MATH. SCAND. 29 (1971), 259–271

IDEALS IN ORDERED LOCALLY CONVEX SPACES

RAINER J. NAGEL

Ideals in Banach lattices have proved very useful in investigating the structure of such spaces (Lotz [8]) and the spectral properties of positive operators (Schaefer [14], [15]). So it is quite natural to ask for a generalization of the notion of ideal for an arbitrary ordered locally convex space. This has already been done for some special spaces, like simplex spaces (Effros [5]) and C^*-algebras (Størmer [16]). Developing a concept of A. J. Ellis [6] we define *fully perfect ideals* in an ordered locally convex space, which seem to have all of the properties one can reasonably expect in the general case, and which, in a locally convex vector lattice, coincide with the closed lattice ideals. In this paper we study the structure theory of these ideals. Applications of these results to operator theory will follow later.

Section 1 of the paper is concerned with terminology and notation.

In section 2 the class of fully perfect ideals of an ordered locally convex space E is defined. After studying the elementary properties we prove the main theorem (2.8) which gives a dual characterization for fully perfect ideals. Moreover, we get a complete symmetry for fully perfect ideals in E and E_σ', in which their positive part is total (2.10). In a locally convex vector lattice, finally, the fully perfect ideals coincide with the closed lattice ideals.

The third section is devoted to the more special situation of an ordered Banach space E. We prove a new characterization of support points of the positive cone in E and show that the fully perfect ideals in E separate points (trivial cases excluded) (3.2). The final theorem states that if the dual norm is additive on the dual positive cone, then every proper fully perfect ideal in E is contained in a maximal fully perfect ideal.

I am very grateful to Professor Dr. H. H. Schaefer for drawing my attention to this subject and for his helpful comments.

1. Preliminaries.

In what follows, E is a vector space over R and $(E,\mathfrak{U},C)$ denotes the ordered locally convex space (o.l.c.s.) E with 0-neighbourhood base $\mathfrak{U}$

Received January 29, 1970.

and ordered by the cone C. For a subset M of E, $[M]=(M+C)\cap(M-C)$ is the C-saturated hull of M and $[\mathfrak{U}]$ denotes the set of all $[U]$ for $U\in\mathfrak{U}$. Moreover, the polar of M is

$$M^\circ = \{f\in E' : f(x)\leqq 1, \forall x\in M\}.$$

On E' we consider the topology $\sigma(E',E)$ and the order defined by

$$C' = \{f\in E' : f(x)\geqq 0, \forall x\in C\}.$$

Finally, if A is a (linear) subspace of E, we write $\hat{M}$ for the canonical image of M in $\hat{E}=E/A$. For further terminology see [13] and [11].

(1.1) DEFINITION. Let (E,C) be an ordered vector space. A subspace A of E is called an *order ideal* if one of the following equivalent conditions is satisfied:

(a) For all $x\in A$, $[0,x]=\{y\in E : 0\leqq y\leqq x\}\subset A$.

(b) $A=[A]$.

(c) $A\cap C$ is an *extreme subset* of C, that is, if

$$x = \lambda x_1+(1-\lambda)x_2\in A\cap C \quad \text{for } 0<\lambda<1 \text{ and } x_1,x_2\in C,$$

then $x_1,x_2\in A$.

(d) $\hat{E}=E/A$ is an ordered vector space for the cone $\hat{C}$.

REMARK. A subspace A in E is a maximal order ideal if and only if $A=f^{-1}(0)$ for some positive linear form f on E. Moreover, every subspace B in E with $B\cap C=(0)$ is an order ideal.

(1.2) DEFINITION. A subspace A in an ordered vector space (E,C) is *positively generated* if $A=A^+-A^+$ for $A^+=A\cap C$.

(1.3) DEFINITION. (Ellis [6]). A subspace A in an o.l.s.c. $(E,\mathfrak{U},C)$ is called *perfect*, if for every 0-neighborhood $U\in\mathfrak{U}$ and for every $x\in A$ there exist $y\in A$ and $\mu,\nu\in U$ such that $-\mu-y\leqq x\leqq y+\nu$.

2. Fully perfect ideals in ordered locally convex spaces.

The main purpose of this section is to find an appropriate notion of ideals in an o.l.c.s. $(E,\mathfrak{U},C)$. It is clear that both the topology and the order structure of E have to be involved. This is the case for closed order ideals, but the remark in (1.1) already shows that this relationship is not strong enough to ensure a relevant theory. So we define the following hull-operation which relates the topology and the order structure of E.

