$R(X)$ AS A DIRICHLET ALGEBRA AND REPRESENTATION OF ORTHOGONAL MEASURES BY DIFFERENTIALS
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1. Introduction.

Let $X$ be a compact subset of the complex plane $\mathbb{C}$, and let $R(X)$ denote the uniform closure on $X$ of the rational functions with poles off $X$. Let $A(X)$ be the continuous functions on $X$ which are analytic in the interior $X^\circ$ of $X$. In this paper we will always assume that $\mathbb{C} \setminus X^\circ$ is connected. We say that $R(X)$ is a Dirichlet algebra if $\text{Re}(R(X))$, the real parts of the functions in $R(X)$, are uniformly dense on the boundary $bX$ of $X$ in the real continuous functions on $bX$. This occurs if and only if there are no non-zero real measures on $bX$ orthogonal to $R(X)$. If $R(X)$ is a Dirichlet algebra, then $\mathbb{C} \setminus X^\circ$ is connected [6, section 4]. For information on Dirichlet algebras, see [17], [9] and [6].

In this paper we treat a problem, raised by Bishop in [2] and [3]: When can every measure $\mu$ on $bX$ which is orthogonal to $R(X)$ be represented by its analytic differential $(2\pi i)^{-1} \hat{\mu}(z)dz$, where

$$\hat{\mu}(z) = \int (\zeta - z)^{-1} d\mu(\zeta) \quad \text{for } z \in X^\circ?$$

(See definitions below.) In section 2 we give a necessary and sufficient condition for this to be true, theorem 2.2, and then we use this result to prove that if $R(X)$ is a Dirichlet algebra, then every orthogonal measure $\mu$ on $bX$ is represented by its differential $(2\pi i)^{-1} \hat{\mu}(z)dz$ (theorem 2.4). It is an interesting question whether the converse of this is also true.

In section 3 we prove three results on when $R(X)$ is a Dirichlet algebra. The first states, roughly speaking, that if $R(X)$ "locally" is a Dirichlet algebra, then $R(X)$ is a Dirichlet algebra. Using the same technique we then prove that if $R(bX) = C(bX)$ and every bounded analytic function on $X^\circ$ can be approximated pointwise by a bounded sequence in $R(X)$, then $R(X)$ is a Dirichlet algebra. The same proof, without claiming $R(bX) = C(bX)$, works for $A(X)$. The third result states that if $X$ is an

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intersection of compact sets $X_n$ such that $R(X_n)$ is a Dirichlet algebra, then $R(X)$ is a Dirichlet algebra. These results are contained in recent work done by Gamelin, Garnett and Davie (see [7], [10] and [11]) but our proofs are entirely different.

At the end of section 3 we combine these results with theorem 2.4 to obtain a generalization of a result of Bishop in [3].

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2. Representation of orthogonal measures by differentials.

2.1. Definitions. (See [2] and [3].) A sequence $\Gamma = \{\Gamma_n\}$ of compact subsets of $X^\circ$ is said to converge to bX provided:

(i) Each $\Gamma_n$ is the union of a finite number of disjoint piecewise smooth simple closed curves lying in $X^\circ$, no two of which belong to the same component of $X^\circ$, and

(ii) If $S$ is any compact subset of $X^\circ$, then for all $n$ sufficiently large, $S$ will lie in the union of the bounded components of $C \setminus \Gamma_n$.

If $\mu$ is a measure on bX and $dw = f(z)dz$ is an analytic differential in $X^\circ$, we say that $dw$ represents $\mu$ (with respect to $\Gamma$) if there exists a sequence $\Gamma = \{\Gamma_n\}$ converging to bX such that for all continuous functions $h$ on X, we have

$$\int_{bX} h(t) d\mu(t) = \lim_n \int_{\Gamma_n} h(z) f(z) \, dz .$$

Let $\{a_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be fixed sequences of non-negative numbers such that

(i) $a_1 = 0$, $r_1 = 1$,

(ii) $a_n \uparrow a < \infty$,

(iii) $r_n \downarrow 0$,

(iv) $a_{n+1} - r_{n+1} > a_n + r_n$, $n = 1, 2, \ldots$.

Put $\Delta_n = \{z; |z - a_n| < r_n\}$. Let $V_1, V_2, \ldots$ denote the components of $X^\circ$, and let $m$ denote the number of such components (which may be countably infinite). Then we define $K = K_X$ to be the closure of $\bigcup_{n=1}^{m} \Delta_n$. We let $\Phi$ denote a fixed conformal mapping from $X^\circ$ onto $K^\circ$ and let $\Psi: K^\circ \to X^\circ$ be its inverse. By a theorem of Fatou (see [15]) $\Psi$ has
non-tangential boundary values $\Psi^*\ a.e.\ on\ bK$ with respect to the measure $d\theta = \sum_n 2^{-n} d\theta_n$, where $d\theta_n$ denotes the normalized Lebesgue measure on $bA_n$.

