ON THE HOMOTOPY GROUPS OF COMPLEX
PROJECTIVE ALGEBRAIC MANIFOLDS

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0. Introduction.

In this note we study an algebraic manifold $A$ embedded in some complex-projective space $\mathbb{P}_n$ of dimension $n$, small compared with $\dim A$. In [3] the first author gave a relation between the rational homology $H_{\ast}(A, \mathbb{Q})$ of $A$ and the dimension $n$. This relation provides intermediate results between the well-known properties of hypersurfaces and the elementary fact that $A \subseteq \mathbb{P}_n$ is connected if $\dim_x A \geq \frac{1}{4}n$ at all points $x \in A$.

Here we want to generalize these intermediate results to homotopy groups. The best generalization would be the

**Theorem.** If $A \subseteq \mathbb{P}_n$ is closed algebraic, nonsingular, of dimension $a$ at each of its points, and if $2a \geq n + s$, then the relative homotopy groups $\pi_i(\mathbb{P}_n \setminus A)$ vanish for $i = 1, \ldots, s + 1$.

We do not know whether this theorem holds. Our paper contains only the following two steps towards it:

**Theorem I.** If $A \subseteq \mathbb{P}_n$ is as above, and if $2a \geq n + 1$, then $\pi_1(A) = 0$.

**Theorem II.** If $A \subseteq \mathbb{P}_n$ is as above, and if $2a \geq n + s$, then the relative homotopy groups $\pi_i(\mathbb{P}_n \setminus A)$ are finite for $1 \leq i \leq s + 1$. In particular, the groups $\pi_3(A), \ldots, \pi_s(A)$ are finite.

Theorem II is easily reduced to theorem I. Theorem I is proved using Andreotti–Grauert [1] to extend sections in unramified coverings.

1. Preliminaries.

Here we are going to state our notational conventions, and to collect the analytical tools we use.

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P_n is the complex-projective space of dimension n. We put G := U(n+1), the unitary group, and G^1 := SU(n+1), the special unitary group. G and G^1 operate on P_n by \( x \rightarrow \sigma x \) for \( x \in P_n \), \( \sigma \in G \). The letter A will always denote a connected nonsingular closed algebraic subset of \( P_n \), with dimension at least a. We assume \( 2a \geq n+1 \). Later on we shall use the maps

\[
\varphi: G \times A \to P_n, \quad p_G: G \times A \to G, \quad p_A: G \times A \to A,
\]

where \( p_G \) and \( p_A \) are projections and \( \varphi \) is the differentiable fiber bundle defined by \( \varphi(\sigma, x) = \sigma x \).

We need some properties of tubular neighbourhoods:

(a) If \( B \subseteq G \) is an open subset containing \( 1 \in G \), then \( Bx \) is open for all \( x \in P_n \).

We choose \( B \) in the following way: Since \( G \) is compact, there exist local differentiable coordinates \( l_i \) for \( G \), centered at \( 1 \), such that the function \( \Sigma l_i^2 \) is invariant under all inner automorphisms of \( G \). If \( c \in \mathbb{R} \), \( c > 0 \), is small enough, then the set \( B_c = \{ \sigma \in G \mid \Sigma l_i^2(\sigma) < c \} \) has the following properties:

i) \( B_c \) is connected;

ii) \( \sigma B_c \sigma^{-1} = B_c \) for all \( \sigma \in G \).

These imply, just as in [2, lemmata 3, 4 and 5]: there exists some \( b = b(c) \) such that for all \( x \in P_n \)

\[
B_c x = \{ y \in P_n \mid \text{dist}(x, y) < b \},
\]

where dist denotes the usual Fubini-Study metric on \( P_n \). Thus, for small \( c \), the set \( B_c A \) is a tubular neighbourhood of \( A \). We fix one such \( c \) once for all and put \( B := B_c \), \( TA := BA \). Then obviously \( T \sigma A = \sigma TA \) for \( \sigma \in G \) is a tubular neighbourhood of \( \sigma A \).