(2.1) DEFINITION. For any subspace A in E, $\underline{G}(A) = \bigcap_{U \in \mathfrak{U}} [A + U]$. A is called a *full ideal* if $A = \underline{G}(A)$.

LEMMA. *For any subspace A in E, $\bar{A} \subset \underline{G}(A) = \underline{G}(\bar{A}) = \overline{\underline{G}(A)} = \underline{G}(\underline{G}(A))$.*

The proof is an easy computation using the fact that

$$\underline{G}(A) = \{x \in E : \forall U \in \mathfrak{U}, \exists \mu, \nu \in U \text{ and } y, z \in A \text{ such that } -\mu - y \leqq x \leqq \nu + z\}.$$

As a consequence of the lemma we remark: $\underline{G}(A)$ is always a full ideal, and each full ideal is a closed order ideal. Examples of full ideals are provided by the intersection of kernels of continuous positive linear forms on E. (2.3) below will show that, conversely, any full ideal is of this type.

(2.2) PROPOSITION. *A subspace A in E is a full ideal if and only if $(\hat{E} = E/A, [\hat{\mathfrak{U}}], \hat{C})$ is an* o.l.c.s.

PROOF. $\hat{C}$ is a proper cone in $\hat{E}$ if and only if A is an order ideal. Hence it remains to prove that the topology on $\hat{E}$ generated by $[\hat{\mathfrak{U}}]$ is separated if and only if A is a full ideal. But

$$(\underline{G}(A))\hat{} = \bigcap_{U \in \mathfrak{U}} [A + U]\hat{} = \bigcap_{U \in \mathfrak{U}} [(A + U)\hat{}\,] = \bigcap_{U \in \mathfrak{U}} [\hat{U}]$$

shows that $\bigcap_{U \in \mathfrak{U}} [\hat{U}] = (0)$ is equivalent to $A = \underline{G}(A)$.

REMARK. The following properties of $\hat{E} = E/A$, where A is a full ideal, are immediate: $\hat{C}$ is a normal cone in $(\hat{E}, [\hat{\mathfrak{U}}])$ and consequently

$$(\hat{E}, [\hat{\mathfrak{U}}])' = A^\circ \cap C' - A^\circ \cap C'.$$

Moreover, the closure in $(\hat{E}, [\hat{\mathfrak{U}}])$ of $\hat{C}$ is a proper cone.

(2.3) THEOREM. *A subspace A in E is a full ideal if and only if A is the intersection of all maximal closed order ideals in E containing A:*

$$A = (A^\circ \cap C' - A^\circ \cap C')^\circ.$$

PROOF. We show that $\underline{G}(A) = (A^\circ \cap C' - A^\circ \cap C')^\circ$ for any subspace A in E. Observe, that $f \in A^\circ \cap C'$ implies $\underline{G}(A) \subset f^{-1}(0)$ and hence

$$\underline{G}(A) \subset (A^\circ \cap C' - A^\circ \cap C')^\circ.$$

For the converse inclusion, it is enough to show that $A^\circ \cap C' - A^\circ \cap C'$ is $\sigma(E',E)$-dense in $(\underline{G}(A))^\circ$. Consider $\hat{E} = E/\underline{G}(A)$. We get $(\hat{E},\hat{\mathfrak{U}})' = (\underline{G}(A))^\circ$, and by the remark in (2.2), $(\hat{E},[\hat{\mathfrak{U}}])' = F'$ where

$$F' = (\underline{G}(A))^\circ \cap C' - (\underline{G}(A))^\circ \cap C' .$$

But $\underline{G}(A)$ is a full ideal. Hence $\langle \hat{E},F'\rangle$ and $\langle \hat{E},(\underline{G}(A))^\circ\rangle$ are dual pairs, which implies that F' is $\sigma\big((\underline{G}(A))^\circ,\hat{E}\big)$-dense in $(\underline{G}(A))^\circ$. By [13, IV, 4.1, Cor. 1], F' is also $\sigma(E',E)$-dense in $(\underline{G}(A))^\circ$. The assertion follows now from $(\underline{G}(A))^\circ \subset A^\circ$.

(2.4) In a Banach lattice, $f^{-1}(0)$ is far from being a lattice ideal for every continuous positive linear form f. Hence, the above characterization shows that the class of full ideals is too large for our purpose. In the following definition, which is also valid for arbitrary cones and locally convex topologies, we combine (1.3) and (2.1).

Definition. For any subspace A in E,

$$\underline{H}_{\mathfrak{U},C}(A) = \underline{H}(A) = \textstyle\bigcap_{U\in\mathfrak{U}}[(A+U)\cap(C+U)-(A+U)\cap(C+U)] .$$

A is called a *fully perfect ideal* if $A = \underline{H}(A)$.

