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ON THE HOMOTOPY TYPE OF CERTAIN SPACES OF DIFFERENTIABLE MAPS

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0. Introduction.

In this paper we shall establish certain relations in the homotopy theory of the classical function spaces in differential topology. If M and X are smooth manifolds with M compact the spaces we have in mind are $C^r(M,X)$, $C^r(M,X;k)$ and $\mathrm{Emb}^r(M,X)$, which denote respectively the space of differentiable maps of class C^r with C^r-topology, $2 \leqq r \leqq \infty$ and its subspaces of k-mersions and embeddings.

Our main motivation to look for such relations is that comparatively little is known about the homotopy theory of the space $\mathrm{Emb}^r(M,X)$ and much more is known about that of $C^r(M,X)$, since the latter space by a theorem of Palais [9, Theorem 13.14] is homotopy equivalent to the space of continuous maps $C^0(M,X)$, which in many special cases has been studied extensively by topologists. To indicate how little is known about the homotopy theory of the space $\mathrm{Emb}^r(M,X)$, it suffices to mention that it is an open question when it has finitely generated homotopy groups.

The relations in homotopy theory that we shall establish are expressed in terms of homotopy properties of the inclusion maps

i) $C^r(M,X;k) \to C^r(M,X)$,

ii) $\mathrm{Emb}^r(M,X) \to C^r(M,X)$.

After some preliminary material in sections 1 and 2 we show in Theorem 3.1 that the map i) is a $q(n,m,k)$-equivalence and that ii) is an $(m-2n-1)$-equivalence. Here n and m denote respectively the dimension of M and X and

$$q(n,m,k) = m-2k+(n-k)(m-k) .$$

This indicates that the maps ought to be homotopy equivalences, when X is an infinite dimensional smooth manifold. In Theorem 3.2 we show that this is indeed the case for $2 \leqq r < \infty$. For $r = \infty$ we need to assume that $C^\infty(M,X)$ is an ANR.

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If we consider an expanding sequence of finite dimensional smooth manifolds of increasing dimension, say

$$X_{n_0} \subset X_{n_0+1} \subset \ldots \subset X_n \subset \ldots ,$$

then it is an easy consequence of Theorem 3.1 that the following induced maps between the naturally induced direct limit spaces

i) $C^r(M,\boldsymbol{X};\, k)_\infty \to C^r(M,\boldsymbol{X})_\infty$,

ii) $\mathrm{Emb}^r(M,\boldsymbol{X})_\infty \to C^r(M,\boldsymbol{X})_\infty$

are homotopy equivalences. Theorems concerning such limit spaces are contained in section 4.

Finally, in section 5 we prove that the functors $C^r(M,\cdot\,;k)$ and $\mathrm{Emb}^r(M,\cdot)$ commute with smooth homotopy direct limits, thereby proving a conjecture in [4].

1. Definitions and preliminaries.

Throughout this paper M shall denote a finite dimensional, paracompact (usually compact) smooth manifold with or without boundary, and X shall denote a paracompact (equivalent metrizable) smooth manifold modelled on a Banach space of finite or infinite dimension. The manifold X is always without boundary. The infinite dimensional Banach spaces are sometimes supposed to be C^∞-smooth, that is, they admit partitions of unity of class C^∞ subordinated to any open covering. It is easy to see that X then also admits partitions of unity of class C^∞.

For $0 \leqq r \leqq \infty$ we denote by $C^r(M,X)$ the space of differentiable maps of class C^r from M into X. We equip $C^r(M,X)$ with the C^r-topology, which is a metrizable topology. Assume now that M is compact. Then it is known that $C^r(M,X)$ for $0 \leqq r < \infty$ is a metrizable smooth manifold modelled on Banach spaces. If X is either finite dimensional or modelled on an infinite dimensional C^∞-smooth Banach space, then it is also known that $C^\infty(M,X)$ is a metrizable manifold modelled on Fréchet spaces. For proofs of these statements see Krikorian [6] or Penot [10]. Therefore $C^r(M,X)$ is an ANR (absolute neighbourhood retract) for the class of metrizable spaces for all $0 \leqq r < \infty$ and for $r = \infty$ if the model for X is C^∞-smooth. See Palais [8, Theorem 5] for this conclusion.

(It is unknown whether $C^\infty(M,X)$ is a manifold without any assumptions on X or as here on the model for X.)

