \( \varphi \) defined by \( (\mathcal{G}) \) and the secondary operation \( \bar{\varphi} \) defined by \( (\bar{\mathcal{G}}) \) will satisfy the conditions described in the following propositions 2.2 and 2.3.

Let \( BS\tilde{p}^{(k)} \) be the Postnikov approximation of \( B\tilde{p} \) in \( \dim \leq k \):

\[ \pi_m(B\tilde{p}) \xrightarrow{\sim} \pi_m(BS\tilde{p}^{(k)}) \quad \text{for} \ m \leq k \]

and \( \pi_m(BS\tilde{p}^{(k)}) = 0 \) for \( m > k \). Denote \( B = BS\tilde{p}^{(14)} \), let \( j : B \rightarrow B\tilde{p}^{(13)} \) and let \( \theta : \tilde{B} \rightarrow B \) be the \( K(\mathbb{Z}, 3) \) principal fibration induced by \( \tilde{g}_1 : B \rightarrow K(\mathbb{Z}, 4) \), where \( \tilde{g}_1^* t_4 \) is a generator.

**2.2. Proposition.** For every choice of a non-decomposable generator \( z \) in \( H^8(B\tilde{p}^{(13)}, \mathbb{Z}_4) \), \( h_2 \) and \( h_3 \) in \( (\mathcal{G}) \) can be so chosen that one gets a commutative diagram

\[ \begin{array}{ccc}
\tilde{B} & \xrightarrow{\sim} & K_3 \\
\downarrow \theta & & \downarrow h_3 \\
B & \xrightarrow{g_3} & E_3 \\
\downarrow j & & \downarrow h_3 \\
BS\tilde{p}^{(13)} & \xrightarrow{g_1} & E_2 \\
\downarrow g_1 & & \downarrow h_3 \\
E_1 = K(\mathbb{Z}_4, 8) & & \\
\end{array} \]

with \( g_1^* t_1 = z \), and \( E_i, h_i \) and \( r_i \) from \( (\mathcal{G}) \) for \( n = 8 \).
2.3. Proposition. With respect to the choices of \( h_i \) in \((\mathcal{G})\) made in 2.2, \( \overline{h}_2 \) in \((\mathcal{G})\) can be chosen so that the following commutative diagram is obtained:

\[
\begin{array}{ccccccccc}
K(\mathbb{Z}_2, n + 4) &=& \overline{K}_2 & \xleftarrow{\overline{h}_2} & \overline{E}_2 & \xrightarrow{\varphi_e} & E_3 & \xrightarrow{h_2} & K_3 = K(\mathbb{Z}_2, n + 8) \\
(\mathcal{G}_2) & & \downarrow{r_1} & & \downarrow{r_2} & & \downarrow{r_1} & & \downarrow{r_1} \\
\overline{E}_1 &=& K(\mathbb{Z}_4, n) & \xrightarrow{\varphi_1} & E_2 & \xrightarrow{r_1} & E_1 = K(\mathbb{Z}_4, n)
\end{array}
\]

with \( \varphi_1 * r_1 * \iota_1 = 2 \iota_1 \) and \( \varphi_e * h_2 * \iota_2 = \text{Sq}^4 \overline{h}_2 * \iota_2 + \text{Sq}^8 \varphi_2 \iota_1 * \iota_1 \), where \( \iota_1, \iota_1, \iota_2 \), and \( \iota_3 \) are the fundamental vectors.

Proof of 2.2. One can obviously obtain the following part of \((\mathcal{G}_1)\):

\[
\begin{array}{ccccccccc}
\hat{B} & \xrightarrow{\theta} & B \\
\downarrow & & \downarrow \\
(\mathcal{G}_1)\text{part} & & \\
BSp^{(13)} & \xrightarrow{g_3} & E_2 & \xrightarrow{r_1} & E_1
\end{array}
\]

Note that \( H^k(\text{BSp}, \mathbb{Z}_2) \approx H^k(\text{BSp}^{(13)}, \mathbb{Z}_2) \) for \( k \leq 14 \) and hence \( \text{Sq}^{2}z = 0, \text{Sq}^{6}z = 0 \).

