MATH. SCAND. 29 (1971), 161–174

BILINEARITY AND CARTESIAN CLOSED MONADS

ANDERS KOCK

Introduction.

The observation that universal algebra can be treated "in a coordinate free way", using monads, together with the observation that many techniques available in the category of sets are available in any symmetric monoidal closed category $\mathscr{V}$, makes it possible, by combined use of the monad language and the closed category language, to describe some of the elementary notions of universal algebra in an element-free way. Syntactically, such a description no longer has variables ranging over the set of elements in an algebra, but rather over the class of algebras itself. In particular, though the monad-theoretic treatment allows for infinitary algebras, the notions formulated here are all formulated in a finitary way. For example, one can formulate the notion of "algebra, where every operation is a homomorphism with respect to any operation". This was done in [3], [4], and [5].

The notion to be described here is that of a "function in two variables which is a homomorphism in each variable separately."

This concept of "bilinearity" is used in the main Section 2, which deals with cartesian closed categories. The main theorem is Theorem 2.6 which says that a monad on a cartesian closed category, whose functor part commutes with products satisfies:

(i) it carries a unique structure as *strong* monad; and
(ii) this structure makes the monad *commutative* in the sense of [3]; and
(iii) the category of algebras for it is cartesian closed.

Monads of this kind we call cartesian closed (Definition 2.7). They can also be characterized as commutative monads for which a map "is bilinear if and only if it is linear" (Theorem 2.5). The two implications in this "if and only if" are investigated separately in Theorem 2.1 and Proposition 2.3, the first describing the notion of "affine monad".

A remark on the scope of universal algebra in the setting of *strong* monads on symmetric monoidal closed categories $\mathscr{V}$: This scope is quite

Received February 3, 1971; in revised form May 11, 1971.

limited, if $\mathscr{V}$ is not cartesian closed; for instance, "tensor algebra construction" (with respect to the $\otimes$ of $\mathscr{V}$) will, if it exists, not in general be a *strong* monad (but will probably distribute over any strong commutative monad). On cocomplete *cartesian* closed categories, however, there will be a strong monad corresponding at least to each finitary algebraic theory. A hint of this fact may be found in [1].

Note that in Sections 1 and 2, the symmetric monoidal closed category $\mathscr{V}$ is not assumed to be cartesian closed. Thus we write $\otimes$ for what later is replaced by $\times$.

We often suppress mention of which T-structure we have in mind; we just write "f: $A \otimes B \to C$ is bilinear" instead of "f is bilinear with respect to the T-structures α: $AT \to A$, β: $BT \to B$, γ: $CT \to C$ on A, B, and C, respectively".

1. Partial linearity and bilinearity.

Let $\mathscr{V}$ be any symmetric monoidal closed category [2], and let T be a $\mathscr{V}$-functor ("strong functor") from the $\mathscr{V}$-category $\mathscr{V}$ to itself,

$$T: \mathscr{V} \to \mathscr{V}$$

that is, T is a functor $\mathscr{V}_0 \to \mathscr{V}_0$, and there is given a family of maps ("the strength of T"):

$$\mathrm{st}_{A,B}: A \pitchfork B \to AT \pitchfork BT .$$

These data satisfy the axions VF1, VF2 of [2]. Recall [3] that out of these data we can construct a natural transformation

$$t''_{A,B}: A \otimes BT \to (A \otimes B)T .$$

In fact, st can be reconstructed from $T: \mathscr{V}_0 \to \mathscr{V}_0$ and t'', by [6].

Using the symmetry c of $\mathscr{V}$, we also construct the natural

$$t'_{A,B} = c \cdot t''_{B,A} \cdot cT: AT \otimes B \to (A \otimes B)T .$$

Now assume that T is the functor part of a strong monad $\mathsf{T} = (T, \eta, \mu)$ (meaning that η and μ are $\mathscr{V}$-natural transformations in the sense of [2, p. 466]). Then according to [3] we get two monoidal structures ψ and $\tilde{\psi}$ on T, with

$$\psi_{A,B}: AT \otimes BT \xrightarrow{t'} (A \otimes BT)T \xrightarrow{t''T} (A \otimes B)T^2 \xrightarrow{\mu} (A \otimes B)T$$

and with

$$\tilde{\psi}_{A,B} = c \cdot \psi_{B,A} \cdot cT: AT \otimes BT \to (A \otimes B)T$$

or equivalently, $\tilde{\psi}_{A,B} = t'' \cdot t'T \cdot \mu$. Recall [3] that the monad is called *commutative* if $\psi_{A,B} = \tilde{\psi}_{A,B}$ for all A, B.

