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ON DOMINATED EXTENSION OF CONTINUOUS AFFINE FUNCTIONS ON SPLIT FACES

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The purpose of this note is to prove a theorem on dominated extension of continuous affine functions from split faces of compact convex sets (Theorem 1) which improves the result in [2, Th. 3.3]. From this theorem we are able to derive a result of J.-E. Bjørk [5] on interpolation of closed subspaces of $C_{\mathbf{R}}(X)$ (Theorem 3) which generalizes the Bishop–Carleson–Rudin theorem [4], [6], [7] in the real case.

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In this note K denotes a compact convex subset of a real locally convex vector space and $A(K)$ the Banach space of real-valued continuous affine functions on K. If F is a closed face of K, we denote by F' the union of all faces disjoint from F. Then F' is a G_δ [2, Cor. 1.3]. If F is a closed face of K, we call F a *split face*, if F' is a face and each point in $K \setminus (F \cup F')$ admits a unique representation as a convex combination of a point in F and a point in F'.

The following proposition is proved analogously to Theorem 3.3 of [2], so we shall just give a sketch of proof. It is also proved in slightly different form in [3].

PROPOSITION 1. *Let F be a proper closed split face of a compact convex set K. Let $a_1,\ldots,a_n,b \in A(K)$ and $a_0 \in A(F)$ such that*

$$(1)\qquad a_i \leqq b, \quad a_i|_F \leqq a_0 \leqq b|_F, \quad i=1,\ldots,n .$$

Then for every $\varepsilon > 0$ there is a $c \in A(K)$ such that

$$(2)\qquad a_i - \varepsilon \leqq c \leqq b, \quad c|_F = a_0 .$$

PROOF. Let $a = a_1 \vee \ldots \vee a_n$, where the supremum is formed pointwise.

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By Lemma 3.2 of [2] we can choose $c_1 \in A(K)$ such that

$$a < c_1 < b+\varepsilon, \qquad a_0+\varepsilon 2^{-1} < c_1|_F < a_0+\varepsilon .$$

By induction and application of the dual version of Lemma 3.2 of [2] we construct $\{c_m\}_{m=1}^{\infty} \subseteq A(K)$ such that

$$c_m-\varepsilon 2^{-m} < c_{m+1} < c_m \wedge (b+\varepsilon 2^{-m}) ,$$
$$a_0+\varepsilon 2^{-(m+1)} < c_{m+1}|_F < a_0+\varepsilon 2^{-m} .$$

Then $c=\lim c_m \in A(K)$ will satisfy the requirements.

COROLLARY 1. *Let F be a proper closed split face of a compact convex set K. Let g be a real continuous concave function on K, $b \in A(K)$ and $a_0 \in A(F)$ such that*

$$b < g, \quad b|_F \leqq a_0 < g|_F .$$

Then there is a $c \in A(K)$ such that

$$b \leqq c < g, \quad c|_F = a_0 .$$

PROOF. By continuity and compactness we can choose $\varepsilon > 0$ such that

$$b < g-\varepsilon, \quad a_0 < g-\varepsilon|_F .$$

By the Hahn–Banach theorem

$$g(x) = \inf \{a(x) \mid g<a,\ a\in A(K)\}$$

and hence

$$g(x) = \inf \{(a_1\wedge \ldots \wedge a_n)(x) \mid g<a_i,\ a_i\in A(K)\}$$

for all x in K. By Dini's theorem for decreasing nets we can choose $b_1,\ldots,b_n \in A(K)$, $g<b_i$ such that

$$(b_1\wedge \ldots \wedge b_n)(x)-g(x) < \tfrac{1}{2}\varepsilon, \text{ all } x \text{ in } K .$$