For $0 \leqq k \leqq \min\{\dim M, \dim X\}$ and $1 \leqq r \leqq \infty$, $C^r(M,X;\,k)$ shall denote the open subspace of k-mersions in $C^r(M,X)$. We recall that $f \in C^r(M,X)$ is a k-mersion if it has $\mathrm{rank} \geqq k$ everywhere. Thus $C^r(M,X;\,0)$ is just the space $C^r(M,X)$ itself, and if $\dim M \leqq \dim X$ then

$$C^r(M,X;\dim M) = \mathrm{Imm}^r(M,X)$$

is the space of immersions. Again provided $1 \leqq r \leqq \infty$ we have also the open subspace of embeddings $\mathrm{Emb}^r(M,X)$ in $C^r(M,X)$. If $C^r(M,X)$ is an ANR then the spaces $C^r(M,X;k)$ and $\mathrm{Emb}^r(M,X)$ are also ANR's since they are open subspaces of an ANR.

For convenience we make also the following almost standard

DEFINITION. Let X and Y be non-empty topological spaces, and let $f: X \to Y$ be a continuous map. Let also $q \geqq 0$ be an integer.

We call f a *0-equivalence* if the induced map between path components $f_*: \pi_0(X) \to \pi_0(Y)$ is onto.

For $q \geqq 1$ we call f a *q-equivalence* if $f_*: \pi_0(X) \to \pi_0(Y)$ is a bijection and if for any base point $x \in X$ the induced map $f_*: \pi_i(X,x) \to \pi_i(Y,f(x))$ is an epimorphism for $0 \leqq i \leqq q$ and a monomorphism for $0 \leqq i \leqq q-1$.

Finally, we call f a *weak homotopy equivalence* if it is a q-equivalence for all $q \geqq 0$.

In later sections we shall use extensively that a weak homotopy equivalence $f: X \to Y$ automatically is a homotopy equivalence if X and Y are ANR's. This is a theorem of J. H. C. Whitehead (see Palais [8]).

2. A transversality theorem.

In section 3 we shall need a corollary to a parametrized version of Thom's tranversality theorem. Before stating the theorem we explain some terminology.

Let Q, M and X be finite dimensional smooth manifolds. Denote by $J^s(M,X)$ the space of s-jets of maps from M into X. For $f \in C^r(Q \times M, X)$ and $0 \leqq s \leqq r$ we define the partial s-jet of f after M as the map

$$j_M{}^s(f): Q \times M \to J^s(M,X),$$

which maps $(q,x) \in Q \times M$ into the usual s-jet of $f_q: M \to X$ at $x \in M$, that is $j_M{}^s(f)(q,x) = j^s(f_q)(x)$.

In the following when we talk about approximations of maps in $C^r(Q \times M, X)$, we will always mean approximations with respect to a metric defining the C^r-topology on $C^r(Q \times M, X)$.

The transversality theorem we need now reads as follows:

THEOREM 2.1. *Let Q, M and X be finite dimensional smooth manifolds, and let $A \subset Q$ and $K \subset M$ be closed subsets. Let also $W \subset J^s(M,X)$ be a smooth submanifold with closed image and suppose that*

$$r > \max\{\dim(Q \times M) - \mathrm{codim}(W), s\}.$$

Then any map $f \in C^r(Q \times M, X)$ *such that* $j_M{}^s(f)$ *is transversal to* W *on* $A \times K$ *can be arbitrarily close approximated by a map* $g \in C^r(Q \times M, X)$ *such that* $g|A \times K = f|A \times K$ *and such that* $j_M{}^s(g)$ *is transversal to* W *on all of* $Q \times M$.

If we put Q equal to a point in this theorem we get of course the classical Thom transversality theorem. The proof of the version with a parameter space Q stated here can be modelled on the proof of the special case given in for example Morlet [7]. The theorem is also a consequence of the general transversality theorem of Abraham (see for example Abraham and Robbin [1]). As usual the restriction on the degree of differentiability r is caused by the application of Sard's theorem.

The theorem has this useful

COROLLARY 2.2. *Let* Q^i, M^n and X^m *be finite dimensional smooth manifolds and let* $A \subset Q$ *and* $K \subset M$ *be closed subsets. Suppose also that* $2 \leqq r \leqq \infty$, $0 \leqq k \leqq \min\{n, m\}$ *and* $0 \leqq i \leqq m - 2k + (n-k)(m-k)$.