If \( \mu_B \) and \( \Delta_B \) denote the multiplication \( B \times B \rightarrow B \) and diagonal \( B \rightarrow B \times B \), put \( \lambda_2 = \mu_B \Delta_B \). As \( \pi_{15}(\text{BSp}) = 0 \) and as the \( k \)-invariant of \( \text{BSp} \) in dim 17 is integral of order divisible by 4,

\[
H^*(B, \mathbb{Z}_4) = H^*(\text{BSp}^{(15)}, \mathbb{Z}_4) = H^*(\text{BSp}, \mathbb{Z}_4) = \mathbb{Z}_4[w_4, w_8, w_{12}, w_{16}]
\]

in dim \( \leq 16 \) and

\[
\mu_B * w_{4k} = \sum_{j=0}^{k} w_{4j} \otimes w_{4(k-j)}, \quad w_0 = 1,
\]

it follows that

\[
\begin{align*}
\lambda_2 * w_4 &= 2w_4, \\
\lambda_2 * w_8 &= 2w_8 + w_4^2, \\
\lambda_2 * w_{12} &= 2w_{12} + 2w_8 w_4, \\
\lambda_2 * w_{16} &= 2w_{16} + 2w_4 w_{12} + w_8^2.
\end{align*}
\]

If \( j*z = \pm w_8 + kw_4^2 \), \( k \in \mathbb{Z}_4 \), then \( \lambda_2 * j*z = 2w_8 \pm w_4^2 \) is primitive. Consequently \( g_1 \varphi_1 \lambda_2 \) is an \( H \)-mapping, and as \( H^k(B \wedge B, \mathbb{Z}_2) = 0 \) for \( k = 9, 13, \ldots \),
it follows that \([B \wedge B, \Omega K_1] = 0\) and \(g_2 \circ j \circ \lambda_2\) is an \(H\)-mapping too. As \(PH^k(B, \mathbb{Z}_2) = 0\), \(k = 11, 15\), and \(PH^{16}(B, \mathbb{Z}_2)\) is generated by \(\epsilon_2 w_4^4\), it follows that
\[
(g_2 \circ j \circ \lambda_2)^* h_2^* \iota_2 = (0, 0, \epsilon_2 w_4^4), \quad \epsilon \in \mathbb{Z}_2.
\]
Changing \(h_2\) in (\(\mathcal{D}\)) if necessary so that \(h_2^* \iota_2\) is altered by \(Sq^8 \tilde{\iota}_1^* \iota_1 = Sq^8 \tilde{\theta}_2 \tilde{\iota}_1^* \iota_1\) (and see remark 2.1.1) one may assume that \((g_2 \circ j \circ \lambda_2)^* h_2^* \iota_2 = 0\).

Let \(r_2': E_3' \rightarrow E_2\) be the fibration induced by
\[
p_{12} \circ h_2: E_2 \rightarrow K(\mathbb{Z}_2, n+3, n+7),
\]
where \(p_{12}: K(\mathbb{Z}_2; n+3, n+7, n+8) = K_2 \rightarrow K(\mathbb{Z}_2, n+3, n+7)\) is the projection. Now, if \(h_2^* \iota_2 = (v_1, v_2, v_3)\), then \((p_{12} \circ h_2)^* = (v_1, v_2)\), and hence \(v_1, v_2 \in \ker r_2'*\). Further, as
\[
0 = B_3 h_2^* \iota_2 = Sq^4 v_1 + Sq^2 v_2 + Sq^1 v_3,
\]
\(Sq^1 v_3 \in \ker r_2'\). It follows that \(r_2'^* h_2^* \iota_2 = (0, 0, \epsilon_2 v)\) for some class \(v \in H^{n+8}(E_3, \mathbb{Z}_4)\).

One gets the following diagrams:
As $h_3 \circ j_2'' \sim h_3 \circ j_2 \circ \Omega i_3 \sim h_3 \circ j_2''$ if $t_3 \in H^*(K_3, Z_2)$ is the fundamental class, then

$$(h_3'' - h_3^*)t_3 \in \ker j_2''^*, \quad (h_3'' - h_3^*)t_3 = r_2'' \circ v_1'$$

where $v_1' \in H^{n+8}(E_3', Z_2)$. Altering $\varphi_v$ (and hence $v$) by $i_2 \varphi_{v_1'}$, where $\varphi_{v_1'}: E_3' \to K_3$ and $\varphi_{v_1'} t_3 = v_1'$, $e_2 v$ is not altered and one may assume $h_3' = h_3$. Moreover, any further alteration of $v$ by an element in $r_2''(\ker e_2)$ will correspond to a change of $h_3 = h_3'$ in $\mathcal{G}$ without changing the relation $j_2''h_3 = B_3$.