Let $a_i'=b_i-\frac{1}{2}\varepsilon$. Then

$$a_1' \wedge \ldots \wedge a_n' < g, \quad g-\tfrac{1}{2}\varepsilon < a_i' .$$

Hence by the choice of ε,

$$b < a_i', \quad a_0 < a_i'|_F .$$

Let $a_i=a_i'-\frac{1}{2}\varepsilon$. Then

$$b < g-\varepsilon < a_i, \quad a_0 < g-\varepsilon|_F < a_i|_F .$$

We apply the dual version of Proposition 1 and find $c \in A(K)$ such that

$$b \leqq c \leqq a_i+\tfrac{1}{2}\varepsilon, \quad c|_F = a_0 .$$

Then

$$c(x) \leqq a_i(x)+\tfrac{1}{2}\varepsilon = a_i'(x), \quad i=1,\ldots,n\,,$$

and hence

$$c(x) \leqq (a_1' \wedge \ldots \wedge a_n')(x) < g(x) \quad \text{for all } x \text{ in } K\,.$$

Now we shall use a variant of a technique which appeared in Pelczynski's paper [8] to avoid one of the requirements of strict inequality in Corollary 1.

THEOREM 1. *Let F be a closed split face of a compact convex set K. Let g be a real continuous concave function on K, $b \in A(K)$ and $a_0 \in A(F)$ such that*

$$b < g, \quad b|_F \leqq a_0 \leqq g|_F\,.$$

Then there is an $a \in A(K)$ such that

$$b \leqq a \leqq g, \quad a|_F = a_0\,.$$

PROOF. By subtraction of b we may assume $b=0$, and normalizing we may assume $g \leqq 1$.

Since $2g > g$ we may apply Corollary 1 to choose $a_1 \in A(K)$ such that

$$a_1|_F = a_0, \quad 0 \leqq a_1 < 2g\,.$$

Let

$$b_2 = 2^2(g-2^{-1}a_1)$$

which is a concave and strictly positive continuous function. If $k \in F$ and $a_0(k)=0$, then

$$b_2(k) = 4g(k) > 0 = a_0(k)\,.$$

If $k \in F$ and $a_0(k) > 0$, then

$$b_2(k) = 4(g(k)-2^{-1}a_0(k)) \geqq 4\big(a_0(k)-2^{-1}a_0(k)\big) = 2a_0(k) > a_0(k)\,.$$

Hence $b_2|_F > a_0$, and so $b_2 \wedge 2$ is concave and continuous and

$$0 < b_2 \wedge 2, \quad a_0 < b_2 \wedge 2|_F\,.$$

By Corollary 1 we can choose $a_2 \in A(K)$ such that

$$0 \leqq a_2 < 2 \wedge b_2, \quad a_2|_F = a_0\,.$$

Assume now by induction that $a_1,\ldots,a_n \in A(K)$ are constructed such that

$$0 \leqq a_p < 2 \wedge 2^p(g-\textstyle\sum_{r=1}^{p-1}2^{-r}a_r), \quad a_p|_F = a_0, \quad p=2,\ldots,n\,.$$

Let

$$b_{n+1} = 2^{n+1}(g - \textstyle\sum_{r=1}^{n} 2^{-r} a_r) .$$

Then b_{n+1} is concave, continuous and strictly positive by induction hypothesis. Moreover

$$b_{n+1}|_F = 2^{n+1}(g|_F - (1-2^{-n})a_0) .$$

If $k \in F$ and $a_0(k)=0$, then

$$b_{n+1}(k) = 2^{n+1} g(k) > 0 = a_0(k) .$$

If $k \in F$ and $a_0(k)>0$, then

$$\begin{aligned} b_{n+1}(k) &= 2^{n+1}(g(k) - (1-2^{-n})a_0(k)) \\ &\geqq 2^{n+1}(a_0(k) - (1-2^{-n})a_0(k)) = 2a_0(k) > a_0(k) . \end{aligned}$$

Hence $b_{n+1}|_F > a_0$, and so $b_{n+1} \wedge 2$ is concave, continuous and

$$0 < b_{n+1} \wedge 2, \quad a_0 < b_{n+1} \wedge 2|_F .$$

Again by Corollary 1: choose $a_{n+1} \in A(K)$ such that

$$0 \leqq a_{n+1} < b_{n+1} \wedge 2, \quad a_{n+1}|_F = a_0 .$$

The sequence $\{a_n\}_{n=1}^{\infty} \subseteqq A(K)$ so constructed satisfies

(i) $0 \leqq a_n < 2$, $\quad n=1,2,\ldots,$
(ii) $0 \leqq \sum_{r=1}^{n} 2^{-r} a_r < g$, $\quad n=1,2,\ldots,$
(iii) $a_n|_F = a_0$, $\quad n=1,2,\ldots,$