Then any map $f \in C^r(Q \times M, X)$ *such that* f_q *has rank* $\geqq k$ *on* K *for* $q \in A$ *can be arbitrarily close approximated by a map* $g \in C^r(Q \times M, X)$ *such that* $g|A \times K = f|A \times K$ *and such that* g_q *has rank* $\geqq k$ *on* M *for all* $q \in Q$.

PROOF. Let $W(p) \subset J^1(M, X)$ be the subset of 1-jets of rank p. Then $W(p)$ is a submanifold with closed image of codimension $c(p) = (n-p)(m-p)$ in $J^1(M, X)$. Observe now that a map $g \in C^r(Q \times M, X)$ will satisfy the condition $\operatorname{rank}(g_q) \geqq k$ on M for all $q \in Q$ if and only if $j_M{}^1(g)$ avoids $W = W(0) \cup \ldots \cup W(k-1)$. If $c(p) - (i+n) \geqq 1$ for all $0 \leqq p \leqq k-1$ then it is clear that $j_M{}^1(g)$ will avoid W if and only if $j_M{}^1(g)$ is transversal to $W(p)$ for $0 \leqq p \leqq k-1$. Since $k \leqq \min\{n, m\}$ it follows from the formula $c(p) = (n-p)(m-p)$ that $c(p) \geqq c(k-1)$ for all $0 \leqq p \leqq k-1$ and therefore that

$$c(p) - (i+n) \geqq c(k-1) - (i+n) = (n-k+1)(m-k+1) - (i+n) .$$

Therefore $c(p) - (i+n) \geqq 1$ for all $0 \leqq p \leqq k-1$ if and only if

$$i \leqq m - 2k + (n-k)(m-k) .$$

Remark also that $r \geqq 2$ is the degree of differentiability we need in this case in order to apply Theorem 2.1 since $(i+n) - c(p) \leqq -1$ for $0 \leqq p \leqq k-1$.

With these observations at our disposal the corollary is an immediate consequence of Theorem 2.1.

3. Homotopy properties of the inclusion maps of $C^r(M,X;k)$ and $\mathrm{Emb}^r(M,X)$ into $C^r(M,X)$.

The purpose of this section is to provide proofs of the following theorems announced in a slightly different form as Lemma 2.2 and Lemma 2.3 in [5].

For any integers n, m and k we put $q(n,m,k)=m-2k+(n-k)(m-k)$.

THEOREM 3.1. *Let M^n and X^m be finite dimensional smooth manifolds with M compact, and let k and r be integers satisfying $0 \leqq k \leqq \min\{n,m\}$ and $2 \leqq r \leqq \infty$.*

i) *If $0 \leqq q(n,m,k)$, then $C^r(M,X;k) \neq \emptyset$ and the inclusion map*

$$C^r(M,X;k) \to C^r(M,X)$$

is a $q(n,m,k)$-equivalence.

ii) *If $0 \leqq m-2n-1$, then $\mathrm{Emb}^r(M,X) \neq \emptyset$ and the inclusion map*

$$\mathrm{Emb}^r(M,X) \to C^r(M,X)$$

is an $(m-2n-1)$-equivalence.

THEOREM 3.2. *Let M^n be a compact smooth manifold and let X be a smooth manifold modelled on an infinite dimensional Banach space E. Let also k and r be integers satisfying $0 \leqq k \leqq n$ and $2 \leqq r \leqq \infty$.*

Then $C^r(M,X;k)$ and $\mathrm{Emb}^r(M,X)$ are both non-empty. Furthermore, if we for $r=\infty$ assume that $C^\infty(M,X)$ is an ANR then the following inclusion maps are homotopy equivalences:

i) $C^r(M,X;k) \to C^r(M,X)$

ii) $\mathrm{Emb}^r(M,X) \to C^r(M,X)$.

REMARK 3.3. $C^\infty(M,X)$ is an ANR if E is C^∞-smooth and is expected to be so in general. (Again, observe that it is unknown, whether $C^\infty(M,X)$ is a metrizable manifold, in particular an ANR, without any assumptions on X.)