(b) For every \( \sigma \in G \) the set \( \varphi^{-1}(A) \cap (B \sigma \times A) \) is connected.

In order to prove (b) it is enough to show that \( \varphi^{-1}(A) \cap (\overline{B}_o \sigma \times A) \) is connected whenever \( B_o \subseteq B \) is an open connected subset. Now \( \varphi^{-1}(A) \) is a closed submanifold of \( G \times A \), so \( \varphi^{-1}(A) \cap (\overline{B}_o \sigma \times A) \) contains at most finitely many connected components \( K_1, \ldots, K_r \). Since these \( K_i \) are compact, their images \( p_G(K_i) \subseteq \overline{B}_o \sigma \) are closed. Now we use

i) \( A \cap \sigma'A \) is never empty for \( \sigma' \in G \), since by assumption \( 2a \geq n+1 \);

ii) \( A \cap \sigma'A \) is always connected (cf. [3, prop. 4]).

Property i) shows \( \overline{B}_o \sigma = \bigcup_{i=1}^r p_G K_i \). Since \( \overline{B}_o \sigma \) is connected, \( p_G K_i \cap p_G K_j \not= \emptyset \) for some \( i \not= j \) if \( r \geq 2 \). Then, for \( \sigma' \in p_G K_i \cap p_G K_j \), the set

\[
\varphi^{-1}(A) \cap (\{ \sigma' \} \times A) \cong A \cap \sigma'A
\]
must have more than one connected component. So \( r \geq 2 \) contradicts ii).

(c) Tubular neighbourhoods are pseudoconcave [2, Satz 3].

We need the following consequence [1, thm. 10] of this fact:

If \( F \) is a coherent analytic sheaf over \( P_n \), subject to the condition

\[
dih F > n - a,
\]

then for every point \( q \in \partial(B_d A) \), \( d < c \), there exists an arbitrarily small neighbourhood \( U \subseteq P_n \), such that the restriction

\[
H^0(U, F) \rightarrow H^0(U \cap (B_d A), F)
\]

is bijective.

Obviously, \( U_q \cap (B_d A) \) has to be connected if \( U_q \) is the connected component of \( q \) in \( U \cap (\text{support of } F) \).

2. Reduction of theorem II to theorem I.

Here we assume \( \pi_1(A) = 0 \). There is the general Hurewicz homomorphism for relative groups:

\[
\pi_i(P_n, A) \rightarrow H_i(P_n, A; Z).
\]

If \( 2a \geq n + s \), then \( H^i(P_n, A; R) = 0 \) for \( 1 \leq i \leq s + 1 \) [3, thm. III]. Therefore the groups \( H_i(P_n, A; Z) \) are finite in this range. From [4, thm. 21, p. 511] we deduce, that the Hurewicz homomorphism is an isomorphism modulo the class of abelian torsion groups. This implies that \( \pi_i(P_n, A) \) is finite if \( 1 \leq i \leq s + 1 \). Next we use the relative homotopy sequence

\[
\ldots \rightarrow \pi_i(A) \rightarrow \pi_i(P_n) \rightarrow \pi_i(P_n, A) \rightarrow \ldots
\]

which shows that the kernels of the homomorphisms \( \pi_i(A) \rightarrow \pi_i(P_n) \), \( 1 \leq i \leq s \), are finite. It is well known that \( \pi_i(P_n) \) vanishes for \( 3 \leq i \leq 2n \). So theorem II is proved under the assumption \( \pi_1(A) = 0 \).