Return now to the case $n = 8$ and diagram $(\mathcal{G}_1)_{\text{part}}$. As $H^k(B, Z_2) = 0$, $k = 11, 15$, it follows that $g_2 \circ j$ lifts to

$$(\mathcal{G}_4) \quad \begin{array}{ccc}
B & \xrightarrow{g_2 \circ j} & E_3' \\
\downarrow & & \downarrow r_2' \\
B & \xrightarrow{g_2 \circ j} & E_3 \\
\end{array}$$

As $[B \wedge B, K(Z_2, 10, 14)] = 0$, $g_3' \circ \lambda_2$ is an $H$-mapping. As $v$ is primitive, $\lambda_2^*g_3'v = 2aw_4^4$ ($a = 0$ or $1$ in $Z_4$).

If $t_0 \in H^n(E_1, Z_4) = H^n(K(Z_4, n), Z_4)$ is the fundamental class $Sq^8g_2r_1^*t_0 = 0$ for all $n$, then

$$Sq^8g_2r_1^*t_0 = \varrho_2 \bar{v} \quad \text{for some} \quad \bar{v} \in H^{n+8}(E_2, Z_4),$$

and for $n = 8$,

$$Sq^8g_2r_1^*t_0 = \varrho_2(r_1^*t_0)^2 = g_2 \bar{v}.$$

It follows that $2(r_1^*t_0)^2 = 2\bar{v}$. Now,

$$\lambda_2^*g_3' r_2^*(r_1^*t_0)^2 = \lambda_2^*j g_1^*t_0^2 = \lambda_2^*(\pm w_8 + \lambda w_4^2)^2 = w_4^4$$

and

$$\lambda_2^*g_3' r_2^*2aw_4^4 = \lambda_2^*g_3' r_2^*2a(r_1^*t_0)^2 = 2aw_4^4.$$

Replacing $v$ by $v + 2ar_2^* \bar{v}$ if necessary, we may assume

(a) \quad $\lambda_2^*g_3'v = 0$.

As $g_3'v \in \ker \lambda_2^*$, $\dim v = 16$, it follows that $g_3^*v$ must be in the ideal generated by $w_4$ and hence, $g_3^*v \in \ker \theta^*$. Using this fact, diagrams $(\mathcal{G}_3)$ and $(\mathcal{G}_4)$ for $n = 8$ yield

$$\begin{array}{cccc}
K(Z_2, 15) & \xrightarrow{Sq^1} & K(Z_2, 16) \\
\downarrow j_2'' & & \downarrow r_2'' \\
\hat{B} & \xrightarrow{g_2} & E_3 & \xrightarrow{h_3} & K(Z_2, 16) \\
\downarrow \theta & & \downarrow \theta \\
B & \xrightarrow{g_2'} & E_3' & \xrightarrow{\varphi_v} & K(Z_4, 16),
\end{array}$$
with \( g_3 \circ g_3' \circ \theta \sim \ast \). As \( h_3 \circ g_3 \) can be lifted to \( D: \tilde{B} \to K(\mathbb{Z}_2, 15) \), a change in \( g_3 \) by \( j_2'' \circ D \) yields \( h_3 \circ g_3 \sim \ast \), and 2.2 follows.

As a side result of this proof one gets (a) and therefore one has

\[
\begin{array}{ccc}
E_3 & \xrightarrow{\tilde{g}_3} & B \\
\mathcal{g}_3 & \downarrow & \mathcal{g}_1 \circ j_2 \\
K(\mathbb{Z}_4, 8) = E_1 & \xrightarrow{r_1 \circ r_2} & K(\mathbb{Z}_2, 8) = E_1 .
\end{array}
\]

As it was done for \( g_3 \), the map \( \tilde{g}_3 \) can be chosen so that \( h_3 \circ \tilde{g}_3 \sim \ast \). If \( i: BS^3 \to B \) is induced by

\[ BS^3 = BSp(1) \subset BSp \to B \]

one gets \( i^* \lambda_2 \ast \lambda_2 \ast \tilde{g}_3 \ast h_3 \ast t_3 = 0 \). As

\[ i^* \lambda_2 \ast \lambda_2 \ast j^* g_1 \ast t_3 = 2w^2 \]

where \( w \in H^4(BS^3, \mathbb{Z}_4) \) is a generator, one has:

2.4. Corollary. If \( \varphi \) is the third order operation induced by \( (\mathcal{G}) \) (with the 2.2 choices of \( h_i \)), then \( 0 = \varphi(2w^2) \) (with 0 indeterminancy).