Let (A,α) and (C,γ) be algebras for T, and let B be arbitrary. A map $f\colon A\otimes B \to C$ is called 1-linear if the following pentagon commutes

$$\begin{tikzcd} AT\otimes B \arrow[r, "t'_{A,B}"] \arrow[d, "\alpha\otimes 1"'] & (A\otimes B)T \arrow[r, "fT"] & CT \arrow[d, "\gamma"] \\ A\otimes B \arrow[rr, "f"'] & & C \end{tikzcd}$$

If β is a T-structure on B, one defines similarly the notion of a 2-linear map using t''. Finally, call f ψ-bilinear if the following pentagon commutes:

$$\begin{tikzcd} AT\otimes BT \arrow[r, "\psi_{A,B}"] \arrow[d, "\alpha\otimes\beta"'] & (A\otimes B)T \arrow[r, "fT"] & CT \arrow[d, "\gamma"] \\ A\otimes B \arrow[rr, "f"'] & & C \end{tikzcd} \tag{1.1}$$

Such a definition of bilinearity has been suggested by Linton [7].

Using $\tilde{\psi}_{A,B}$ instead of $\psi_{A,B}$ in the above diagram gives similarly the notion of f being $\tilde{\psi}$-bilinear. It turns out, however, that ψ-bilinearity is the same notion as ψ-bilinearity (Theorem 1.1 below); so we may just describe the notion by the word "bilinearity".

THEOREM 1.1. *For a map $f\colon A\otimes B \to C$, the following three conditions are equivalent*:

(i) *f is ψ-bilinear,*
(ii) *f is 1-linear and 2-linear,*
(iii) *f is $\tilde{\psi}$-bilinear.*

PROOF. We shall prove (i) $\Leftrightarrow$ (ii); since the condition (ii) is "left-right symmetric", the proof that (iii) $\Leftrightarrow$ (ii) will then be similar. So let f be ψ-bilinear. Multiply the commutative pentagon (1.1) on the left by

$$1\otimes\eta_B\colon AT\otimes B \to AT\otimes BT\,.$$

The lower composite of (1.1) yields $\alpha\otimes 1\cdot f$. To see that the upper composite yields $t'\cdot fT\cdot\gamma$, we have to prove that

$$1\otimes\eta_B\cdot\psi = t'\,. \tag{1.2}$$

But this is an easy consequence of the definition of ψ and of $1_A\otimes\eta_B\cdot t'' = \eta_{A\otimes B}$ (Lemma 2.2 in [3]). So 1-linearity of f is proved. Similarly, 2-linearity is proved using

$$\eta_A\otimes 1\cdot\psi = t''\,,$$

which again is a consequence of Lemma 2.2 in [3].

Conversely, assume that f is 1-linear and 2-linear. By definition of ψ, to prove (1.1) commutative means proving

$$t' \cdot t''T \cdot \mu \cdot fT \cdot \gamma = \alpha \otimes \beta \cdot f .$$

Now the left hand side here equals

$$\begin{aligned}
& t' \cdot t''T \cdot fT^2 \cdot \mu \cdot \gamma && \text{by naturality of } \mu \\
= \; & t' \cdot t''T \cdot fT^2 \cdot \gamma T \cdot \gamma && \text{by ``associative law'' for } \gamma \\
= \; & t' \cdot (1 \otimes \beta)T \cdot fT \cdot \gamma && \text{by 2-linearity of } f \\
= \; & 1 \otimes \beta \cdot t' \cdot fT \cdot \gamma && \text{by naturality of } t' \\
= \; & 1 \otimes \beta \cdot \alpha \otimes 1 \cdot f && \text{by 1-linearity of } f ,
\end{aligned}$$

which is the right hand side of the desired equation. The theorem is proved.

Remark 1.2. There is an element-free version of Theorem 1.1 provided $\mathscr{V}_0$ has equalizers; we can then define subobjects Lin_1, Lin_2, Bilin, and $\mathrm{Bilin}^{\sim}$ of $(A \otimes B) \pitchfork C$, whose "elements" are the 1-linear, 2-linear, ψ-bilinear, $\tilde{\psi}$-bilinear maps $A \otimes B \to C$; the sharpened form of the theorem says:

$$\mathrm{Lin}_1 \cap \mathrm{Lin}_2 = \mathrm{Bilin} = \mathrm{Bilin}^{\sim} .$$

The proof of this is somewhat more complicated, and we omit it, since the sharpened form plays no role in the present paper.

In the following, we assume that $\mathscr{V}_0$ has equalizers. Since B and C have T-algebra structures (β and γ, respectively), we may, as is well known (by [4], say), form a subobject $B \Pitchfork C$ of $B \pitchfork C$, »the subobject of homomorphisms", namely as the equalizer of $\mathrm{st} \cdot 1 \pitchfork \gamma$ and $\beta \pitchfork 1$, st being the strength of T. Also, we may make $B \pitchfork C$ into a T-algebra by means of

$$(B \pitchfork C)T \xrightarrow{\lambda} B \pitchfork CT \xrightarrow{1 \pitchfork \gamma} B \pitchfork C ,$$

where λ is the combinator constructed out of the strength of T as in [4] or [5].

Consider $f \colon A \otimes B \to C$ as before, and let $\check{f}$ denote the transpose, $\check{f} \colon A \to B \pitchfork C$. We then have

Proposition 1.3. *The map f is bilinear if and only if $\check{f}$ is a T-homomorphism and factors through $B \Pitchfork C$.*

Proof. We shall in fact argue that

(i) f is 1-linear iff $\check{f}$ is a T-homomorphism and
(ii) f is 2-linear iff $\check{f}$ factors through $B \Pitchfork C$,

from which the Proposition will follow, using Theorem 1.2.