A path in $C^r(M,X)$ is also called a regular homotopy. If we put $k=n$ in Theorem 3.1 i) it follows therefore that any differentiable map is regular homotopic to an immersion when $m-2n \geqq 0$, and that any two immersions which are regular homotopic are regular homotopic through immersions when $m-2n-1 \geqq 0$. Similar results holds by Theorem 3.1 ii) for embeddings when $m-2n-1 \geqq 0$ and $m-2n-2 \geqq 0$ respectively. This is of course the classical results of Whitney. Theorem 3.1 can therefore be seen as a generalization of Whitney's results.

REMARK 3.4. For $r=\infty$ the result in Theorem 3.1 ii) follows also from a stronger theorem of Dax [2], which takes into account connectedness properties of M and X in the spirit of Haefliger [3].

PROOF OF THEOREM 3.1. Let Q^i be a compact smooth manifold with the compact submanifold $A \subset Q^i$ and the base point $q_0 \in A \subset Q^i$.

i) It is well-known (and follows in fact immediately from Corollary 2.2) that $C^r(M,X;k) \neq \emptyset$ when $q(n,m,k) \geqq 0$. Suppose now that $0 \leqq i \leqq q(n,m,k)$ and let $f_0 \in C^r(M,X;k)$ be an arbitrary k-mersion. Consider then the homotopy class of a map

$$f:(Q^i,A,q_0) \to (C^r(M,X),C^r(M,X;k),f_0)$$

with the associated map

$$\hat{f}: Q^i \times M^n \to X^m .$$

We can assume without loss of generality that $\hat{f}$ is of class C^r. Since $f(A) \subset C^r(M,X;k)$, the map $\hat{f}_q = f(q)$ has of course rank $\geqq k$ on M for all $q \in A$. By Corollary 2.2, $\hat{f}$ can then be approximated arbitrarily close in the C^r-topology by a map $\hat{g}: Q^i \times M^n \to X^m$ such that $\hat{g}|A \times M = \hat{f}|A \times M$ and such that $\hat{g}_q$ has rank $\geqq k$ on M for all $q \in Q^i$. Using a tubular neighbourhood for X in a Banach space E we can therefore also homotope $\hat{f}$ into a map $\hat{g}$ as above by connecting them linearly in the tubular neighbourhood and then projecting onto X. This homotopy will now induce a homotopy of f into a map $g: Q^i \to C^r(M,X;k)$ which is constantly equal to $f|A$ over A. Hence f represents relatively the zero class. Consider now the induced map

$$\pi_i(C^r(M,X;k),f_0) \to \pi_i(C^r(M,X),f_0) .$$

If we put $Q^i = S^i$ and $A = q_0$ in the analysis above, we conclude that this map is epic for $0 \leqq i \leqq q(n,m,k)$. If we put $Q^{i+1} = D^{i+1}$ and $A = S^i$ it follows from the analogous analysis with $\dim Q = i+1$ that this map is monic for $i+1 \leqq q(n,m,k)$. This is, however, exactly what we had to prove.

ii) It is again well-known that $\mathrm{Emb}^r(M,X) \neq \emptyset$ when $m-2n-1 \geqq 0$. Suppose now that $0 \leqq i \leqq m-2n-1$ and let $f_0 \in \mathrm{Emb}^r(M,X)$ be an arbitrary embedding. Consider then the homotopy class of a map

$$f:(Q^i,A,q_0) \to (C^r(M,X),\ \mathrm{Emb}^r(M,X),f_0)$$

with the associated map

$$\hat{f} : Q^i \times M^n \to X^m ,$$

which we again without loss of generality can assume to be of class C^r. Since $0 \leqq i \leqq m-2n-1 \leqq m-2n$ we can as in the proof of i) homotope f into a map $g : Q^i \to \mathrm{Imm}^r(M,X)$ fixing everything on A. We observe now that $\mathrm{Emb}^r(M,X)$ in this case is equal to the space of 1-1 immersions, since M is compact. It is easy to see that $g(q) = \hat{g}_q$ is 1-1 for all $q \in Q^i$ if and only if the map defined by the diagram

$$\begin{array}{ccc} Q^i \times M^n \times M^n & \xrightarrow{\Delta_Q \times 1_M \times 1_M} & Q^i \times Q^i \times M^n \times M^n & & \\ & & \Big\downarrow{\scriptstyle 1_Q \times \mathrm{Twist} \times 1_M} & & \\ & & Q^i \times M^n \times Q^i \times M^n & \xrightarrow{\hat{g} \times \hat{g}} & X^m \times X^m \end{array}$$