Proof of 2.3. First consider the following commutative ladder:

\[
\begin{array}{ccc}
\Omega \tilde{K}_1 = K(\mathbb{Z}_2, n, n+1, n+5) & \xrightarrow{\tilde{j}_1} & K(\mathbb{Z}_2, n) \\
\mathcal{j}_1 & \downarrow & \mathcal{r}_1 \\
K(\mathbb{Z}_2, n) & \xrightarrow{i_n} & K(\mathbb{Z}_4, n) \xrightarrow{\mathcal{e}_n} K(\mathbb{Z}_2, n) \\
& & \xrightarrow{\mathcal{s}_q} K(\mathbb{Z}_2, n+1)
\end{array}
\]

where \( \mathcal{e}_2, \mathcal{h}_1, E_2 = \tilde{E}_2, \mathcal{r}_2 \) and \( \tilde{j}_1 \) are the same as in diagram \( (\mathcal{G}) \) of 2.1. Further \( j_1^* h_2 * t_2 = \tilde{B}_2 \mathcal{r}_2 \) with \( \tilde{B}_2 \) as in the proof of 2.1. If \( i_1: K(\mathbb{Z}_2, n) \to \Omega \tilde{K}_1 \) is the injection, then \( \tilde{\varphi}_1 = \tilde{j}_1 \circ i_1 \) is a lifting of \( i_2 \), and

\[
\varphi_1 \ast \tilde{h}_2 \ast t_2 = i_1 \ast j_1 \ast h_2 \ast t_2 = i_1 \ast \tilde{B}_2 \mathcal{r}_2 = \left( \frac{S_q^3}{S_q^7 + S_q^{4,2,1}}, \frac{S_q^6,2}{} \right) t_0 .
\]

Put \( \varphi_1 = \tilde{\varphi}_1 \circ \mathcal{g}_2 \) and note that

\[ \varphi_1 \ast \mathcal{r}_1 \ast t_1 = \mathcal{g}_2 \ast \tilde{\varphi}_1 \ast \mathcal{r}_1 \ast t_1 = \mathcal{g}_2 \ast i_2 \ast t_1 = 2\mathcal{t}_1 . \]
Now consider the following commutative diagram:

\[ \Omega \tilde{K}_1 \xrightarrow{(\text{inj})} \Omega \tilde{K}_1 \xrightarrow{\theta_3} \Omega \tilde{K}_2 \]

\[ \Omega \tilde{K}_1 \xrightarrow{\tilde{\varphi}_3} \Omega \tilde{K}_2 \]

\[ E_1 = K(Z, n) \xrightarrow{\theta_3} \tilde{E}_1 \]

\[ \tilde{E}_1 \xrightarrow{\tilde{r}_1} \tilde{E}_2 \]

\[ \tilde{E}_1 = K(Z, n) \xrightarrow{\theta_3} \tilde{E}_1 \]

\[ \tilde{r}_1 \]

\[ \tilde{r}_1 \]

\[ \tilde{r}_2 \]

\[ \tilde{r}_2 \]

\[ \tilde{E}_1 = K(Z, n + 2, n + 4) \]

\[ K(Z, n + 2, n + 4) \]

\[ K(Z, n + 3, n + 7, n + 8) = K_2 \]

\[ \tilde{\varphi}_3^* = \tilde{B} = \begin{pmatrix} 0 & \frac{\Sigma q^1}{0} & \frac{\Sigma q^3}{0} \\ \frac{\Sigma q^4}{0} & \frac{\Sigma q^2}{0} & \frac{\Sigma q^3}{0} \\ \frac{\Sigma q^4}{0} & \frac{\Sigma q^2}{0} & \frac{\Sigma q^4}{0} \end{pmatrix} \]

\[ \tilde{h}_1^* = \begin{pmatrix} \frac{\Sigma q^1}{0} \\ \frac{\Sigma q^2}{0} \end{pmatrix} \]

Put \( \tilde{\varphi}_3 \circ (\text{inj}) = \varphi_3 \). There exists

\[ z \in H^{n+4}(\tilde{E}_2, Z), \quad \tilde{r}_1^* z = (\Sigma q^4, \Sigma q^2, 1, \Sigma q^4) \varphi^* \tilde{r}_1, \]

and one can choose \( \tilde{h}_2 \) so that \( z = \tilde{h}_2^* \tilde{r}_2 \) and hence

\[ \tilde{r}_1^* \tilde{h}_2^* = (\Sigma q^4, \Sigma q^2, 1, \Sigma q^4), \]

since

\[ \tilde{r}_1^* \varphi_3^* \tilde{h}_3^* = (\Omega \varphi_3)^* \tilde{r}_1^* \tilde{h}_3^* \tilde{r}_1^* = (\text{inj})^* B_3 \tilde{B} \varphi^* \tilde{r}_1 \]

\[ = \Sigma q^4 (\Sigma q^2, 1, \Sigma q^4) \sigma^* \tilde{r}_1 = \Sigma q^4 \tilde{r}_1^* \tilde{h}_2^* \tilde{r}_2. \]