The proof of (i) depends of course on knowing the interrelations between t' and λ. From Lemma 1.2 in [4], one can deduce, by $\otimes$-$\pitchfork$ adjointness, that $\lambda\colon (B \pitchfork C)T \to B \pitchfork CT$ can be expressed as the following composite (where u and ev denote front and end adjunctions for the $\otimes$-$\pitchfork$ adjointness):

$$(1.3) \qquad (B \pitchfork C)T \xrightarrow{u} B \pitchfork [(B \pitchfork C)T \otimes B] \to \xrightarrow{1 \pitchfork t'} B \pitchfork [(B \pitchfork C)\otimes B)T \xrightarrow{1 \pitchfork (\mathrm{ev})T} B \pitchfork CT .$$

So if f is 1-linear, we have

$$t' \cdot fT \cdot \gamma = \alpha \otimes 1 \cdot f\colon \ AT \otimes B \ \to \ C;$$

by $\otimes$-$\pitchfork$ adjointness we get

$$u \cdot 1 \pitchfork t' \cdot 1 \pitchfork fT \cdot 1 \pitchfork \gamma = \alpha \cdot \check{f}\colon \ AT \ \to \ B \pitchfork C .$$

The left hand side here is easy to rewrite as $\check{f}T$ followed by $1 \pitchfork \gamma$, and so equals $fT \cdot \lambda \cdot 1 \pitchfork \gamma$ which proves (i).

The proof of (ii) does not involve information about λ, but only the equality

$$u^{BT} \cdot 1 \pitchfork t'' \ = \ u^{B} \cdot \mathrm{st} ,$$

which is immediate from the definition of t'' in terms of st, as in [3]. We omit the details.

REMARK 1.4. In case T is commutative, $B \pitchfork\!\!\pitchfork C$ is a sub- T-algebra of $B \pitchfork C$, according to [4]. So we can make sense to the expression $A \pitchfork\!\!\pitchfork (B \pitchfork\!\!\pitchfork C)$. In the terminology of Remark 1.2, one can then prove that the subobject Bilin of $(A \otimes B) \pitchfork C$ corresponds under $\otimes$-$\pitchfork$ adjointness to $A \pitchfork\!\!\pitchfork (B \pitchfork\!\!\pitchfork C)$. Also, there will be induced a canonical isomorphism $A \pitchfork\!\!\pitchfork (B \pitchfork\!\!\pitchfork C) = B \pitchfork\!\!\pitchfork (A \pitchfork\!\!\pitchfork C)$, reflecting the fact that Bilin$=$ Bilin$^{\sim}$.

We conjecture that, as soon as a reasonable definition of "(non-monoidal) *symmetric* closed category" (as hinted at in [5]) has been found, the isomorphism $A \pitchfork\!\!\pitchfork (B \pitchfork\!\!\pitchfork C) \cong B \pitchfork\!\!\pitchfork (A \pitchfork\!\!\pitchfork C)$ will make the closed category generated by a commutative monad T (as in [4]) into a symmetric closed category.

We list here some examples of 1-, 2-, and bilinear maps. Let X and Y be arbitrary objects in $\mathscr{V}_0$, and make XT, YT, $(X \otimes Y)T$ into algebras for T by means of the appropriate μ. Then

$$t'_{X,Y}\colon \ XT \otimes Y \ \to (X \otimes Y)T \quad \text{is 1-linear}$$
$$t''_{X,Y}\colon \ X \otimes YT \to (X \otimes Y)T \quad \text{is 2-linear}$$

(1.4) $\psi_{X,Y}\colon XT\otimes YT \to (X\otimes Y)T$ is 1-linear

(1.5) $\tilde{\psi}_{X,Y}\colon XT\otimes YT \to (X\otimes Y)T$ is 2-linear .

These facts are easy to see using Lemma 1.1 and Lemma 1.2 in [3].

PROPOSITION 1.5. *Let* T *be a strong monad and* $\psi, \tilde{\psi}$ *the two associated monoidal structures on the functor part* T. *Then for any objects* X, Y, *the following two conditions are equivalent*:

(i) $\psi_{X,Y} = \tilde{\psi}_{X,Y}$;
(ii) $\psi_{X,Y}$ *is bilinear or* $\tilde{\psi}_{X,Y}$ *is bilinear.*

Further, the following conditions are equivalent

(iii) T *is commutative* (*that is,* $\psi = \tilde{\psi}$);
(iv) $\psi_{X,Y}$ *is bilinear for all* X, Y, *or* $\tilde{\psi}_{X,Y}$ *is bilinear for all* X, Y;
(v) μ *is a* ψ-*monoidal transformation or a* $\tilde{\psi}$-*monoidal transformation.*