By (i): $\sum_{r=1}^{\infty} 2^{-r} a_r$ is uniformly convergent and hence determines an element a of $A(K)$. By (ii): $0 \leqq \sum_{r=1}^{\infty} 2^{-r} a_r \leqq g$, and by (iii) we get $a|_F = \sum_{r=1}^{\infty} 2^{-r} a_0 = a_0$. This completes the proof.

COROLLARY 2. *Let F be a closed split face of a compact convex set K. Let $a_0 \in A(F)$. Then a_0 admits an extension a in $A(K)$ such that $\|a_0\|_F = \|a\|_K$.*

To apply the preceding results to function spaces we need more notation. If X is a compact Hausdorff space, $M(X)$ denotes the Banach space of all signed Radon measures on X, $M^+(X)$ denotes the positive measures and $M_1^+(X)$ the probability measures. If F is a Borel subset of X and $\mu \in M(X)$, we denote by $\mu|_F \in M(X)$ the measure defined by

$$\mu|_F(S) = \mu(F \cap S) \text{ for all Borel sets } S \text{ in } X .$$

When $\mu|_F$ is considered as a measure on F, we denote it by μ_F. If F is a compact subset of X and $\nu \in M(F)$, we denote by $\nu^X \in M(X)$ the measure defined by

$$\nu^X(S) = \nu(S \cap F) \text{ for all Borel sets } S \text{ in } X .$$

If K is a compact convex subset of a locally convex Hausdorff space, we denote by $\partial_e K$ the set of extreme points of K. A signed measure μ on K is called a boundary measure if the total variation $|\mu|$ is a maximal measure in Choquet's ordering of positive measures [1], [9]. The boundary measures form a normclosed subspace of $M(K)$ [1, Prop. I.4.5]. A boundary measure is supported by $\overline{\partial_e K}$ [1, Prop. I.4.6].

By r we denote the barycenter map from $M_1{}^+(K)$ to K, that is, for $\mu \in M_1{}^+(K)$, $r(\mu)$ is the unique point in K satisfying $a(r(\mu)) = \mu(a)$ for all $a \in A(K)$. Choquet's theorem states that each point in K is the barycenter of a maximal (boundary) probability measure.

If $f: K \to \mathsf{R}$ is a bounded function, $\hat{f}: K \to \mathsf{R}$ is defined by

$$\hat{f}(k) = \inf \{a(k) \mid a > f,\ a \in A(K)\}, \quad \text{all } k \in K .$$

If X is a compact Hausdorff space, we let B be a closed subspace of $C_{\mathsf{R}}(X)$, which separates the points of X and contains the constants. We define

$$B^{\perp} = \{\mu \in M(X) \mid \mu(b) = 0, \forall b \in B\} .$$

We let

$$S_B = \{\varphi \in B^* \mid \varphi(1) = 1 = \|\varphi\|\}$$

which is convex and compact in the w^*-topology. Since B separates points in X we have a homeomorphic embedding Φ of X into S_B defined by

$$\Phi(x)(b) = b(x), \quad \text{all } b \in B .$$

Then $\partial_e S_B \subseteq \Phi(X)$ [9, Lem. 6.1]. We define the Choquet boundary $\partial_B X$ as the set

$$\partial_B X = \{x \in X \mid \Phi(x) \in \partial_e S_B\} .$$

If Φ carries a measure μ on X to S_B, we call the carried measure $\mu \circ \Phi^{-1}$. We let

$$M(\partial_B X) = \{\mu \in M(X) \mid \mu \circ \Phi^{-1} \text{ is a boundary measure on } S_B\} .$$

Notice that boundary measures on S_B can be carried to X as $\mu \circ \Phi$ such that $(\mu \circ \Phi) \circ \Phi^{-1} = \mu$, since boundary measures are supported by $\overline{\partial_e S_B} \subseteq \Phi(X)$.

