ON A THEOREM OF DIXMIER

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A well-known theorem of Alaoglu (cf. [3, p. 84]) tells us that the closed unit ball in the Banach dual space of a normed space is compact with respect to the \( w^* \)-topology. In [1], Dixmier showed that this property is characteristic for Banach dual spaces. In this note, we shall give a short proof of a variant of Dixmier's theorem. This variant appears to be more convenient for applications [2]. Our argument is inspired by Edwards' paper [2] and is strictly elementary (in particular, we do not use the Krein–Smulian theorem).

**Theorem 1.** Let \((X, \| \cdot \|)\) be a normed space with closed unit ball \( \Sigma \). Suppose there exists a (Hausdorff) locally convex topology \( \tau \) for \( X \) such that \( \Sigma \) is \( \tau \)-compact. Then \( X \) itself is a Banach dual space, that is, there exists a Banach space \( V \) such that \( X \) is isometrically isomorphic to the dual space \( V' \) of \( V \) (in particular, \( X \) is complete).

**Proof.** Let \((X, \tau)'\) and \((X, \| \cdot \|)'\) denote the dual spaces of \( X \) under \( \tau \) and \( \| \cdot \| \) respectively. Let \( V \) be the space of all linear functionals \( f \) on \( X \) such that \( f \) is \( \tau \)-continuous on \( \Sigma \). Then

\[
(1) \quad (X, \tau)' \subseteq V \subseteq (X, \| \cdot \|)' .
\]

The first inequality is obvious, and to see the second, let \( f \in V \). Then \( f(\Sigma) \) is the continuous image of the \( \tau \)-compact set \( \Sigma \), so is compact and hence bounded. Therefore \( f \) is continuous on \((X, \| \cdot \|)\), and (1) is proved. Now it is easily seen that \( V \) is a closed subspace of the Banach space \((X, \| \cdot \|)'\). Thus, \( V \) may be regarded as a Banach space in its own right.

For each \( x \) in \( X \), define \( \varphi(x) \) by the rule

\[
(\varphi(x))(v) = v(x), \quad v \in V .
\]

Then it is easy to see that \( \varphi \) is a 1-1 continuous (in fact norm-reducing) map from \( X \) into the Banach dual space \( V' \) of \( V \). Also, since each \( v \) in \( V \) is \( \tau \)-continuous on \( \Sigma \), the restriction \( \varphi|\Sigma \) of \( \varphi \) to \( \Sigma \) is continuous with respect to the relative \( \tau \)-topology and the \( w^* \)-topology \( \sigma(V', V) \). Since

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$\Sigma$ is $\tau$-compact, it follows that $\varphi(\Sigma)$ is $\sigma(V', V)$-compact. Also, this set $\varphi(\Sigma)$ is convex. By the bipolar theorem (cf. [3, 126]), it is precisely its bipolar $[\varphi(\Sigma)]^{\text{ba}}$ with respect to the duality $(V', V)$. Note that

$$[\varphi(\Sigma)]^{\text{ba}} = \{ v \in V : (\varphi(x))(v) \leq 1, \forall x \in \Sigma \} = \{ v \in V : v(x) \leq 1, \forall x \in \Sigma \},$$

which is just the unit ball in $V$, and hence $[\varphi(\Sigma)]^{\text{ba}}$ (that is, $\varphi(\Sigma)$) is the unit ball in $V'$. In other words, $\varphi$ maps $\Sigma$ onto the unit ball in $V'$. Therefore $\varphi$ is an isometry and onto the space $V'$. The proof of theorem 1 is thus completed.

This theorem implies immediately the theorem of Dixmier referred to at the beginning:

**Theorem 2.** Let $(X, \| \cdot \|)$ be a Banach space with closed unit ball $\Sigma$. Suppose there exists a total subspace $V$ of $(X, \| \cdot \|)'$ such that $\Sigma$ is $\sigma(X, V)$-compact. Then $X$ itself is a Banach dual space.

**REFERENCES**


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