PROOF. Assume (i). Then by (1.4) and (1.5), $\psi_{X,Y}$ is 1-linear and 2-linear, hence bilinear by Theorem 1.1, proving (ii). Assume (ii), for instance, assume that $\psi_{X,Y}$ is bilinear. By Theorem 1.1, we may take this to mean "$\tilde{\psi}$-bilinear", which means that

$$\tilde{\psi}_{XT,YT}\cdot\psi_{X,Y}T\cdot\mu_{X\otimes Y} = \mu_X\otimes\mu_Y\cdot\psi_{X,Y} \cdot$$

If we multiply this equation on the left by $\eta_X T\otimes\eta_Y T$, the right hand side gives $\psi_{X,Y}$. The left hand side gives, by naturality of $\tilde{\psi}$,

$$\tilde{\psi}\cdot(\eta_X\otimes\eta_Y)T\cdot\psi T\cdot\mu \ .$$

Now the proof of Theorem 3.2 in [3] shows that (without assumption of commutativity of the monad) $\eta_X\otimes\eta_Y\cdot\psi_{X,Y} = \eta_{X\otimes Y}$, so that the above becomes

$$\tilde{\psi}_{X,Y}\cdot(\eta_{X\otimes Y})T\cdot\mu_{X\otimes Y}$$

which by monad laws is just $\tilde{\psi}_{X,Y}$. This proves (i). (Similarly, if our assumption in (ii) were "$\tilde{\psi}_{X,Y}$ bilinear", we would take "bilinear" to mean "ψ-bilinear".)

The proof of (iii) $\Leftrightarrow$ (iv) is trivial from (i) $\Leftrightarrow$ (ii). Finally, (iv) $\Leftrightarrow$ (v) is immediate, noting that the diagram saying that $\psi_{X,Y}$ is ψ-bilinear is the same as a general one out of the family establishing that μ is ψ-monoidal. This proves Proposition 1.5.

When the tensor product of more than two algebras are involved in the domain, e.g.

(1.6) $f\colon A_0\otimes\ldots\otimes A_{n-1} \to A_n$, where (A_c, α_c) is a T-algebra, the notions of 1-linearity, 2-linearity, and bilinearity split into many different notions, according to how the left hand side is bracketed. There is, however, probably a coherence theorem saying that "the bracketing does not matter"; at least for $n=3$, this is true because of Propositions 1.5, 1.6, and (2.4) in [3].

2. Monads on cartesian closed categories.

In the following, $\mathscr{V}$ is a cartesian closed category (see e.g. [2, IV.2]), so we write 1 instead of I for the unit object (which now is terminal), and $A \times B$ for $A \otimes B$ (which is now a categorical product:

$$A \xleftarrow{\text{proj}_0} A \times B \xrightarrow{\text{proj}_1} B) .$$

Any functor $T: \mathscr{V}_0 \to \mathscr{V}_0$ comes equipped with the well-known combinator

$$\varkappa_{A,B}: (A \times B)T \to AT \times BT$$

given by $\varkappa_{A,B} \cdot \text{proj}_i = (\text{proj}_i)T$, $i=0,1$. (One can actually show that $\varkappa$ is $\mathscr{V}$-natural if T is a $\mathscr{V}$-functor.) In particular, if T is the functor part of a monad T, and $(A,\alpha),(B,\beta)$ are algebras for it, then $A \times B$ carries an algebra structure

$$(3.1) \qquad (A \times B)T \xrightarrow{\varkappa} AT \times BT \xrightarrow{\alpha \times \beta} A \times B .$$

If also (C,γ) is an algebra, a map

$$(3.2) \qquad f: A \times B \to C$$

may or may not be a T-homomorphism (or "linear" in the terminology to be adopted here). If T is a $\mathscr{V}$-monad, f may or may not be bilinear.

THEOREM 2.1. *Let* $(T, \text{st}), \eta, \mu$ *be a* $\mathscr{V}$*-monad on the cartesian closed category* $\mathscr{V}$, *with associated monoidal structures* $\psi, \tilde{\psi}$. *The following conditions are equivalent (and describe the notion of affine monad):*

(i) $1T \cong 1$;

(ii) $AT \times BT \xrightarrow{\psi_{A,B}} (A \times B)T \xrightarrow{\varkappa_{A,B}} AT \times BT = \text{id}$;

(iii) *as* (ii), *but with* $\tilde{\psi}_{A,B}$ *instead of* $\psi_{A,B}$;

(iv) *any map* $f: A \times B \to C$ *which is linear is also bilinear.*

PROOF. Assume (i). To prove (ii), it suffices to prove

$$\psi_{A,B} \cdot \varkappa_{A,B} \cdot \text{proj}_i = \text{proj}_i. \quad i=0,1 .$$

Let us do it for $i=0$. Let $k: B \to 1$ denote the unique such map. Then it is easy to see, using naturality of $\psi \cdot \varkappa$ with respect to k that

$$(2.3) \qquad \begin{aligned}\psi\cdot\varkappa\cdot\mathrm{proj}_0 &= \psi\cdot\varkappa\cdot 1\times kT\cdot\mathrm{proj}_0 \\ &= 1\times kT\cdot\psi\cdot\varkappa\cdot\mathrm{proj}_0 = 1\times kT\cdot\psi\cdot(\mathrm{proj}_0)T\,.\end{aligned}$$