3. Reformulation of the problem.

By a covering over a complex space \( S \) we understand a map \( \gamma: C \rightarrow S \) of a topological space \( C \) onto \( S \) such that for each \( s \in S \) there is a neighbourhood \( U \) of \( s \) with \( \gamma^{-1}(U) \) a disjoint union of open sets on each of which \( \gamma \) is homeomorphic. An isomorphism between two coverings \( \gamma_1: C_1 \rightarrow S \) and \( \gamma_2: C_2 \rightarrow S \) is a bijective map \( h: C_1 \rightarrow C_2 \) such that the diagram
is commutative. We will denote this $h: (C_1 \simeq C_2) \mid S$. A covering is called trivial if it is isomorphic to some projection $S \times J \rightarrow S$. If $f: T \rightarrow S$ is a continuous map and $\gamma: C \rightarrow S$ is a covering, then we denote by $f^{\ast}C$ the fiber product of $f$ and $\gamma$, which is a covering over $T$.

For a connected complex space $S$, $\pi_1(S) = 0$, if and only if every covering over $S$ is trivial.

So to prove theorem I we have to show: every covering $\gamma_1: C_1 \rightarrow A$ is trivial. If any $\gamma_1$ is fixed, we denote the covering $\text{id} \times \gamma_1: G \times C_1 \rightarrow G \times A$ over $G \times A$ by $\gamma: C \rightarrow G \times A$. We denote by $C_\sigma$ the covering $\sigma^{-1}C_1$ over $\sigma A$. Because of

$$\varphi \mid \{\sigma\} \times A = \sigma \circ p_\sigma \mid \{\sigma\} \times A,$$

we have

$$(\varphi^{\ast} C_\sigma \simeq C) \mid \{\sigma\} \times A.$$

Let $B$ and $T \sigma A$ be as in (a) of section 1. Let $\tau: T \sigma A \rightarrow \sigma A$ be a tubular retraction. Then we have:

**Lemma 1.** There exists a unique isomorphism $(\varphi^{\ast} \tau^{\ast} C_\sigma \simeq C) \mid B \sigma \times A$ extending the natural one over $\{\sigma\} \times A$.

**Proof.** Trivial. Existence, because $\tau \varphi$ induces the identity on the fundamental groups. Uniqueness, because the base space is connected.

Now, let us look at $C \mid \varphi^{-1}(A) \cap G^1 \times A$ and $\varphi^{\ast}(C_1)$. These are isomorphic if restricted to $\{1\} \times A$. If we can show that they are isomorphic all over $\varphi^{-1}(A) \cap G^1 \times A$, then we get theorem I. Because then, $C$ restricted to any fiber $F = \varphi^{-1}(x) \cap G^1 \times A$ for some $x \in A$, and hence any $x \in P_n$, is trivial.

From the fibering

$$F \hookrightarrow G^1 \times A \rightarrow P_n,$$

we get the exact sequence [4, thm. 10, p. 377]

$$\pi_1(F) \rightarrow \pi_1(G^1 \times A) \rightarrow \pi_1(P_n) = 0.$$

So a covering over $G^1 \times A$ is trivial if the restriction to $F$ is trivial. This means that $C$ is trivial, and therefore $C_1$, with which we started, has to be trivial.
4. An extension lemma.

**Lemma 2.** Let $\sigma \in G$ be arbitrary and $\delta : D \to A \cap T \sigma A$ a covering. Then every continuous cross-section $s : A \cap \sigma A \to D$ can be uniquely extended to a cross-section over $A \cap T \sigma A$.

**Proof.** The uniqueness part is trivial, since by (b) of section 1 the set $\varphi^{-1}(A) \cap (B \sigma \times A)$ and therefore also

$$\varphi(\varphi^{-1}(A) \cap (B \sigma \times A)) = A \cap \varphi(B \sigma \times A) = A \cap T \sigma A$$

is connected. We put

$$d := \sup \{c' : \text{there exists a cross-section } s_{c'} \text{ over } \ A \cap (B_{c'} \sigma A) \text{ extending } s \} ,$$

and have to show $d = c$.

i) $d > 0$: By assumption, $\delta | s(A \cap \sigma A)$ is bijective. We cover $s(A \cap \sigma A)$ by open sets $U_i \subseteq V_i$ such that $V_i$ is path connected and

a) $\delta | V_i$ is bijective;

b) if $\delta U_i \cap \delta U_j \neq \emptyset$, then $\delta U_i \subseteq \delta V_j$;

c) $\delta^{-1} A \cap (U \setminus V_i) = s(A \cap \sigma A)$;

d) $U_i \cap s(A \cap \sigma A) \neq \emptyset$ for all $i$.