If $\nu$ is a measure on $bK$ orthogonal to $R(K)$, that is, $\nu \in R(K)^4$, then by the F. and M. Riesz theorem $\nu \ll d\theta$, and so we can define $\Psi^*(\nu)$ as the measure on $bX$ whose value on the Borel set $E$ is $\nu(\Psi^*E)$.

With these notations we get the following characterisation of measures $\mu$ on $bX$ that is represented by its differential:

2.2. Theorem. Let $\mu$ be a measure on $bX$. The following are equivalent:

(i) $\mu$ is represented by its analytic differential $(2\pi i)^{-1} \hat{\mu}(z)dz$.

(ii) There exists a measure $\nu \in R(K)^4$ carried on $bK$ with $\mu = \Psi^*(\nu)$.

Proof. If there exists a sequence $\Gamma = \{\Gamma_n\}$ converging to $bX$ such that $(2\pi i)^{-1} \hat{\mu}(z)dz$ represents $\mu$ with respect to $\Gamma$, then we can find a subsequence $\{\Gamma_{n_k}\}$ such that $\delta = \{\delta_k\} = \{\Phi(\Gamma_{n_k})\}$ converges to $bK$, and the differential

$$\Phi((2\pi i)^{-1} \hat{\mu}(z)dz) = (2\pi i)^{-1} \hat{\mu}(\Psi(z)) \Psi'(z) dz$$

represents a measure $\nu$ on $bK$ with respect to $\delta$. The proof of this is word for word the same as the proof of theorem 2 in [2]. As remarked on page 283 in [2] we then have

$$\hat{\mu}(\Psi(z)) \Psi'(z) = \hat{\nu}(z) \quad for \ z \in K^o.$$ 

Using Cauchy's theorem we get $\nu \in R(K)^4$.

Let $\sigma = \Psi^*(\nu)$. We will prove that $\sigma = \mu$. If $z \in K^o$ we have

$$\hat{\sigma}(\Psi(z)) = \int \frac{d\sigma(s)}{s - \Psi(z)} = \int \frac{d\nu(t)}{\Psi^*(t) - \Psi(z)}$$

and

$$\hat{\nu}(\Psi(z)) = \frac{1}{\Psi'(z)} \hat{\nu}(z) = \frac{1}{\Psi'(z)} \int \frac{d\nu(t)}{t - z}.$$ 

Now since

$$G(t) = \frac{1}{\Psi(t) - \Psi(z)} - \frac{1}{\Psi'(z)} = \frac{1}{t - z}$$

is bounded and analytic in $K^o$, we have $\int G^*(t) d\nu(t) = 0$, where $G^*(t)$ denotes the non-tangential boundary values of $G$, such that $\hat{\mu}(w) = \hat{\sigma}(w)$ for all $w \in X^o$. Since both $\mu$ and $\sigma$ are orthogonal to $A(X)$, lemma 1.1 in [7] gives $\sigma = \mu$.

Suppose $\mu = \Psi^*(\nu)$, where $\nu \in R(K)^4$ is a measure on $bK$. Then, essen-
tially by the F. and M. Riesz theorem (see [2, p. 283]), there exists a 
sequence \( \delta = \{ \delta_n \} \) converging to \( bK \) such that \( (2\pi i)^{-1} \hat{v}(z)dz \repr{v} \) with respect to \( \delta \). Again, by the proof of theorem 2 in [2], there exists a subsequence \( \{ \delta_{n_k} \} \) such that \( \Gamma = \{ \Gamma_k \} = \{ \Psi(\delta_{n_k}) \} \) converges to \( bX \) and the differential 
\[
\Psi((2\pi i)^{-1} \hat{v}(z)dz) = (2\pi i)^{-1} \hat{v}(\Phi(z)) \Phi'(z)dz
\]
represents a measure \( \sigma \) on \( bX \) with respect to \( \Gamma \). By the remark p. 283 in [2], we have \( \hat{\sigma}(z) = \hat{v}(\Phi(z)) \Phi'(z) \), so it is enough to prove that \( \sigma = \mu \). 
Since both \( \sigma \) and \( \mu \) are orthogonal to \( A(X) \), it is enough to prove, by lemma 1.1 in [7], that \( \hat{\mu}(w) = \hat{\sigma}(w) \) for all \( w \in X^\circ \). Letting \( z = \Phi(w) \) we have 
\[
\hat{\sigma}(w) = \Phi'(w) \hat{v}(\Phi(w)) = \frac{1}{\Psi'(z)} \hat{v}(z) = \frac{1}{\Psi'(z)} \int \frac{d\nu(t)}{t-z}
\]
and 
\[
\hat{\mu}(w) = \int \frac{d\mu(s)}{s-w} = \int \frac{d\nu(t)}{\Psi^*(t) - \Psi(z)}.
\]
Hence, since \( G(t) \) is bounded and analytic in \( K^\circ \), the result follows.