We shall prove

$$\psi\cdot\mathrm{proj}_0 T = \mathrm{proj}_0\colon AT\times 1T\to AT\,.$$

(which clearly will imply that (2.3) is just proj_0). Now this equation says that * in the following diagram commutes:

$$\begin{tikzcd}
AT\times 1 \arrow[r, "1\times\eta_1"] \arrow[rr, bend left, "t'"] \arrow[ddrr, "\mathrm{proj}_0=r_{AT}"'] & AT\times 1T \arrow[r, "\psi"] \arrow[ddr, "\mathrm{proj}_0"] \arrow[d, phantom, "**"] & (A\times 1)T \arrow[dd, "r_AT=\mathrm{proj}_0 T"] \arrow[d, phantom, "*"] \\
& & \\
& & AT
\end{tikzcd}$$

From (1.2), the top diagram here is known to commute; the "triangle" ** obviously commutes; the outer diagram commutes by the unit law for t'. (See Lemma 1.8 in [3] for a proof of the symmetric statement about t''.) To conclude that * commutes, we thus need that $1\times\eta_1$ is epic. But, by (i), $1T$ is terminal, and thus $\eta_1\colon 1\to 1T$ is an isomorphism, so $1\times\eta$ is an isomorphism. The proof of (i) $\Rightarrow$ (iii) is similar. To prove (ii) $\Rightarrow$ (iv) (or (iii) $\Rightarrow$ (iv)): let $f\colon A\times B\to C$ be linear. This means

$$\varkappa\cdot\alpha\times\beta\cdot f = fT\cdot\gamma\,.$$

If we multiply this on the left by ψ, we get by assumption (ii)

$$\alpha\times\beta\cdot f = \psi\cdot fT\cdot\gamma\,,$$

saying that f is ψ-bilinear, thus bilinear. (To prove (iii) $\Rightarrow$ (iv), replace ψ by $\tilde\psi$ and conclude that f is $\tilde\psi$-bilinear, thus bilinear.)

We shall now prove (iv) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i). If (iv) holds, then $\mathrm{id}\colon AT\times BT\to AT\times BT$ is bilinear with respect to the canonical structures μ_A, μ_B, $\varkappa_{A,B}\cdot\mu_{A\times B}$. This means

$$\psi_{AT,BT}\cdot\varkappa_{AT,BT}\cdot\mu_A\times\mu_B = \mu_A\times\mu_B\,.$$

Multiplying this equation on the left by $\eta_A T\times\eta_B T$ and using naturality of ψ and $\varkappa$ gives

$$\psi_{A,B}\cdot\varkappa_{A,B}\cdot(\eta_A T\times\eta_B T)\cdot(\mu_A\times\mu_B) = (\eta_A T\times\eta_B T)\cdot(\mu_A\times\mu_B)\,,$$

whence (ii) follows from monad laws. Finally assume (ii). Since $\psi\cdot\varkappa=\mathrm{id}$,

$$\psi_{1,1}\colon 1T\times 1T\to(1\times 1)T$$

is monic. Since $(\mathrm{proj}_0)T\colon(1\times 1)T\to 1T$ is an isomorphism, we conclude that $\psi\cdot\mathrm{proj}_0 T$ is monic. But

$$\psi_{1,1}\cdot\mathrm{proj}_0 T = \psi_{1,1}\cdot\varkappa\cdot\mathrm{proj}_0 = \mathrm{proj}_0\colon 1T\times 1T \to 1T\,,$$

so $\mathrm{proj}_0\colon 1T\times 1T\to 1T$ is monic. This together with the fact that there always is at least one map $X\to 1T$, namely

$$X \to 1 \xrightarrow{\eta} 1T\,,$$

enables us to conclude that $1T$ is terminal, hence $1\cong 1T$. This proves Theorem 2.1.

The condition (ii) in the following Proposition was pointed out to me by Gavin Wraith; $A=(A,\alpha)$, $B=(B,\beta)$, and $C=(C,\gamma)$ are algebras as usual.

PROPOSITION 2.2. *The following conditions are equivalent*:

(i) *for every* A,B

(2.4) $$1_{(A\times B)T}=\varkappa_{A,B}\cdot\psi_{A,B}\colon (A\times B)T\to AT\times BT\to (A\times B)T;$$

(ii) *for every* C, *the diagram*

(2.5) $$\begin{tikzcd} CT \arrow[r, "\Delta_{CT}"] \arrow[dr, "(\Delta_C)T"'] & CT\times CT \arrow[d, "\psi_C"] \\ & (C\times C)T \end{tikzcd}$$

commutes, Δ *being the diagonal.*

Further, (2.4) *implies that a map*

$$f\colon A\times B\to C$$

which is bilinear is also linear.