Let $x \in [(A+U)\cap(C+U)-(A+U)\cap(C+U)]$ for $U\in\mathfrak{U}$. This is equivalent to $x \leqq (y_1+u_1)-(y_2+u_2)$ and a similar inequality for $-x$, where $y_i\in A$, $u_i\in U$ and $(y_i+u_i)\in C+U$ for $i=1,2$. Hence there exist $u_i'\in U$ such that $(y_i+u_i+u_i')\in C$. By adding the appropriate positive elements at the right side of the above inequalities we finally get $x\leqq y+\nu$ and $-x\leqq y+\mu$ for $y\in A$ and $\mu,\nu\in 3U$. Conversely, if $-\mu-y\leqq x\leqq \nu+y$ for $y\in A$ and $\mu,\nu\in U$, then $0\leqq y+\frac{1}{2}(\mu+\nu)$. Hence $(\mu+y)$, $(\nu+y)\in U+C$ and we have proved

$$(*)\quad \underline{H}(A) = \{x\in E:\ \forall\, U\in\mathfrak{U},\ \exists\,\mu,\nu\in U,\ y\in A \text{ such that } -\mu-y\leqq x\leqq \nu+y\} .$$

Hence every fully perfect ideal (fp-ideal) is "almost" positively generated, that is, perfect. It is a full ideal by the following lemma.

Lemma. *If $A\subset\underline{H}(A)$, then $\bar{A}\subset\underline{H}(A)=\underline{H}(\bar{A})=\overline{\underline{H}(A)}=\underline{H}(\underline{H}(A))=\underline{G}(A)$.*

Proof. For any subspace A in E, $\underline{H}(A)$ is again a subspace and hence $\underline{H}(A)\subset\underline{G}(\underline{H}(A))$. For the reverse inclusion assume $x\in\underline{G}(\underline{H}(A))$ and $U\in\mathfrak{U}$. This means

$$(1) \qquad -\mu_1 - y \leqq x \leqq \nu_1 + z$$
$$(2) \qquad -\mu_2 - \bar{y} \leqq y \leqq \nu_2 + \bar{y}$$
$$(3) \qquad -\mu_3 - \bar{z} \leqq z \leqq \nu_3 + \bar{z}\,,$$

where $\mu_i, \nu_i \in U$, $y, z \in \underline{H}(A)$ and $\bar{y}, \bar{z} \in A$. Using (2) and (3) we get from (1)

$$-\mu_1 - \nu_2 - \bar{y} - \left(\tfrac{1}{2}(\mu_3 + \nu_3) + \bar{z}\right) \leqq x \leqq \nu_1 + \nu_3 + \bar{z} + \left(\tfrac{1}{2}(\mu_2 + \nu_2) + \bar{y}\right).$$

Since U was arbitrary, this shows $x \in \underline{H}(A)$ and hence $\underline{H}(A) = \underline{G}(\underline{H}(A))$. By a similar computation one proves $\underline{H}(\underline{H}(A)) \subset \underline{H}(A)$. Suppose now $A \subset \underline{H}(A)$. Then $\underline{H}(A) \subset \underline{H}(\underline{H}(A)) \subset \underline{H}(A)$ and $\underline{G}(A) \subset \underline{G}(\underline{H}(A)) = \underline{H}(A) \subset \underline{G}(A)$. This shows $\underline{H}(A) = \underline{H}(\underline{H}(A)) = \underline{G}(A)$. The remaining statements follow from (2.1).

Proposition. *Let A be a subspace of E such that the positive part $A^+ = A \cap C$ is total in A, that is, $A \subset \overline{A^+ - A^+}$. Then $\underline{H}(A)$ is a* fp-*ideal.*

Proof. $A^+ - A^+ \subset \underline{H}(A^+ - A^+) = \underline{H}(A) = \underline{H}(\underline{H}(A))$ by the above lemma.