maps $Q^i \times (M^n \times M^n \setminus \Delta_M)$ into $X^m \times X^m \setminus \Delta X$. This last condition is a transversality condition when $i + 2n \leqq m-1$ or equivalently $i \leqq m-2n-1$. A transversality argument will therefore allow an arbitraryly close approximation of $\hat{g}$ with a map $\hat{h}$ such that $\hat{h} \mid A \times M = \hat{g} \mid A \times M$ and such that $\hat{h}_q$ is 1-1 for all $q \in Q^i$ provided of course $0 \leqq i \leqq m-2n-1$. Proceeding as in the proof of i) we can therefore homotope g into a map $h : Q^i \to \mathrm{Emb}^r(M,X)$ such that the homotopy is constantly equal to $g \mid A = f \mid A$ on A. f represents therefore relatively the zero class.

The proof of ii) is now finished in analogy with the proof of i).

Proof of Theorem 3.2. A chart on X provides a diffeomorphism $\theta : U \to E$ from an open set $U \subset X$ into the model Banach space E. Let $E = F \oplus \mathsf{R}^m$ be a splitting of E into a Banach space F and a copy of euclidean m-space R^m with $m \geqq 2n+1$. Choose now an arbitrary embedding $f_2 : M^n \to \mathsf{R}^m$ and an arbitrary differentiable map $f_1 : M^n \to F$. Then

$$f = \theta^{-1} \circ (f_1 \times f_2) : M \to X$$

is an embedding. Therefore $\mathrm{Emb}^r(M,X) \neq \emptyset$ and hence also

$$C^r(M,X;k) \neq \emptyset .$$

Let now again Q^i be a compact smooth manifold with the compact submanifold $A \subset Q^i$ and the base point $q_0 \in A \subset Q^i$.

i) Let $f_0 \in C^r(M,X;k)$ be an arbitrary k-mersion and consider for any $i \geqq 0$ the homotopy class of a map

$$f : (Q^i, A, q_0) \to (C^r(M,X), C^r(M,X;k), f_0)$$

with the associated map

$$\hat{f} : Q^i \times M^n \to X .$$

We can again assume without loss of generality that $\hat{f}$ is of class C^r. We want to change $\hat{f}$ by a homotopy constant on $A \times M$ to obtain a map which is a k-mersion for each fixed $q \in Q$. We do that in a sequence of steps, in each step only making changes on a piece of the domain mapped into a chart on the target and keeping fixed what has been obtained after the previous steps. To make this precise we choose open coverings of Q by charts, say $\{V_i\}$ and $\{U_i\}$ with $i = 1, \ldots, l$, such that $V_i \subset \overline{V}_i \subset U_i$. Similarly, we choose for each $i = 1, \ldots, l$, open coverings of M by charts, say $\{V_j{}^i\}$ and $\{U_j{}^i\}$ with $j = 1, \ldots, n_i$ such that

$$\overline{V}_j{}^i \subset V_j{}^i \subset U_j{}^i .$$

Furthermore, all these coverings shall be chosen such that $\hat{f}(U_i \times U_j{}^i)$ is contained in a chart on X. Consider now $U_1 \times U_1{}^1$. Since $\hat{f}$ maps this subset into a chart on X, $\hat{f} | U_1 \times U_1{}^1$ corresponds by a diffeomorphism to a map

$$U_1 \times U_1{}^1 \to E = F \oplus \mathsf{R}^m ,$$

where we choose m sufficiently large in the splitting $E = F \oplus \mathsf{R}^m$ of E. By the technique behind Corollary 2.2 we can alter the component into R^m of this map and thereby construct a homotopy (with support in $U_1 \times U_1{}^1$) from $\hat{f}$ to a map

$$\hat{f}_1{}^1 : Q \times M \to X ,$$

such that the homotopy is constant on $A \times M$ and such that $(\hat{f}_1{}^1)_q$ has rank $\geqq k$ on $\overline{V}_1{}^1$ for each $q \in A \cup \overline{V}_1$. Consider then $\hat{f}_1{}^1$ on $U_1 \times U_2{}^1$. By the same method as before we can change $\hat{f}_1{}^1$ inside $U_1 \times U_2{}^1$ and thereby obtain a map