PROOF. The last statement can be proved with the same technique as the one used for (ii) $\Rightarrow$ (iv) in the previous theorem. Now assume (2.4) for $A=B=C$, and multiply (2.4) on the left by $(\Delta_C)T$ to get the first equation in

$$(\Delta_C)T = (\Delta_C)T\cdot\varkappa_{C,C}\cdot\psi_{C,C} = \Delta_{CT}\cdot\psi_{C,C};$$

the last equation being obvious from the definition of $\varkappa$ and Δ. Conversely, if (2.5) commutes for $C=A\times B$, we multiply it on the right by T of the map

$$\mathrm{proj}_0\times\mathrm{proj}_1\colon (A\times B)\times(A\times B)\to A\times B$$

("projecting onto first and fourth factor"); the equation (2.4) now is easily derived, using naturality of ψ with respect to proj_0 and proj_1, and obvious properties of proj, $\varkappa$, and Δ.

The proposition also holds, if we replace ψ by $\tilde{\psi}$ everywhere. If the monad is commutative, that is, $\psi = \tilde{\psi}$, then a converse of Proposition 2.2 holds, namely (iv) $\Rightarrow$ (i) in

PROPOSITION 2.3. *Let* T *be a commutative monad. Then the following conditions are equivalent*:

(i) $\varkappa_{A,B} \cdot \psi_{A,B} = 1_{(A\times B)T}$ *for every* A, B,

(ii) $\Delta_{CT} \cdot \psi_C = (\Delta_C)T$ *for every* C;

(iii) *Any map from a binary product of algebras which is bilinear is also linear*;

(iv) ψ *is linear*: $AT \times BT \to (A \times B)T$ *(with respect to the structures* μ_A, μ_B, *and* $\mu_{A\times B}$).

PROOF. We have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) by the preceding Proposition. By Proposition 1.5 and commutativity of T, the map ψ is bilinear, so if (iii) holds, ψ is linear as well, proving (iv). Finally assume that (iv) holds; this means that

$$\varkappa_{AT,BT} \cdot \mu \times \mu \cdot \psi = \psi T \cdot \mu \,. \tag{2.6}$$

Multiply this equation on the left by $(\eta_A \times \eta_B)T$. The left hand side gives $\varkappa \cdot \psi$ by naturality of $\varkappa$ and monad laws. Theorem 3.2 in [3] gives that η is a monoidal transformation. So the right hand side of (2.6) above yields, by left multiplication by $(\eta_A \times \eta_B)T$,

$$(\eta_A \times \eta_B)T \cdot \psi T \cdot \mu = (\eta_{A\times B})T \cdot \mu = 1 \,.$$

This proves that $\varkappa \cdot \psi = 1$, which is (i). Proposition 2.3 is proved.

REMARK 2.4. To prove η monoidal one need not use the commutativity of T (as can be seen by inspecting the proof of Theorem 3.2 of [3]), so that, without assuming commutativity of T, we can prove that "$\psi_{A,B}$ linear" implies that (2.4) holds.

THEOREM 2.5. *Let* T *be a strong monad on a cartesian closed category, with associated monoidal structures* ψ *and* $\tilde{\psi}$. *Then the following conditions are equivalent*:

(i) T *is commutative and a map* $f\colon A \times B \to C$ *is linear if and only if it is bilinear*;

(ii) $\varkappa_{A,B}$ *and* $\psi_{A,B}$ *are inverse to each other for all* A, B;

(iii) $\varkappa_{A,B}$ *and* $\tilde{\psi}_{A,B}$ *are inverse to each other for all* A, B;

(iv) *the functor part* T *of* T *commutes with finite products.*

PROOF. Assume (i). By Theorem 2.1 and Proposition 2.3, $\psi_{A,B} = \tilde{\psi}_{A,B}$ is a two-sided inverse for $\varkappa_{A,B}$, proving (ii) and (iii). If (ii) or (iii) holds, $\varkappa_{A,B}$ is always an isomorphism, whence T commutes with binary products. Also, using Theorem 2.1 again, $1T \cong 1$, whence T commutes with empty products, thus with all finite products, that is, (iv) holds. Finally, if (iv) holds, then $1T \cong 1$, whence by Theorem 2.1, $\psi_{A,B}$ as well as $\tilde{\psi}_{A,B}$ are one-sided inverses for $\varkappa_{A,B}$, which however is an isomorphism, so that $\psi_{A,B} = \tilde{\psi}_{A,B}$ is a two sided inverse. By $\psi_{A,B} = \tilde{\psi}_{A,B}$, the monad T is commutative, and "linear $\Leftrightarrow$ bilinear" follows from (ii) $\Rightarrow$ (iv) in Theorem 2.1 and (i) $\Rightarrow$ (iii) in Propisition 2.3. So (i) holds, proving Theorem 2.5.

THEOREM 2.6. *Let T, η, μ be a monad on a cartesian closed category $\mathscr{V}$, and assume that T commutes with finite products. Then the monad carries a unique structure of strong monad, and as such, it is a cartesian closed (in particular commutative) monad. If $\mathscr{V}$ has equalizers, the category of algebras for (T, η, μ) is cartesian closed.*

PROOF. To construct a (commutative) strength st on T, η, μ it suffices by [6, Theorem 2.3], to construct a monoidal structure on T, making η and μ monoidal transformations.