Then $\delta$ is bijective on $U = \bigcup U_i$. Otherwise there would exist $p \in U_i, q \in U_j$ such that $p \neq q$, but $\delta p = \delta q$. Because of a), we have $i \neq j$. Because of b), $\delta U_i \subseteq \delta V_j$. So c) and d) show $U_i \cap V_j \neq 0$, and this implies $U_i \subseteq V_j$. Since $\delta | V_j$ is bijective, we get $p = q$.

ii) There is a cross-section $S : A \cap (B_\sigma \sigma A) \to D$ extending $s$: All sets $A \cap B_{c'} \sigma A$ are connected, since $\varphi^{-1}(A) \cap (B_{c'} \sigma \times A)$ is connected according to (b) of section 1, and

$$\varphi(\varphi^{-1}(A) \cap (B_{c'} \sigma \times A)) = A \cap \varphi(B_{c'} \sigma \times A) = A \cap (B_{c'} \sigma A) .$$

So for $c' < d$, the cross-section $s_{c'}$ over $A \cap B_{c'} \sigma A$ is uniquely determined by $s$. Thus $s_{c'} | B_{c'} \sigma A = s_{c''}$ for $c'' < c'$. This means that the collection $\{s_{c'} c' < d\}$ determines a cross-section $S$ over $A \cap (B_\sigma \sigma A)$.

iii) Denote by $R$ the closure of $S(A \cap B_\sigma \sigma A)$ in $D$. Then $\delta | R$ is bijective: Since $S$ is a cross-section, $\delta | R$ is bijective. If there are $p_1, p_2 \in R$, $p_1 \neq p_2$, with $q = \delta(p_1) = \delta(p_2)$, then $q \in \delta(B \sigma \sigma A)$. Now take an open neighbourhood $U_q \subseteq A$ of $q$ as in (c) of section 1 using $O_\mathcal{A}$ for $F$. This is possible, since by assumption $A$ is nonsingular and so

$$\text{dih} O_\mathcal{A} = \dim A = a > n - a .$$

We may assume

$$U_q = \delta U_1 = \delta U_2, \quad U_i \subseteq D \text{ open} ,$$
where \( p_i \in U_i \) and \( \delta | U_i \) is bijective. We may further assume \( U_1 \cap U_2 = \emptyset \).

So

\[
\delta(U_1 \cap \hat{R}) \cap \delta(U_2 \cap \hat{R}) = \emptyset ,
\]

since \( \delta | \hat{R} \) is bijective. Because of

\[
\delta \hat{R} \cap U = \delta(\hat{R} \cap \delta^{-1} U) ,
\]

the sets \( \delta(U_i \cap \hat{R}) \) are connected components of \( \delta \hat{R} \cap U \), which cannot be different in view of (c) of section 1. This contradicts \( p_1 \neq p_2 \).

iv) In the same way as in i), we show: \( \delta \) is even bijective on an open neighbourhood of \( R \).

If \( d < c \), this would contradict the choice of \( d \). So \( d = c \), and the lemma is proved.

5. The extension method.

Here we give a proposition, which is the heart of the proof of theorem I.