Finally, the map $\psi: B \to A(S_B)$, where

$$\psi(b)(\varphi) = \varphi(b), \quad \text{all } \varphi \text{ in } S_B ,$$

is an order-isomorphism of B *onto* $A(S_B)$ [1, Th. II. 1.8]. Notice that

$$\psi(b)(\Phi(x)) = b(x), \quad \text{all } b \in B \text{ and all } x \in X .$$

We shall need the following measure theoretic characterization of split faces due to Alfsen [1, Th. II.6.12].

THEOREM 2. *If F is a closed face of a compact convex set K, then the following conditions are equivalent:*

(i) *F is a split face.*

(ii) *If $\mu \in A(K)^{\perp}$ and μ is concentrated on $F \cup F'$, then $\mu|_F \in A(K)^{\perp}$.*

(iii) *If $\mu \in A(K)^{\perp}$ and μ is a boundary measure, then $\mu|_F \in A(K)^{\perp}$.*

LEMMA 1. *Let K be a compact convex set.*

(i) *If F_0 is a compact subset of $\partial_e K$ and ν is a boundary measure on $\overline{\text{co}}\,(F_0)$, then ν^K is a boundary measure on K.*

(ii) *If F is a closed convex subset of K and μ is a boundary measure on K, then μ_F is a boundary measure on F.*

PROOF. (i) Since ν is a boundary measure on $\overline{\text{co}}\,(F_0)$, ν is supported by the closure of $\partial_e\, \overline{\text{co}}\,(F_0)$. But by Milman's theorem $\partial_e\, \overline{\text{co}}\,(F_0) = F_0$. Hence if $f \in C_{\mathbf{R}}(K)$,

$$|\nu^K|\,(\hat{f}-f) = \int_{F_0} (\hat{f}-f)\, d|\nu| = 0 ,$$

since $\hat{f}=f$ on $\partial_e K$ [9; Prop. 3.1]. From [9; Prop. 9.3] it follows that ν^K is a boundary measure on K.

(ii) It suffices to consider the case where μ is a positive measure. Now

$$\mu = \mu|_F + \mu|_{K \setminus F} .$$

Choose a maximal measure ν on F such that $\mu_F < \nu$.

If f is a convex and continuous real function, then $f|_F$ is convex and continuous and hence

$$\mu(f) = \mu_F(f) + \mu|_{K\setminus F}(f) \leqq \nu(f) + \mu|_{K\setminus F}(f) = (\nu^K + \mu|_{K\setminus F})(f) .$$

But μ is a maximal measure on K and therefore

$$\mu = \nu^K + \mu|_{K\setminus F} ,$$

that is, $\mu|_F = \nu^K$. Finally $\mu_F = (\nu^K)_F = \nu$, and μ_F is maximal.

PROPOSITION 2. *Let K be a compact convex set and F_0 a compact subset of $\partial_e K$. If*

$$\mu \in A(K)^{\perp},\ \mu \text{ boundary} \Rightarrow \mu|_{F_0} \in A(K)^{\perp}\,,$$

then $F = \overline{\mathrm{co}}\,(F_0)$ *is a split face.*

PROOF. First we prove that $\overline{\mathrm{co}}\,(F_0)$ is a face of K. Let $x \in F$ and let

$$x = \alpha y + (1-\alpha)z\,,$$

where $0 < \alpha < 1$ and $y,z \in K$.

Assume $y \notin F$. Let $x = r(\mu_1)$, where $\mu_1 \in M_1{}^+(F)$ and boundary on F. Then $\mu_1(F_0) = 1$. Let $y = r(\mu_2)$, $z = r(\mu_3)$, where $\mu_2,\mu_3 \in M_1{}^+(K)$ and boundary on K. Then $\mu_2(F) < 1$, for otherwise $y \in F$.