We take the monoidal structure ψ to be given by the inverse of

$$\varkappa_{A,B}\colon (A \times B)T \to AT \times BT;$$

the transformation id $\to T$ we take to be the given η, which is easily seen to be monoidal. Also, μ can be seen to be monoidal, by multiplying the desired commutativity on the left and right by suitable composites of the invertible $\varkappa$.

It then follows from Theorem 2.5 that the strong monad $((T, \mathrm{st}), \eta, \mu)$ is in fact cartesian closed. The statement about the algebra category being cartesian closed will be proved in the next section. Finally, we prove the uniqueness of a strength on T, η, μ: if $\mathrm{st}_0, \mathrm{st}_1$ are two strengths, they give rise to the same monoidal structure $\psi\colon AT \times BT \to (A \times B)T$, since this has to be an inverse for the invertible $\varkappa_{A,B}$, by Theorem 2.5 (ii). However, the tensorial strengths $t_0'', t_1''\colon A \times BT \to (A \times B)T$ corresponding to $\mathrm{st}_0, \mathrm{st}_1$ under the one-one correspondence of [6, Theorem 1.3], can be reconstructed from ψ as $\eta_A \times 1 \cdot \psi$; thus $t_0'' = t_1''$, and therefore $\mathrm{st}_0 = \mathrm{st}_1$.

DEFINITION 2.7. A monad, which satisfies the condition of Theorem 2.6 above is called *cartesian closed.*

3. Algebras for cartesian closed monads.

Recall that if T is a strong commutative monad on a symmetric monoidal closed category $\mathscr{V}$ with equalizers, then, by [4], the category of algebras for T can be made into a closed category. In fact, if $A=(A,\alpha)$ and $B=(B,\beta)$ are algebras, the sub-object $A \pitchfork\!\!\pitchfork B$ of $A \pitchfork B$ which equalizes $\alpha \pitchfork 1$ and $\mathrm{st}\cdot 1 \pitchfork \beta$ (where st is the strength of T) can be made into a T-algebra, in fact, a subalgebra of $A \pitchfork B$ equipped with the structure $\lambda_{A,B}\cdot 1 \pitchfork \beta$. The main result of [4, Theorem 2.1] states, then, that $\pitchfork\!\!\pitchfork$ makes the category $\mathscr{V}_0^{\mathsf{T}}$ of T-algebras into a closed category.

For the particular case of a cartesian closed monad (Definition 2.7), $\pitchfork\!\!\pitchfork$ will turn the closed category of algebras $\mathscr{V}_0^{\mathsf{T}}$ into a cartesian closed category, as we shall now prove.

Denote the underlying - functor $\mathscr{V}_0^{\mathsf{T}} \to \mathscr{V}_0$ by U. If (B,β) is an algebra (by abuse denoted B), then

$$-\times B\colon \mathscr{V}_0^{\mathsf{T}} \to \mathscr{V}_0^{\mathsf{T}}$$

is a functor; we propose to show that it has the functor

$$B \pitchfork\!\!\pitchfork -\colon \mathscr{V}_0^{\mathsf{T}} \to \mathscr{V}_0^{\mathsf{T}} \tag{3.1}$$

as a right adjoint. (The fact that (3.1) is indeed a functor with $\mathscr{V}_0^{\mathsf{T}}$ as codomain follows from the commutativity of T together with the results of [4]).

For (A,α) another algebra, we have maps in $\mathscr{V}_0$,

$$\overline{\mathrm{ev}}_A\colon (B \pitchfork\!\!\pitchfork A)\times B \xrightarrow{e\times 1} (B \pitchfork A)\times B \xrightarrow{\mathrm{ev}} A\,,$$

e being the equalizer defining $B \pitchfork\!\!\pitchfork A$. This family is natural in (A,α).

Lemma 3.1. *Each $\overline{\mathrm{ev}}_A$ is a homomorphism.*

Proof. By condition (i) of Theorem 2.5, we just have to prove $\overline{\mathrm{ev}}_A$ bilinear. Since $\mathrm{ev}\colon (B \pitchfork A)\times B \to A$ is the transpose of $1\colon B \pitchfork A \to B \pitchfork A$, the map $\overline{\mathrm{ev}}_A$ is the transpose of $e\colon B \pitchfork\!\!\pitchfork A \to B \pitchfork A$. However, e is a homomorphism by construction of structure on $B \pitchfork\!\!\pitchfork A$, and it factors through $e\colon B \pitchfork\!\!\pitchfork A \to B \pitchfork A$, whence Proposition 1.3 gives the bilinearity of $\overline{\mathrm{ev}}_A$. This proves the lemma.