Hence it follows that

\[ \varphi_2^* \tilde{h}_3^* \tilde{r}_3 + \Sigma q^4 \tilde{h}_2^* \tilde{r}_2 \in \tilde{r}_1^* P H^{n+8}(\tilde{E}_1, Z) \]

and, consequently,

\[ \varphi_2^* \tilde{h}_3^* \tilde{r}_3 = \Sigma q^4 \tilde{h}_2^* \tilde{r}_2 + \Sigma q^8 \varphi \tilde{r}_1^* \tilde{r}_0, \quad \in Z_2, \quad \tilde{r}_0 \in H^*(\tilde{E}_1, Z_2). \]

If one shows that \( \bar{\varphi}(w^2) = \varphi_3 w_3 \), where \( w \in H^4(BS^3, Z_4) \) is a generator, then by 2.4, one obtains

\[ 0 = \varphi(2w^2) = \Sigma q^4 \bar{\varphi}(w^2) + \Sigma q^8 \varphi \varphi_2 w_4^2 = (1 + \varepsilon) \varphi \varphi_2 w_4^2. \]

Hence \( \varepsilon = 1 \), and 2.3 follows.

Now, the relation \( \bar{\varphi}(w^2) = \varphi_3 w_3 \) follows from general calculations carried out independently by Kristensen [6], Gitler and Milgram [3] and others, but as the adaptation of their calculations to this simple
case is not simpler than a direct proof, we give here an outline of the proof of the following:

2.5. Lemma. Let \( \langle x, Y, u \rangle \) be the universal example for the (unique) \( Z(Z_2) \) secondary operation induced by the (non-stable) relation

\[
e[(S^2_1 S^2_1) \circ \varrho] > 4.
\]

Then \( \varrho_4 x^2 \in D(\bar{\varphi}) \) and \( \varrho_2 x^3 \in \bar{\varphi}(x^2) \), where \( \varrho_4 \): \( H^*(, Z) \to H^*(, Z_4) \) is the reduction.

Proof. One has the (once deloopable) commutative diagram:

\[
\begin{array}{ccccc}
K(Z_2, 5) & \xrightarrow{(S^2_1, 0, \varrho_4^* \bar{f})} & K(Z_2, 8, 9, 11) \\
\downarrow j_0 & & \downarrow j_1 \\
Y & \xrightarrow{\bar{f}} & \bar{E}_2 \\
\downarrow r_0 & & \downarrow \bar{r}_1 \\
K(Z_4, 4) & \xrightarrow{S^2_1 \varrho_2} & K(Z_2, 8) = \bar{E}_1 \\
\downarrow S^2_1 \varrho_2 & & \downarrow \bar{r}_1 \\
K(Z_2, 6) & \xrightarrow{(S^2, 0, \varrho^*)} & K(Z_2, 9, 10, 12) \\
\end{array}
\]

\[
\begin{pmatrix} S^1_4 \\ S^2_4 \\ S^4_4 \end{pmatrix} = \bar{r}_1^* \]

\[
j_0^* \bar{f}^* h_2^* \tau_2 = S^2_4 S^2_1 \tau_5 = j_0^* S^4_4 u,
\]

where \( j_0^* u = S^2_4 \tau_5 \). Now, if \( \mu_X \) is the multiplication in \( Y \), one has

\[
\mu_X^* u = u \otimes 1 + 1 \otimes u + \varrho_2 x \otimes \varrho_2 x, \quad x = r_0^* \tau_4.
\]

It follows that

\[
S^2_4 u + \varrho_2 x^3 + \bar{f}^* h_2^* \tau_2 \in PH_{12}(Y, Z_2) \cap \ker j_0^* = r_0^* PH_{12}(K(Z, 4), Z_2) = 0.
\]

Changing \( \bar{f} \) so that \( \bar{f}^* h_2^* \) is altered by \( S^2_4 u \), that is, altering \( \bar{f} \) by \( \alpha: Y \to K(Z_2, 8, 9, 11) \), \( \alpha^* \tau_8 = u \), \( \alpha^* \tau_j = 0 \), \( j = 9, 11 \), one gets \( \varrho_2 x^3 \in \bar{\varphi}(x^2) \).

3. The proof of the main theorem.

Let \( X, \mu \) be an \( H \)-space satisfying (1) and (2).