Now μ_{2F} is a boundary measure on F by Lemma 1 (ii), and since $\partial_e F = F_0$ is compact, μ_{2F} is supported by F_0, and so

$$\mu_2(F_0) = \mu_{2F}(F_0) = \mu_{2F}(F) = \mu_2(F) < 1\,.$$

By definition of barycenters,

$$\mu_1{}^K - \alpha\mu_2 - (1-\alpha)\mu_3 \in A(K)^{\perp}$$

and a boundary measure on K, since $\mu_1{}^K$ is a boundary measure on K by Lemma 1 (i). But

$$(\mu_1{}^K - \alpha\mu_2 - (1-\alpha)\mu_3)|_{F_0}(1) = 1 - \alpha\mu_2(F_0) - (1-\alpha)\mu_3(F_0) > 0\,,$$

which is a contradiction. Hence $y \in F$ and analogously $z \in F$, and F is a face.

If $\mu \in A(K)^{\perp}$ and μ is a boundary measure on K, then $\mu|_F = \mu|_{F_0}$ by Lemma 1 (ii), and the conclusion follows from Theorem 2.

Now we are able to state and prove the theorem of Bjørk [5] in the non-metrizable case.

THEOREM 3. (Bjørk). *Let X be a compact Hausdorff space and B a norm-closed linear subspace of $C_{\mathbf{R}}(X)$ such that B contains the constants and separates the points of X. Let F_0 be a compact subset of the Choquet boundary $\partial_B X$ which satisfies*

$$\mu \in M(\partial_B X) \cap B^{\perp} \Rightarrow \mu|_{F_0} \in B^{\perp}\,.$$

Then each $b_0 \in B|_{F_0}$ is the restriction to F_0 of an element $b \in B$ with $\|b\|_X = \|b_0\|_{F_0}$.

PROOF. 1. Firstly we prove that $F = \overline{\mathrm{co}}\,(\Phi(F_0))$ is a split face of S_B. Since $\Phi(F_0) \subseteq \partial_e S_B$, it suffices by Proposition 2 to prove that $\mu \in A(S_B)^{\perp}$ and μ boundary measure imply $\mu|_{\Phi(F_0)} \in A(S_B)^{\perp}$.

Hence let $\mu \in A(S_B)^\perp$ and boundary. If $b \in B$, then

$$\int_X b\, d(\mu \circ \Phi) = \int_{\Phi(X)} b \circ \Phi^{-1}\, d\mu = \int_{\Phi(X)} \psi(b)\, d\mu = \int_{S_B} \psi(b)\, d\mu = 0 .$$

Hence $\mu \circ \Phi \in B^\perp$, and it is evident that $\mu \circ \Phi \in M(\partial_B X)$. By assumption $\mu \circ \Phi|_{F_0} \in B^\perp$. If $a \in A(S_B)$, then $\psi^{-1}(a) \in B$ and so

$$0 = \int_{F_0} \psi^{-1}(a)\, d\mu \circ \Phi = \int_{\Phi(F_0)} \psi^{-1}(a) \circ \Phi^{-1}\, d\mu = \int_{\Phi(F_0)} a\, d\mu .$$

Hence $F = \overline{\mathrm{co}}(\Phi(F_0))$ is a split face.

2. Next, let $b_0 \in B|_{F_0}$, $b_0 = b_1|_{F_0}$, where $b_1 \in B$. Then $\psi(b_1) \in A(S_B)$ and by Corollary 2 there is a $c \in A(S_B)$ such that

$$c|_F = \psi(b_1)|_F, \quad \|c\|_{S_B} = \|\psi(b_1)\|_F .$$

Now the norm of a continuous affine function is obtained at the extreme boundary, and so

$$\begin{aligned} \|\psi^{-1}(c)\|_X = \|c\|_{\Phi(X)} = \|c\|_{S_B} = \|\psi(b_1)\|_F &= \|\psi(b_1)\|_{\Phi(F_0)} \\ &= \|b_1 \circ \Phi^{-1}\|_{\Phi(F_0)} = \|b_1\|_{F_0} , \end{aligned}$$

and if $k \in F_0$, then

$$\psi^{-1}(c)(k) = c(\Phi(k)) = \psi(b_1)(\Phi(k)) = b_1(k) = b_0(k) .$$

Hence $b = \psi^{-1}(c) \in B$ will do, and the proof is complete.

Remark. With the same assumptions as in Theorem 3 one can get a stronger version corresponding to Theorem 1 with g a B-concave function.

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