Consider the transpose $u\colon A \to B \pitchfork (A\times B)$ of $1_{A\times B}$. Since $1_{A\times B}$ is linear, it is bilinear by (i) of Theorem 2.5, whence by Proposition 1.3, u is a homomorphism and factors across $B \pitchfork\!\!\pitchfork (A\times B)$ as $\bar{u}_A$,

$$\begin{tikzcd}
A \arrow[d, "\bar{u}_A"'] \arrow[dr, "u"] & \\
B \pitchfork\!\!\pitchfork (A\times B) \arrow[r, "e"'] & B \pitchfork (A\times B)\,.
\end{tikzcd}$$

By commutativity of the monad, e can be made into a homomorphism; the fact that u is a homomorphism now implies that $\bar{u}_A$ also is. Again, $\bar{u}_A$ is natural in (A,α). We thus have homomorphism, natural in (A,α):

$$\overline{\mathrm{ev}}_A\colon (B \pitchfork A) \times B \to A, \quad \bar{u}_A\colon A \to B \pitchfork (A \times B) .$$

The fact that these satisfy the equations making $- \times B \dashv B \pitchfork -$ now easily follows from the fact that $U\colon \mathscr{V}_0{}^{\mathsf{T}} \to \mathscr{V}_0$ is faithful, and from the corresponding equations between ev and u, whence

THEOREM 3.2. *If* T *is a cartesian closed monad on a cartesian closed category with equalizers, the category of algebras for* T *is again a cartesian closed category.*

REMARK. The closed structure on the algebra category mentioned in the theorem is (as the proof above shows) the one existing by [4] because of commutativity of T. The fact that a commutative monad whose functor part commutes with products has a cartesian closed category of algebras was originally conjectured by Lawvere and Tierney.

AN EXAMPLE. Let $\overline{PO}$ denote the category of partially ordered sets. To a partially ordered set A, let $(A)T$ denote the set of "directed non-empty filters" of A, that is, the set of such nonempty subsets $X \subseteq A$ so that

(i) $(a \leq x) \wedge (x \in X) \Rightarrow a \in X$,
(ii) $(x_0 \in X) \wedge (x_1 \in X) \Rightarrow (\exists x_2 \in X)(x_0 \leq x_2) \wedge (x_1 \leq x_2)$;

$(A)T$ is partially ordered by inclusion. Then clearly, T is a functor $\overline{PO} \to \overline{PO}$; for an order preserving map f, fT takes a directed nonempty filter into the filter generated by its set-theoretic direct image.

For $a \in A$, let $(a)\eta_A \in (A)T$ be $\{x \in A \mid x \leq a\}$; for $Z \in (A)TT$, let $(Z)\mu_A \in (A)T$ be the set-theoretic union. These data together form a monad on $\overline{PO}$.

To see that it is a cartesian closed monad, we just have to prove that T commutes with finite products. By the nonemptyness condition, $1T \cong 1$. By Theorem 2.1, then,

$$\varkappa\colon (A \times B)T \to AT \times BT$$

is split epi in $\overline{PO}$; so it is an isomorphism if we can show that it is 1-1 as a set mapping. Let X, X' be directed nonempty filters with $(X)\varkappa = (X')\varkappa$. Assume $\langle a,b \rangle \in X$. Then $a \in (X)\varkappa_0$, $b \in (X)\varkappa_1$, hence by assumption $a \in (X')\varkappa_0$, $b \in (X')\varkappa_1$, meaning that there exist a', b', so that

$$\langle a,b' \rangle \in X', \quad \langle a',b \rangle \in X' .$$

By condition (ii), there exists a common upper bound $\langle a'',b''\rangle$ for them in X', that is

$$a \leqq a'', \quad b' \leqq b'',$$

and

$$a' \leqq a'', \quad b \leqq b''.$$

But then $\langle a,b\rangle \leqq \langle a'',b''\rangle$, whence also $\langle a,b\rangle \in X'$. So $X \subseteqq X'$; similarly, $X' \subseteqq X$.

The category of algebras for the monad here is the category of partially ordered sets such that every directed nonempty filter has a least upper bound (and the maps are those that preserve this bound).

REFERENCES

1. J. Beck, *On H-spaces and infinite loop spaces*, Catory Theory, Homology Theory and their Applications III (Lecture Notes in Mathematics 99), Springer-Verlag, Berlin · Heidelberg · New York, 1969, 139–153.
2. S. Eilenberg and M. G. Kelly, *Closed categories*, Proc. Conf. Categorical Algebra (La Jolla 1965). Springer-Verlag, 1966, 421–562.
3. A. Kock, *Monads on symmetric monoidal closed categories*, Arch. Math. (Basel) 21 (1970), 1–10.
4. A. Kock, *Closed categories generated by commutative monads*, J. Austral. Math. Soc., 12 (1971), 405–424.
5. A. Kock, *On double dualization monads*, Math. Scand. 27 (1970), 151–165.
6. A. Kock, *Strong functors and monoidal monads*, Aarhus Universitet, Various Publications Series No. 11, August 1970. (Mimeographed.)
7. F. E. J. Linton, *Coequalizers in categories of algebras*, Seminar on Triples and Categorical Homology Theory (Lecture Notes in Mathematics 80), Springer-Verlag, Berlin · Heidelberg · New York, 1969, 75–90.

DALHOUSIE UNIVERSITY, HALIFAX, CANADA,
AND
AARHUS UNIVERSITET, DENMARK