$$\hat{f}_2{}^1 : Q \times M \to X ,$$

such that $\hat{f}_2{}^1$ is homotopic to $\hat{f}_1{}^1$ through a homotopy constant on $A \times M \cup \overline{V}_1 \times \overline{V}_1{}^1$ and such that $(\hat{f}_2{}^1)_q$ has rank $\geqq k$ on $\overline{V}_1{}^1 \cup \overline{V}_2{}^1$ for each $q \in A \cup \overline{V}_1$. We construct now by induction a sequence of maps

$$\hat{f}, \hat{f}_1{}^1, \hat{f}_2{}^1, \ldots, \hat{f}_{n_1}^1, \hat{f}_1{}^2, \ldots, \hat{f}_{n_2}^2, \ldots, \hat{f}_{n_l}^l ,$$

such that the map $\hat{f}_{n_i}^i$ for each $i = 1, \ldots, l$ is homotopic to $\hat{f}$ through a homotopy constant on $A \times M$ and such that $(\hat{f}_{n_i}^i)_q$ has rank $\geqq k$ on $M = \bigcup_{j=1}^{n_i} \overline{V}_j{}^i$ for each $q \in A \cup \overline{V}_1 \ldots \cup \overline{V}_i$. Since $Q = \bigcup_{i=1}^{l} \overline{V}_i$, the map

$$\hat{g} = \hat{f}^l_{nl} : Q \times M \to X$$

will therefore induce a map $g : Q \to C^r(M,X; k)$ homotopic to f by a homotopy constant on A. This shows that f relatively represents the zero class.

Proceeding now as in the proof of Theorem 3.1 i) we conclude that the induced map

$$\pi_i(C^r(M,X; k),f_0) \to \pi_i(C^r(M,X),f_0)$$

is a bijection for all $i \geqq 0$. Since f_0 was chosen arbitrarily the map

$$C^r(M,X; k) \to C^r(M,X)$$

is therefore a weak homotopy equivalence, and hence a homotopy equivalence, since the spaces involved are ANR's. This completes the proof of i).

ii) The proof of ii) is carried through in a manner similar to that of Theorem 3.1 ii) by reducing transversality questions to finite dimensional known ones as above.

4. Limit spaces of k-mersions and embeddings.

We recall from [4] that a smooth closed expanding system $(\boldsymbol{X},\boldsymbol{f},n_0)$ is a system indexed over the integers $n \geqq n_0$ of smooth manifolds X_n and smooth embeddings $f_{n,n+1} : X_n \to X_{n+1}$ with closed images. As usual we will abbreviate closed expanding system to CES. The limit space for an expanding system is the direct limit $\boldsymbol{X}_\infty = \varinjlim_n \{X_n, f_{n,n+1}\}$ (weak topology). If M^n is a compact smooth manifold then we get for each $0 \leqq r \leqq \infty$ an induced CES $(C^r(M,\boldsymbol{X}),\boldsymbol{f}_*,n_0)$. Similarly, if the dimension of the manifolds X_n is increasing, we get for each $1 \leqq r \leqq \infty$ and each $0 \leqq k \leqq n$ the induced CES's $(C^r(M,\boldsymbol{X}; k),\boldsymbol{f}_*,n_0)$ and $(\mathrm{Emb}^r(M,\boldsymbol{X}),\boldsymbol{f}_*,n_0)$. Observe, that the lower spaces in these last mentioned induced CES's may be the empty set. We shall be particularly interested in the limit spaces for these induced CES's, that is

$$\begin{aligned} C^r(M,\boldsymbol{X})_\infty &= \varinjlim_n \{C^r(M,X_n), (f_{n,n+1})_*\} \\ C^r(M,\boldsymbol{X}; k)_\infty &= \varinjlim_n \{C^r(M,X_n; k), (f_{n,n+1})_*\} \\ \mathrm{Emb}^r(M,\boldsymbol{X})_\infty &= \varinjlim_n \{\mathrm{Emb}^r(M,X_n), (f_{n,n+1})_*\}\,. \end{aligned}$$

If X is a finite dimensional smooth manifold then we can construct a smooth CES $(X \times \boldsymbol{R},\boldsymbol{f},0)$ by taking $X \times \mathsf{R}^n$ as the nth manifold in the system and the embedding $f_{n,n+1} : X \times \mathsf{R}^n \to X \times \mathsf{R}^{n+1}$ induced by the standard inclusion $\mathsf{R}^n \subset \mathsf{R}^{n+1}$ as the nth embedding in the system. We will call $(X \times \boldsymbol{R}, \boldsymbol{f}, 0)$ the smooth CES *generated* by X.