(2.5) Proposition. *For any subspace A in $(E, \mathfrak{U}, C)$,*

$$\underline{H}_{\mathfrak{U},C}(A) = \underline{H}_{[\mathfrak{U}],C}(A) = \underline{H}_{\mathfrak{U},\bar{C}}(A) = \underline{H}_{[\mathfrak{U}],\bar{C}}(A)\,.$$

Proof. The following inclusions prove the assertion:

$$\begin{aligned}&[(A+U) \cap (C+U) - (A+U) \cap (C+U)] \\ &\qquad \subset [(A+[U]) \cap (\bar{C}+[U]) - (A+[U]) \cap (\bar{C}+[U])] \\ &\qquad \subset [(A+2U) \cap (C+2U) - (A+2U) \cap (C+2U)] \text{ for all } U \in \mathfrak{U}\,.\end{aligned}$$

Remark. The previous proposition shows that the set of fp-ideals in $(E, \mathfrak{U}, C)$ remains unchanged if we consider the topology generated by $[\mathfrak{U}]$ and the closure $\bar{C}$ of C with respect to this topology. $\bar{C}$ is a normal cone for $[\mathfrak{U}]$, but need not be a proper cone. Moreover, $[\mathfrak{U}]$ is not separated in general.

(2.6) Proposition. *Let I, I_γ, $\gamma \in \Gamma$, be* fp-*ideals in E.*

(i) *I is the intersection of all maximal closed order ideals containing I.*

(ii) $\underline{H}(0) = \bigcap_{U \in \mathfrak{U}} [U] \subset I \subset \overline{C - C}$.

(iii) *$\underline{H}(\sum_{\gamma \in \Gamma} I_\gamma)$ is an* fp-*ideal.*

Proof. Since I is a full ideal, (i) follows from (2.3). Definition (2.4) shows that $I = \underline{H}(I) \subset \underline{G}(C - C) = \overline{C - C}$, hence (ii) holds. Finally,

$$\sum_{\gamma\in\Gamma} I_\gamma = \sum_{\gamma\in\Gamma} \underline{H}(I_\gamma) \subset \underline{H}\left(\sum_{\gamma\in\Gamma} I_\gamma\right)$$

implies (iii) by lemma (2.4).

(2.7) The following proposition is fundamental for the dual characterization of fp-ideals. It is a generalization of a theorem of A. J. Ellis [6] and has been proved independently and by a quite different method by G. J. O. Jameson [7].

PROPOSITION. *Let A be a subspace of $(E,\mathfrak{U},C)$. $A\subset\underline{H}(A)$ if and only if A° is an order ideal in (E',C').*

PROOF. Since $A^\circ=(\bar{A})^\circ$ and $\underline{H}(A)=\underline{H}(\bar{A})$, we may assume, without loss of generality, that A is closed. Moreover, let $\mathfrak{U}$ be a base of closed, circled, convex 0-neighborhoods and let C be a closed cone.

Assume now

$$x \in A \subset [(A+U)\cap(C+U)-(A+U)\cap(C+U)]$$

for $U\in\mathfrak{U}$. By the computation in (2.4) there exist $\mu,\nu\in 3U$ and $y\in A$ such that $-\mu-y\leqq x\leqq \nu+y$. This implies $0\leqq y+\frac{1}{2}(\nu+\mu)$ and $0\leqq\nu+(y-x)$. Hence

$$x = y-(y-x) \in A\cap(C+3U)-A\cap(C+3U)\,.$$

Conversely, if

$$A \subset \overline{A\cap(C+U)-A\cap(C+U)} \subset A\cap(C+U)-A\cap(C+U)+U\,,$$

then also

$$A \subset [(A+2U)\cap(C+2U)-(A+2U)\cap(C+2U)]$$

and we have proved

(i) $A\subset\underline{H}(A) \Leftrightarrow A\subset\overline{A\cap(C+U)-A\cap(C+U)},\ \forall\, U\in\mathfrak{U}$.

By (1.1), A° is an order ideal iff $A^\circ=[A^\circ]=(A^\circ+C')\cap(A^\circ-C')$. Since $\bigcup_{U\in\mathfrak{U}} U^\circ=E'$, the following equivalence is also valid:

(ii) $A^\circ=[A^\circ] \Leftrightarrow A^\circ\supset(A^\circ+C'\cap U^\circ)\cap(A^\circ-C'\cap U^\circ),\ \forall\, U\in\mathfrak{U}$.

We will prove the equivalence of (i) and (ii) by taking polars with respect to the dual pair $\langle E,E'\rangle$. For this, we recall the following properties: If M,N are closed, convex subsets of E (or E_σ') containing (0), then

(a) $(M+N)^\circ\supset\frac{1}{2}(M^\circ\cap N^\circ)$, (b) $(M\cap N)^\circ\supset\frac{1}{2}(M^\circ+N^\circ)$,

(c) $(M+N)^\circ\subset M^\circ\cap N^\circ$, (d) $(M\cap N)^\circ\subset\overline{(M^\circ+N^\circ)}$.