We want to extend isomorphisms between coverings. This becomes a special case of extending sections: If \( \gamma_i : C_i \to S \), \( i = 1, 2 \) are two coverings, denote by \( \text{Isom}(C_1, C_2) \) the sheaf of germs of isomorphisms \( (C_1 \cong C_2) | U \), where \( U \subseteq S \) is open. Obviously, \( \text{Isom}(C_1, C_2) \) is a covering of \( S \), non-empty if \( \gamma_1 \) and \( \gamma_2 \) have the same degree over \( S \).

**Proposition.** Let \( B \subseteq G \) be an open ball containing \( 1 \) as in section 1. If we are given arbitrarily some \( \sigma \in G \) and an isomorphism

\[
i_{\sigma} : (C \cong \varphi^*C_1) | \varphi^{-1}(A) \cap (\{\sigma\} \times A) ,
\]

then there exists a unique isomorphism

\[
I_{\sigma} : (C \cong \varphi^*C_1) | \varphi^{-1}(A) \cap (B\sigma \times A)
\]

extending \( i_{\sigma} \).

**Proof.** According to section 3, there exists a covering \( C_\sigma \) over \( T\sigma A \) and an isomorphism

\[
J : (C \cong \varphi^*C_\sigma) | B\sigma \times A .
\]

Since \( \varphi | \{\sigma\} \times A \) is a homeomorphism, we obtain an isomorphism

\[
h_\sigma : (C_\sigma \cong C_1) | A \cap \sigma A
\]

with \( \varphi^* h_\sigma = i_{\sigma} \circ J^{-1} \). Now \( h_\sigma \) forms a section over \( A \cap \sigma A \) in the covering \( \text{Isom}(C_\sigma, C_1) \) of \( A \cap T\sigma A \). According to (b) of section 1, \( A \cap T\sigma A \) is con-
nected. Using lemma 2 we find that $h_{\sigma}$ can be extended to a section $H_{\sigma}$ over all of $A \cap T\sigma A$ in the covering Isom$(C_{\sigma}, C_1)$. This means that $h_{\sigma}$ extends to an isomorphism

$$H_{\sigma} \colon (C_{\sigma} \simeq C_1)|A \cap T\sigma A.$$ 

We put $I_{\sigma} = (\varphi^*H_{\sigma}) \circ J|B\sigma \times A$. Obviously this is an isomorphism between $C$ and $\varphi^*C_1$ over $B\sigma \times A$ extending $i_{\sigma}$. That $I_{\sigma}$ is uniquely determined by $i_{\sigma}$ follows from the connectedness of $\varphi^{-1}(A) \cap (B\sigma \times A)$.


Over $S := \varphi^{-1}(A) \cap (G^1 \times A)$ we have the two coverings $C$ and $\varphi^*C_1$. The covering Isom$(C, \varphi^*C_1)$ over $S$ is a sheaf of sets, and we can form the direct image sheaf $E := (p_G)_* \text{Isom}(C, \varphi^*C_1)$ over $G^1$. A germ in $E_{\sigma}$, $\sigma \in G^1$, is represented by an isomorphism $(C \simeq \varphi^*C_1)$ over a neighbourhood of $(\{\sigma\} \times A)$.

By the proposition above, for any $\sigma \in G^1$, the natural map

$$\Gamma(B\sigma, E) \to E_{\sigma}$$

is surjective. This means that the sheaf $E$ is locally constant over $G^1$. Since $G^1$ is simply connected $E$ is constant.

The isomorphism $(C \simeq \varphi^*C_1)|\{1\} \times A$ represents (by the proposition) a germ in $E_1$. This germ can be extended to an element in $\Gamma(G^1, E)$. This means, the isomorphism can be extended to an isomorphism

$$(C \simeq \varphi^*C_1)|\varphi^{-1}(A) \cap (G^1 \times A).$$

Thus, theorem I is proved.

ADDED IN PROOF: Recently, A. Ogus proved by algebraic methods theorem I for the profinite completion $\hat{\pi}_1(A)$ instead of $\pi_1(A)$. (Thesis, Harvard University, 1972).

REFERENCES