Suppose \( PH_{14}(X, Z_2) = 0 \). Let \( \bar{\theta}: \hat{X} \to X \) be the \( K(Z, 2) \)-principal fibra-
tion induced by \( \hat{\vartheta}_0: X \to K(\mathbb{Z}, 3) \), \( \hat{\vartheta}_0 \) inducing isomorphism of \( H^3(\cdot, \mathbb{Z})/\text{torsion} \).

3.1 LEMMA. \( H^*(X, \mathbb{Z}_2) = [H^*(X, \mathbb{Z}_2]/(x_3)] \otimes \mathbb{Z}_2[\hat{\omega}_4] \otimes \Lambda(Sq^1\hat{\omega}_4) \) as a Hopf algebra over the Steenrod algebra.

PROOF. If \( \iota_3 \in H^3(K(\mathbb{Z}, 3), \mathbb{Z}) \) is the fundamental class, then \( \ker \hat{\vartheta}_0^* \) is the \( \mathfrak{a}(2) \) ideal generated by \( Sq^2\hat{\omega}_2\iota_3 \). Hence, \( H^*(X, \mathbb{Z}_2)/\text{im} \hat{\vartheta}^* \) is generated by the algebraic suspension of

\[
Sq^2\hat{\omega}_2\iota_3, \quad Sq^{4, 2}\hat{\omega}_2\iota_3, \ldots, Sq^{2n, 2n-1}, \ldots, 4, 2\hat{\omega}_2\iota_3, \ldots, \quad Sq^3\hat{\omega}_2\iota_3.
\]

Consequently

\[
H^*(X, \mathbb{Z}_2)/\text{im} \hat{\vartheta}^* = \mathbb{Z}_2[\hat{\omega}_4] \otimes \Lambda(Sq^1\hat{\omega}_4),
\]

as \( Sq^2 Sq^2\hat{\omega}_2 = 0, Sq^2\hat{\omega}_4 = 0 \) and, hence, \( Sq^{4, 1}\hat{\omega}_4 = Sq^5\hat{\omega}_4 = 0 \). It follows that

\[
H^*(X, \mathbb{Z}_2) \approx H^*(X, \mathbb{Z}_2)/(x_3) \otimes \mathbb{Z}_2[\hat{\omega}_4] \otimes \Lambda(Sq^1\hat{\omega}_4)
\]

as algebras. Since \( \hat{\omega}_4 \) is obviously primitive, the above is actually an isomorphism of Hopf algebras. The only possible non-trivial extension over \( \mathfrak{a}(2) \) of the above splitting is given by \( Sq^{2, 1}\hat{\omega}_4 = e\hat{\vartheta}^* x_7 \).

Now, \( X \) can be mapped into \( SP^{(6)} \approx SP^{(6)} \). Moreover,

\[
X^{(6)} \approx SP^{(5)} \approx (S^3)^{(5)}
\]

where \( Y^{(k)} \) is the Postnikov approximation of \( Y \). Since

\[
\mathbb{Z}_2 \approx \mathbb{Z}_2(\mathbb{Z}) \oplus \mathbb{Z}_2(SP^{(6)}) = 0,
\]

one has a fibration

\[
X^{(6)} \xrightarrow{f^{(6)}} SP^{(6)} \xrightarrow{h^{(6)}} K(\mathbb{Z}_2, 7).
\]

where \( h^{(6)*}\iota_7 \) restricts to \( \sigma_2 \sigma^* \omega_8 \in H^7(SP^{(6)}, \mathbb{Z}_2) \). Lifting \( f^{(6)} \) to \( \hat{f}^{(6)}: \hat{X} \to \hat{SP}^{(6)} = SP^{(6)} \) made 3-connected, one has

\[
H^*(\hat{SP}^{(6)}, \mathbb{Z}_2) = \mathbb{Z}_2[h^{(6)*}\iota_7] \otimes \mathbb{Z}_2[\hat{\omega}_4] \otimes \Lambda(Sq^1\hat{\omega}_4)
\]

in \( \text{dim} \leq 7 \),

\[
Sq^{2, 1}\hat{\omega}_4 = h^{(6)*}\iota_7 \quad \text{and} \quad \hat{f}^{(6)*}\hat{\omega}_4 = \hat{\omega}_4.
\]

Hence \( Sq^{2, 1}\hat{\omega}_4 = 0 \), and 3.1 follows.