Using these inclusions, we show:

(i) $\Rightarrow$ (ii): Let $A \subset \overline{A \cap (C+U) - A \cap (C+U)}$ for $U \in \mathfrak{U}$. Then

$$\begin{aligned} A^\circ &\supset (A \cap (C+2U))^\circ \cap (A \cap (-C+2U))^\circ \\ &\supset (A^\circ + C^\circ \cap (8U)^\circ) \cap (A^\circ + (-C)^\circ \cap (8U)^\circ) \\ &= (A^\circ - C' \cap \tfrac{1}{8}U^\circ) \cap (A^\circ + C' \cap \tfrac{1}{8}U^\circ) , \end{aligned}$$

which proves (ii).

(ii) $\Rightarrow$ (i): Let $A^\circ \supset (A^\circ + C' \cap U^\circ) \cap (A^\circ - C' \cap U^\circ)$ for $U \in \mathfrak{U}$. Then

$$\begin{aligned} A = A^{\circ\circ} &\subset \overline{(A^\circ + C' \cap U^\circ)^\circ + (A^\circ - C' \cap U^\circ)^\circ} \\ &\subset \overline{A \cap (C' \cap U^\circ)^\circ + A \cap (-C' \cap U^\circ)^\circ} \\ &\subset \overline{A \cap (C+2U) - A \cap (C+2U)} , \end{aligned}$$

which proves (i).

Remark. The proof is valid for any cone in a locally convex space.

(2.8) Theorem. *Let I be a closed subspace of E. Then I is a* fp-*ideal if and only if I° is an order ideal in E' for which the positive part $I^\circ \cap C'$ is $\sigma(E',E)$-total in I°.*

Proof. I is a fp-ideal if and only if $I \subset \underline{H}(I)$ and $I = \underline{G}(I)$. Hence, (2.7) and (2.3) together imply the above theorem.

Corollary 1. *Let I be a subspace of E. There exists a one-to-one correspondence between the* fp-*ideals I in E and the extreme subsets $I^\circ \cap C'$ of C' in E'.*

Corollary 2. *A subspace I is a* fp-*ideal in $(E, \mathfrak{U}, C)$ if and only if it is a* fp-*ideal for any locally convex topology consistent with the duality $\langle E, (E, \mathfrak{U})' \rangle$.*

(2.9) Theorem. *Let J be a* fp-*ideal in $(E, \mathfrak{U}, C)$. There exists a one-to-one correspondence between the* fp-*ideals in $(\hat{E} = E/J, \hat{\mathfrak{U}}, \hat{C})$ and the* fp-*ideals in $(E, \mathfrak{U}, C)$ containing J.*

Proof. Let I be a subspace of E containing J. It is clear that I corresponds to exactly one subspace $\hat{I}$ in $\hat{E}$. Consider now the dual pairs $\langle E, E' \rangle$ and $\langle \hat{E}, J^\circ \rangle$ and recall that every fp-ideal is characterized by the intersection of its polar with the dual cone. The dual cone of $\hat{C}$

coincides with $C' \cap J^\circ$ and we get $I^\circ \cap C' = (\hat{I})^\circ \cap (C' \cap J^\circ)$ because $(\hat{I})^\circ = I^\circ$. Therefore, it remains to show that $I^\circ \cap C'$ satisfies the characterization properties of (2.8) for fp-ideals simultaneously for both dual pairs $\langle E, E' \rangle$ and $\langle \hat{E}, J^\circ \rangle$. The second property follows immediately since $\sigma(E', E)$ coincides with $\sigma(J^\circ, \hat{E})$ on $I^\circ \subset J^\circ$. By hypothesis, J° is an order ideal in E'. Hence, the following simple lemma proves the first property.

LEMMA. *Let F be an ordered vector space and A an order ideal in F. The subspace B of A is an order ideal in F if and only if it is an order ideal in A.*

(2.10) The positive part $A^+ = A \cap C$ of a subspace A in $(E, \mathfrak{U}, C)$ is of special interest. If I is a fp-ideal in E, then I° has $\sigma(E', E)$-total positive part, that is, $I^\circ = \overline{I^{\circ +} - I^{\circ +}}$ in E_σ'. Hence, by (2.4), $\underline{H}_\sigma(I^\circ)$ is a fp-ideal in E_σ'. Let C be a closed cone in E. Now $(E_\sigma', C')' = (E, C)$ and $I^{\circ\circ} = I$ imply that $(\underline{H}_\sigma(I^\circ))^\circ = \overline{I^+ - I^+}$, which proves the following theorem.