DEFINITION 4.1. Let M^n and X^m be finite dimensional smooth manifolds with M^n compact. For each $0 \leqq k \leqq n$ and each $1 \leqq r \leqq \infty$ we will call the limit spaces

$$C_L{}^r(M,X;\,k) = C^r(M,X \times \boldsymbol{R};\,k)_\infty$$

and

$$\mathrm{Emb}_L{}^r(M,X) = \mathrm{Emb}^r(M,X \times \boldsymbol{R})_\infty$$

respectively the *induced limit space* of k-mersions and embeddings.

The following theorem is the main theorem in this section:

THEOREM 4.2. *Let M^n be a compact smooth manifold and let $(\boldsymbol{X}, \boldsymbol{f}, n_0)$ be a smooth* CES *of finite dimensional manifolds of increasing dimension. Suppose also that $0 \leqq k \leqq n$ and $2 \leqq r \leqq \infty$.*

Then the following limits of inclusion maps are homotopy equivalences:

i) $C^r(M,\boldsymbol{X};\,k)_\infty \to C^r(M,\boldsymbol{X})_\infty$,

ii) $\mathrm{Emb}^r(M,\boldsymbol{X})_\infty \to C^r(M,\boldsymbol{X})_\infty$.

PROOF. Since the homotopy functor commutes with the direct limit functor it follows immediately from Theorem 3.1 that both the maps are weak homotopy equivalences. From [4, Corollary 6.4] we know that all the spaces involved have the homotopy type of ANR's. But then the maps are homotopy equivalences by Whitehead's theorem.

From [4] we extract

THEOREM 4.3. *Let M be a compact smooth manifold and let $(\boldsymbol{X}, \boldsymbol{f}, n_0)$ be a smooth* CES *of finite dimensional manifolds.*

Then the following maps are homotopy equivalences:

i) *The limit of natural maps*

$$C^r(M,\boldsymbol{X})_\infty \to C^0(M,\boldsymbol{X})_\infty$$

for each $1 \leqq r \leqq \infty$.

ii) *The map given by the universal property of direct limits*

$$C^0(M,\boldsymbol{X})_\infty \to C^0(M,\boldsymbol{X}_\infty)\,.$$

PROOF. For the proof of i) see [4, section 8, proof of Theorem 5.5 for a CES].

Since both the spaces in ii) have the homotopy type of ANR's (see [4, Theorem 1.1 and Corollary 6.4]), ii) follows from [4, Lemma 7.1 ii)] by Whitehead's Theorem.

Combining Theorem 4.2 and Theorem 4.3 we get

COROLLARY 4.4. *Let M^n be a compact smooth manifold and let $(\boldsymbol{X},\boldsymbol{f},n_0)$ be a smooth* CES *of finite dimensional manifolds of increasing dimension. Let also $0 \leqq k \leqq n$ and $2 \leqq r \leqq \infty$.*

Then all the limit spaces $C^r(M,\boldsymbol{X};k)_\infty$ and $\mathrm{Emb}^r(M,\boldsymbol{X})_\infty$ *have the same homotopy type as $C^0(M,\boldsymbol{X}_\infty)$.*

In particular we get

COROLLARY 4.5. *Let M^n and X^m be finite dimensional smooth manifolds with M^n compact. Let also $0 \leqq k \leqq n$ and $2 \leqq r \leqq \infty$.*

Then all the induced limit spaces $C_L{}^r(M,X;k)$ and $\mathrm{Emb}_L{}^r(M,X)$ *have the same homotopy type as $C^0(M,X)$.*

PROOF. Observe that $(X \times \boldsymbol{R})_\infty = X \times \boldsymbol{R}_\infty$ and that $\boldsymbol{R}_\infty = \mathsf{R}^\infty = \varinjlim_n \mathsf{R}^n$ is contractible. But then $(X \times \boldsymbol{R})_\infty$ is homotopy equivalent to X and hence the corollary follows from Corollary 4.4.

5. A theorem on smooth homotopy direct limits.

The purpose of this section is to prove Theorem 5.2 below, which we conjectured in [4, Remark 9.3]. A special case of Theorem 5.1 was also part of Theorem 2.1 in [5].