3.2 COROLLARY. \( QH^*(X, \mathbb{Z}_2)/PH^*(X, \mathbb{Z}_2) \approx QH^*(\hat{X}, \mathbb{Z}_2)/PH^*(\hat{X}, \mathbb{Z}_2), \)

where \( QA \) stands for the module of indecomposables.
3.3. **Lemma.** Let $B_2(X)$ and $B_2(\hat{X})$ be the projective planes of $X$ and $\hat{X}$, respectively. There exists a commutative diagram

$$
\begin{array}{ccc}
B_2(\hat{X}) & \xrightarrow{f} & \hat{B} \\
\downarrow{B_2(0)} & & \downarrow{j_0} \\
B_2(X) & \xrightarrow{f} & BS\rho(13) \\
\downarrow{g_1} & & \downarrow{g_2} \\
K(\mathbb{Z}_4, 8) & \xrightarrow{\varphi_1} & E_2,
\end{array}
$$

where $g_2$ and $\varphi_1$ are given in 2.2 and 2.3, respectively, and

$$\epsilon_2 \sigma^* \hat{g}_1^* \iota_0 = x_7$$

where $\iota_0 \in H^8(K(\mathbb{Z}_4, 8), \mathbb{Z}_4)$ is the fundamental class.

**Proof.** As in the proof of 3.1 one has a fibration

$$[B_2(X)]^{(7)} \xrightarrow{f^{(7)}} BS\rho^{(7)} \to K(\mathbb{Z}_2, 8)$$

and hence, a diagram

$$
\begin{array}{ccc}
[B_2(X)]^{(7)} & \xrightarrow{f^{(7)}} & BS\rho^{(7)} \\
\downarrow{g_1} & & \downarrow{g_2} \\
K(\mathbb{Z}_2, 8) & \xrightarrow{\times 2} & K(\mathbb{Z}_4, 8) \\
\end{array}
$$

Since $\hat{g}_1^* \iota_0$ is a reduction of a $\mathbb{Z}_4$ class in $H^*(B_2(X), \mathbb{Z}_2)$, one gets

$$
\begin{array}{ccc}
B_2(X) & \xrightarrow{f^{(7)}} & BS\rho^{(7)} \\
\downarrow{g_1} & & \downarrow{g_2} \\
K(\mathbb{Z}_4, 8) & \xrightarrow{\times 2} & K(\mathbb{Z}_4, 8),
\end{array}
$$

where $g_1^{(7)}$ corresponds to $g_1$ in 2.2. Now, consider the mappings

$$B_1(X) = \Sigma X \xrightarrow{i} B_2(X) \xrightarrow{k} B_2(X), B_1(X)$$

and the induced sequence of $\mathbb{Z}_2$ cohomology

$$H^*(B_2(X), B_1(X)) \approx \bar{H}^*(\Sigma X) \otimes \bar{H}^*(\Sigma X) \xrightarrow{k^*} H^*(B_2(X)) \xrightarrow{i} H^*(B_1(X)).$$

It follows that $H^*(B_2(X), \mathbb{Z}_2)$ is generated (as a $\mathbb{Z}_2$ module) in dim $< 16$ by
\[ u_4, u_8, u_4 \cdot u_8, k^* (i^* u_4 \otimes \overline{u}_{11}), k^* (\overline{u}_{11} \otimes i^* u_4) \quad \text{and} \quad (\Sigma^* \overline{u}_{11} = x_3 x_7). \]

Note that
\[ \Sigma^* i^* u_{i+1} = x_i \in H^4(X, \mathbb{Z}_2), \quad i = 3, 7, \quad \text{and} \quad \hat{\theta}_1^* \varrho_2 \iota_8 = u_8. \]

All are reductions of integral classes. Hence,
\[ H^{15}(B_2(X), \mathbb{Z}_2) \subset \ker B_2(\theta)^*, \]
and as \( \pi_{11}(BSp) = 0 \), all \( k \)-invariants of \( BSp \) in \( \dim \leq 14 \) vanish on \( H^*(B_2(X), \pi_* (BSp)) \). It follows that \( f^{(n)} \) can be lifted to \( f : B_2(X) \to BSp^{(13)} \) and as
\[ H^{15}(B_2(X), \mathbb{Z}_2) \subset \ker B_2(\theta)^* \]
one gets \( f' : B_2(\hat{X}) \to BSp^{(14)} = B \) to obtain the commutative diagram:

\[
\begin{array}{cccc}
B_2(\hat{X}) & \xrightarrow{f'} & B = BSp^{(14)} \\
\downarrow B_2(\theta) & & \downarrow j \\
B_2(X) & \xrightarrow{f} & BSp^{(13)} \\
\downarrow \varrho_1 & & \downarrow \varrho_1 \\
K(\mathbb{Z}_4, 8) & \xrightarrow{\times 2} & K(\mathbb{Z}_4, 8). \\
\end{array}
\]