THEOREM. *Let I be a closed subspace of $(E, \mathfrak{U}, C)$ where C is a closed cone. I is a* fp*-ideal with total positive part if and only if I° is a* fp*-ideal in E_σ' with $\sigma(E', E)$-total positive part.*

(2.11) EXAMPLES.

1.) Let E be a locally convex vector lattice. A subspace I in E is a fp-ideal if and only if I is a closed lattice ideal in E. The proof uses the existence of a base of solid 0-neighborhoods and the decomposition property in E.

2.) Let $E = C^{(\infty)}(\mathsf{R}^n)$ be the vector space of all real-valued, infinitely differentiable functions on R^n whose support is compact, endowed with the topology generated by all semi-norms

$$f \to p_n(f) = \sup_{t \in \mathsf{R}^n} |f^{(n)}(t)|$$

and with the natural order. A subspace I in E is a fp-ideal if and only if

$$I = \{f \in E : f(S) = (0) \text{ for some closed subset } S \subset \mathsf{R}^n\}.$$

The proof uses (2.5) and results of [8].

3. Fully perfect ideals in ordered Banach spaces.

From now on, (E, U, C) is always an ordered Banach space with unit ball U and closed cone C. To exclude trivial cases, we assume further that $\dim(C - C) > 1$.

(3.1) DEFINITION (see [11]). An element $x \in C$ is called a *support point of* C if there exists a non-zero positive linear form $f \in E'$ such that $f(x) = 0$.

PROPOSITION. *An element $x \in C$ is a support point of C if and only if*

$$\underline{H}(\langle x \rangle) \neq E \quad \textit{for} \quad \langle x \rangle = \{\lambda x : \lambda \in \mathsf{R}\}.$$

PROOF. $\langle x \rangle$ is positively generated, hence $I = \underline{H}(\langle x \rangle)$ is a fp-ideal by (2.4). The dual characterization theorem (2.8) shows that $I = E$ if and only if $I^\circ \cap C' = (0)$. Thus, the assertion follows from the fact that $I^\circ \cap C' = (\langle x \rangle)^\circ \cap C'$.

REMARK. The proposition gives a characterization of support points not involving duality which is similar to the definition of "quasi-interior points" (see [11]). It is obvious that all of the above is also valid for any o.l.c.s.

(3.2) With the aid of the above proposition and a result of R. R. Phelps [12] we will prove that the fp-ideals in E separate points. This means: if $0 \neq x \in E$, then there exists a non-trivial fp-ideal I in E such that $x \notin I$.

LEMMA 1. *The set $S' = \{f \in C' : \exists\, 0 < y \in E$ such that $f(y) = 0\}$ is $\sigma(E', E)$-dense in the $\beta(E', E)$-boundary of C'.*

PROOF. It is clear that S' is the set of all support points of C' in E_σ'. Therefore, the lemma is only a weaker formulation of [12, theorem 1].

LEMMA 2. *The positive cone C is the convex hull of its (topological) boundary.*

PROOF. Since C is closed, E is archimedean ordered and $E \neq C \cup (-C)$. Let $0 \neq y \in C$ and choose $x \in E$, $x \notin C \cup (-C)$. If we assume

$$\{\lambda x + y : 0 < \lambda \in \mathsf{R}\} \subset C,$$

we get $(x + (y/\lambda)) \in C$ for all $0 < \lambda \in \mathsf{R}$ and hence $x \in \bar{C} = C$, which contradicts the assumption. The same argument for $\{\lambda(-x) + y : 0 < \lambda \in \mathsf{R}\}$ proves the existence of $\lambda_1, \lambda_2 \geqq 0$ such that $(\lambda_1 x + y)$ and $(-\lambda_2 x + y)$ are in the boundary of C. Since the boundary of a cone is invariant under multiplication by positive scalars, it is now easy to find a convex combination for y by elements of the boundary of C.

LEMMA 3. *If I is a* fp-*ideal in* $F = \overline{C - C}$, *then I is also a* fp-*ideal in E.*

PROOF. From (2.6), it follows that $I = \underline{\underline{H}}_F(I) \subset \underline{\underline{H}}_E(I) \subset F$. Now F' and E'/F° are order isomorphic. Hence, each positive continuous linear form on F which vanishes on I has a positive continuous extension to E, which also vanishes on I. Therefore, $x \in \underline{\underline{G}}_E(I)$ implies $x \in \underline{\underline{G}}_F(I)$ and $\underline{\underline{H}}_F(I) = \underline{\underline{H}}_E(I)$.