We recall from [4] that the smooth manifold X is a smooth HDL (homotopy direct limit) of the smooth CES $(\boldsymbol{X},\boldsymbol{f},n_0)$ with respect to the system of smooth maps $(\boldsymbol{g},n_0) = \{g_n\}_{n \geqq n_0}$, if each $g_n : X_n \to X$ is a smooth map such that $g_n = g_{n+1} \circ f_{n,n+1}$ for all $n \geqq n_0$ and such that the map $\boldsymbol{g}_\infty : \boldsymbol{X}_\infty \to X$ induced by the universal property of direct limits is a homotopy equivalence. Smooth HDL's are of course special examples of (continuous) HDL's.

THEOREM 5.1. *Let $(\boldsymbol{X},\boldsymbol{f},n_0)$ be a smooth* CES *of finite dimensional manifolds of increasing dimension and let X be a metrizable smooth manifold modelled on an infinite dimensional C^∞-smooth Banach space. Suppose also that X is a smooth* HDL *of $(\boldsymbol{X},\boldsymbol{f},n_0)$ with respect to the system of smooth* *embeddings* $(\boldsymbol{g}, n_0)$. *Finally, let* M^n *be an arbitrary compact smooth manifold and let* $0 \leqq k \leqq n$ *and* $2 \leqq r \leqq \infty$.

Then $C^r(M,X;k)$ *and* $\mathrm{Emb}^r(M,X)$ *are* HDL's *of the corresponding induced* CES's $(C^r(M,\boldsymbol{X};k), \boldsymbol{f}_*, n_0)$ *and* $(\mathrm{Emb}^r(M,\boldsymbol{X}), \boldsymbol{f}_*, n_0)$ *with respect to the induced systems of continuous embeddings* $(\boldsymbol{g}_*, n_0)$.

For $2 \leqq r < \infty$ *the induced limits will actually be smooth.*

Proof. Consider the following commutative diagrams of natural maps:

$$\begin{array}{ccc} C^r(M,\boldsymbol{X};k)_\infty & \xrightarrow{\boldsymbol{g}_{*\infty}} & C^r(M,X;k) \\ \downarrow & & \downarrow \\ C^r(M,\boldsymbol{X})_\infty & \longrightarrow & C^r(M,X) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Emb}^r(M,\boldsymbol{X})_\infty & \xrightarrow{\boldsymbol{g}_{*\infty}} & \mathrm{Emb}^r(M,X) \\ \downarrow & & \downarrow \\ C^r(M,\boldsymbol{X})_\infty & \longrightarrow & C^r(M,X)\,. \end{array}$$

The vertical maps are homotopy equivalences by Theorem 3.2 and Theorem 4.2. The bottom horizontal map is a homotopy equivalence by [4, Theorem 5.5]. Then $\boldsymbol{g}_{*\infty}$ must be a homotopy equivalence. This is exactly what we should prove.

REFERENCES

1. R. Abraham and J. Robbin, *Transversal mappings and flows*, Benjamin, New York · Amsterdam, 1967.
2. J. P. Dax, *Généralisation des théorèmes de plongement de Haefliger à des familles d'applications dépendant d'un nombre quelconque de paramètres*, C. R. Acad. Sci. Paris Ser. A–B 264 (1967), 499–502.
3. A. Haefliger, *Plongements différentiables de variétés dans variétés*, Comment. Math. Helv. 36 (1961), 47–82.
4. V. L. Hansen, *Some theorems on direct limits of expanding sequences of manifolds*, Math. Scand. 29 (1971), 5–36.
5. V. L. Hansen, *On mapping spaces with infinite dimensional target*, Proc. Adv. Study Inst. Alg. Top. I, Aarhus, 1970, 154–169.
6. N. Krikorian, *Manifolds of maps*, Thesis, Cornell University, 1969.
7. C. Morlet, *Le lemme de Thom et les théorèmes de plongement de Whitney* I–IV, Séminaire Cartan 1961/62, Secretariat mathématique, Paris, 1964.

8. R. S. Palais, *Homotopy theory of infinite dimensional manifolds*, Topology 5 (1966), 1–16.
9. R. S. Palais, *Foundations of global non-linear analysis*, Mathematics lecture note series, Benjamin, New York · Amsterdam, 1968.
10. J. P. Penot, *Diverses méthodes de construction de variétés d'applications*, Thesis (Paris), 1970.

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