Finally
\[
\begin{array}{cccc}
B_2(X) & \xrightarrow{f} & BSp^{(13)} \\
\downarrow \varrho_1 & & \downarrow \varrho_1 \\
K(\mathbb{Z}_4, 8) & \xrightarrow{\varphi_1} & E_2 \\
& \xrightarrow{\times 2} & \downarrow \varphi_1 \\
& & K(\mathbb{Z}_4, 8) \\
\end{array}
\]
is commutative as
\[ [B_2(X), \Omega K_1] = H^9(B_2(X), \mathbb{Z}_2) \oplus H^{13}(B_2(X), \mathbb{Z}_2) = 0, \]
and 3.3 follows.
To complete the proof of the main theorem, we consider the diagram

\[
\begin{array}{ccc}
\quad & E_2 & x_1 \\
\quad & \downarrow r_1 & \quad \\
B_2(\hat{X}) & \quad & E_3
\end{array}
\]

The bottom face of this diagram is commutative by 3.3, the front and right hand faces commutativity follows from 2.2 and 2.3, respectively, and \( g_3^* h_3^* r_3^* = 0 \).

As
\[
\hat{g}_1^* \hat{h}_1^* r_1 = \bar{B}_1 u_3 = (Sq^2, Sq^4) u_3 = (0, \varepsilon u_3 u_4), \quad \varepsilon \in \mathbb{Z}_2,
\]
\( B_2(\theta) \hat{g}_1^* \hat{h}_1^* r_1 = 0 \); and \( \hat{g}_2 \) exists, and the back face of the above diagram is commutative.

Now the difference between \( x_2^* \hat{g}_2 \) and \( g_3^* \hat{f} \) can be measured by an element \( \alpha \) in

\[
[B_2(\hat{X}), \Omega K_2] = H^{10}(B_2(\hat{X}), \mathbb{Z}_2) \oplus H^{14}(B_2(\hat{X}), \mathbb{Z}_2) \oplus H^{15}(B_2(\hat{X}), \mathbb{Z}_2).
\]

Further, \( \hat{k}^*: H^m(B_2(\hat{X}), B_1(\hat{X}); \mathbb{Z}_2) \to H^m(B_2(\hat{X}), \mathbb{Z}_2) \) is onto for \( 10 \leq m < 16 \), and consequently

\[
\hat{g}_2^* x_2^* h_3^* r_3^* + \hat{f}^* g_3^* h_3^* r_3^* = B_3 \alpha = B_3 \hat{k}^* \alpha.
\]

As \( g_3^* h_3^* = 0 \) by 2.2 and

\[
x_2^* h_3^* r_3^* = Sq^4 h_2^* r_2^* + Sq^4 \theta_2 \bar{r}_1^* \alpha
\]

by 2.3, and since \( Sq^4 H^{12}(B_2(\hat{X}), \mathbb{Z}_2) = 0 \), one has

\[
B_3 \hat{k}^* \alpha = \hat{g}_2^* Sq^8 \theta_2 \bar{r}_1^* \alpha = B_2(\theta)^* u_3^2 = \hat{u}_3^2
\]
or as \( \hat{u}_3^2 = \hat{k}^* (t^* \hat{u}_3 \otimes t^* \hat{u}_3) \), where \( t: B_2(\hat{X}) \to B_1(\hat{X}) \),

\[
\hat{k}^* [t^* \hat{u}_3 \otimes t^* \hat{u}_3] + B_3 \alpha = 0.
\]

As \( t^* \hat{u}_3 \notin \overline{u(2)} H^*(B_1(\hat{X}), \mathbb{Z}_2) \),

\[
w = t^* \hat{u}_3 \otimes t^* \hat{u}_3 + B_3 \alpha \neq 0,
\]

Math. Scand. 30 — 14
and there exists $\vartheta_{15} \in H^{15}(B_1(\hat{X}), \mathbb{Z}_2)$, $\delta \vartheta_{15} = w$ with

$$
\delta : H^*(B_1(\hat{X}), \mathbb{Z}_2) \to H^*(B_2(\hat{X}), B_1(\hat{X}), \mathbb{Z}_2).
$$

Observe that

$$
0 \neq \Sigma^* u_{15} \in QH^{14}(\hat{X}, \mathbb{Z}_2)/PH^{14}(\hat{X}, \mathbb{Z}_2) = QH^{14}(X, \mathbb{Z}_2)/PH^{14}(X, \mathbb{Z}_2)
$$

by 3.2, and the main theorem follows.

REFERENCES


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