THEOREM. *The intersection of all non-zero* fp-*ideals in E is* (0).

PROOF. Let $0 \neq x \in E$. We have to show that there exists a non-zero fp-ideal in E not containing x. This is obvious for $x \notin \overline{C - C} = H(C - C)$. In the other case, we may assume, by lemma 3, that $E = \overline{C - C}$. But then, C' is a closed proper cone in the Banach space $E_\beta{}'$ and lemmas 1 and 2 apply to C'. This proves that C' is the $\sigma(E', E)$-closed convex hull of S'. Since C' is $\sigma(E', E)$-total in E', S' is also and, consequently, S' separates points in E. Therefore, for $0 \neq x \in E$, there exist $0 < f \in E'$ and $0 < y \in E$ such that $f(x) \neq 0$ and $f(y) = 0$. This means that y is a support point of C and generates a non-zero fp-ideal

$$I = \underline{\underline{H}}(\langle y \rangle) \neq E .$$

Finally, $f \in I^\circ \cap C'$, but $f(x) \neq 0$ shows that $x \notin I$.

From the proof we conclude the following strengthened version of the above theorem.

COROLLARY. *Let E be an ordered Banach space with closed cone C such that* $\dim(C - C) > 1$. *The* fp-*ideals in E with non-trivial positive part separate points in E.*

REMARK. The last statement suggests that every fp-ideal in an ordered Banach space with closed cone has non-trivial positive part. But an example in [9] shows that this is not true. There even exist maximal fp-ideals in Banach spaces with order unit which have trivial positive part.

(3.3) It is well known that in spaces $C(X)$ each proper lattice ideal is contained in a (proper) maximal lattice ideal [8]. For a generalization of this result, we need the following characterization of (proper) maximal fp-ideals.

THEOREM. *Let I be a subspace in an ordered Banach space $E=\overline{C-C}$. The following assertions are equivalent*:
(a) *I is a maximal* fp-*ideal.*
(b) *I is a* fp-*ideal with* $\dim E/I=1$.
(c) $I=f^{-1}(0)$, *where $f\in C'$ generates an extreme ray of C'.*

PROOF. (*a*) $\Rightarrow$ (b): If I is a maximal fp-ideal, then (2.9) shows that $\hat{E}=E/I$ has no proper fp-ideals. But E is an ordered Banach space in which the positive cone $\hat{C}$ is total. By (2.2), the closure of $\hat{C}$ is still a proper cone. Hence (3.2) applies and shows that $\dim\hat{E}=1$.

(b) $\Rightarrow$ (c): From $\dim\hat{E}=1$, it follows that $I=f^{-1}(0)$ for $f\in E'$. Since I is a full ideal, f can be chosen positive. Finally,

$$I^\circ\cap C' = \{\lambda f:\ 0\leqq\lambda\in \mathsf{R}\}$$

must be an extreme ray, since I is a fp-ideal.

(c) $\Rightarrow$ (a) is obvious from (2.8).

(3.4) LEMMA. *Let (E,U,C) be an ordered Banach space such that the dual norm is additive on C'. A subspace F' is an order ideal in E' if and only if $F'\cap\{0<f\in E' : \|f\|=1\}$ is an extreme subset of $U^\circ\cap C'$.*

PROOF. Let F' be an order ideal in E' and assume $f_0=\lambda f_1+(1-\lambda)f_2$ for

$$f_0\ \in\ F'\cap\{0<f\in E':\ \|f\|=1\},$$

$f_1,f_2\in U^0\cap C'$ and $0<\lambda<1$. Since $0\leqq\lambda f_1$, $(1-\lambda)f_2\leqq f_0$, we get $f_1,f_2\in F'$. Moreover,

$$1 = \|f_0\| = \lambda\|f_1\|+(1-\lambda)\|f_2\| \quad\text{implies}\quad \|f_1\| = \|f_2\| = 1\,,$$

which proves the assertion. To prove the converse implication, let

$$F'\cap\{0<f\in E':\ \|f\|=1\}$$

be an extreme subset of $U^\circ\cap C'$ and let $0<g<f_0\in F'$. Without loss of generality, we assume that $\|f_0\|=1$. From $f_0=g+(f_0-g)$ and the additivity of the norm on C', it follows that $\|g\|=\mu\neq 0$, $\|f-g\|=\lambda\neq 0$ and $\mu+\lambda=1$. Therefore,

$$f_0 = \mu(\mu^{-1}g)+\lambda(\lambda^{-1}(f_0-g))$$

is a convex combination in $U^\circ\cap C'$. The assumption implies that

$$\mu^{-1}g\in F'\cap\{0<f\in E':\ \|f\|=1\}$$

and hence $g\in F'$.

THEOREM. *Let (E, U, C) be an ordered Banach space $(\neq \mathbb{R})$ such that the dual norm is additive on C'. Every* fp-*ideal in E is the intersection of all maximal* fp-*ideals containing it.*

PROOF. By (3.3) and the above lemma, the maximal fp-ideals in E correspond biunivoquely to the non-zero extreme points of $U^\circ \cap C'$. Let I be a proper fp-ideal in E. Then I is determined by $I^\circ \cap (U^\circ \cap C')$ and also, by the Krein–Milman theorem, by the set of extreme points of $I^\circ \cap (U^\circ \cap C')$. But, since $I^\circ \cap (U^\circ \cap C')$ is an extreme subset of $U^\circ \cap C'$, every extreme point f of $I^\circ \cap (U^\circ \cap C')$ is also an extreme point of $U^\circ \cap C'$. Hence, $f^{-1}(0)$ is a maximal fp-ideal in E containing I and the intersection of all these maximal fp-ideals is I.

COROLLARY. *Under the same assumptions, the maximal* fp-*ideals separate points in E.*

PROOF. By the previous considerations, the assertion is equivalent to the fact that the set of extreme points of $U^\circ \cap C'$ is $\sigma(E', E)$-total in E'.

REMARK. Examples of ordered Banach spaces with additive norm on C' are provided by ordered Banach spaces whose open unit ball is directed upwards, especially by ordered Banach spaces whose positive cone has interior points. For more information about such spaces, see [10].

(3.5) EXAMPLES.

1.) If E is the subspace of all hermitean elements of a C^*-algebra A, we have the following one-to-one correspondence between the fp-ideals I in E and the closed left ideals N in A:

$$I^+ = N \cap C \quad \text{and} \quad N = \overline{A \cdot I^+} = \{x \in A : x^*x \in I^+\}$$

(see [4]).

2.) If E is a simplex space, then a subspace of E is a fp-ideal if and only if it is a closed and positively generated order ideal. Moreover, the sum and the intersection of two fp-ideals in E are again fp-ideals (see [5]).

REFERENCES

1. E. Alfsen, *Facial structure of compact sets*, Proc. London Math. Soc. 18 (1967), 385–404.
2. F. F. Bonsall, *Extreme maximal ideals of partially ordered vector space*, Proc. Amer. Math. Soc. 7 (1956), 831–837.

3. F. Combes et F. Perdrizet, *Certains idéaux dans les espaces vectoriels ordonnés*, C. R. Acad. Sci. Paris Sér. A-B 268 (1969), A1552–A1555.
4. E. Effros, *Order ideals in a C^*-algebra and its dual*, Duke Math. J. 30 (1963), 391–412.
5. E. Effros, *Structure in simplexes*, Acta Math. 117 (1967), 103–121.
6. A. J. Ellis, *Perfect order ideals*, J. London Math. Soc. 40 (1965), 288–294.
7. G. J. O. Jameson, *Nearly directed subspaces of partially ordered linear spaces*, Proc. Edinburgh Math. Soc. 16 (1968), 135–144.
8. H. P. Lotz, *Zur Idealstruktur von Banachverbänden*, Tübingen, 1969, preprint.
9. R. J. Nagel, *Idealtheorie in geordneten lokal konvexen Vektorräumen*, Dissertation, Universität Tübingen, 1969.
10. K. F. Ng, *The duality of partially ordered Banach spaces*, Proc. London Math. Soc. 19 (1969), 269–288.
11. A. L. Peressini, *Ordered topological vector spaces*, Harper and Row, New York, 1967.
12. R. R. Phelps, *Weak* support points of convex sets in E^**, Israel J. Math. 2 (1964), 177–182.
13. H. H. Schaefer, *Topological vector spaces*, Macmillan, New York, 1966.
14. H. H. Schaefer, *Invariant ideals of positive operators in $C(X)$ I*, Illinois, J. Math. 11 (1967), 703–715.
15. H. H. Schaefer, *Invariant ideals of positive operators in $C(X)$ II*, Illinois J. Math. 12 (1968), 525–538.
16. E. Størmer, *On partially ordered vector spaces and their duals, with applications to simplexes and C^*-algebras*, Proc. London Math. Soc. 18 (1967), 245–265.

MATHEMATISCHES INSTITUT DER UNIVERSITÄT TÜBINGEN, GERMANY

AND

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND,
COLLEGE PARK, MARYLAND, U.S.A.