If \( z \in X^\circ \), let \( \lambda_z \) be the harmonic measure for \( z \) with respect to \( X^\circ \), and if \( z \in bX \), let \( \lambda_z \) be the pointmass at \( z \). Choose points \( z_n \in V_n \), \( n = 1, 2, \ldots \), (the components of \( X^\circ \)) and put \( \lambda_n = \lambda_{z_n} \) and \( \lambda = \sum_{n=1}^m 2^{-n} \lambda_n \), where \( m \) is, as before, the number of components of \( X^\circ \). If \( f \in L^1(\lambda) \) we let \( \bar{f} \) be the harmonic extension of \( f \) (see [14, ch. 8] for properties of harmonic measures). If \( \sigma \) is a positive measure on \( bX \) with \( \lambda \ll \sigma \) we define \( H^\infty(\sigma) \) to be the weak* closure of \( R(X) \) in \( L^\infty(\sigma) \). We denote by \( H^\infty(X^\circ) \) the set of bounded analytic functions on \( X^\circ \). The map \( S_\sigma: H^\infty(\sigma) \to H^\infty(X^\circ) \) given by \( S_\sigma(f) = \bar{f} \) is a continuous homomorphism of the algebra \( H^\infty(\sigma) \) into the algebra \( H^\infty(X^\circ) \). (See [9, p. 226].) Moreover, we have:

2.3. Lemma. If \( R(X) \) is a Dirichlet algebra, then \( S_\lambda \) is an isometric isomorphism of \( H^\infty(\lambda) \) onto \( H^\infty(X^\circ) \).

Proof. This follows from lemma 2.1 of [7] and theorem VIII.11.1 of [9].

2.4. Theorem. Suppose \( R(X) \) is a Dirichlet algebra. Then every measure \( \mu \in R(X)^\perp \) carried on \( bX \) is represented by its differential \( (2\pi i)^{-1} \hat{\mu}(z)dz \).

Proof. Let \( \mu \) be a measure on \( bX \) orthogonal to \( R(X) \). By theorem
2.2 it is enough to prove that there exists a measure $\nu$ on $bK$ orthogonal to $R(K)$ such that $\mu = \Psi^*(\nu)$. Since $\Phi \in H^\infty(X^\circ)$ there exists, by lemma 2.3, a function $\Phi^* \in H^\infty(\lambda)$ such that $\Phi^* = \Phi$. Let $z \in X^\circ$ and put $x = \Phi(z) \in K^\circ$. Let $\rho_x$ denote the harmonic measure (the Poisson measure) for $x$ with respect to $\Lambda_n$, where $x \in \Delta_n$. Then it is easy to see that $\lambda_x = \Psi^*(\rho_x)$. This gives
\[ x = \Phi(z) = \int \Phi^* d\lambda_x = \int \Phi^* \circ \Psi^* d\rho_x = \int t \, d\rho_x(t) , \]
and hence $\Phi^*(\Psi^*(t)) = t$ a.e. $d\theta$. But then $\Phi^*(\Psi^*(\rho_x)) = \rho_x$ or
\[ \Phi^*(\lambda_x) = \rho_x . \]
This implies that
\[ \int \Psi^*(\Phi^*(t)) \, d\lambda_x(t) = \int \Psi^*(t) \, d\rho_x(t) = \Psi(x) = z , \]
and so $\Psi^*(\Phi^*(t)) = t$ a.e. $\lambda$. Since $R(X)$ is a Dirichlet algebra, we have $|\mu| \leq \lambda$, by the Wermer–Glicksberg theorem [17, Satz 3] and the fact that $R(X)$ has no non-zero completely singular orthogonal measures [18]. Hence $\Psi^*(\Phi^*(t)) = t$ a.e. $\mu$. If we define $\nu = \Phi^*(\mu)$, we have
\[ \int t^k \, d\nu(t) = \int (\Phi^*)^k(\zeta) \, d\mu(\zeta) = 0 , \]
since $\mu \leq \lambda$ and $\Phi^* \in H^\infty(\lambda)$. Therefore $\nu$ is orthogonal to the polynomials, and so $\nu \in R(K)^\perp$ since $C \setminus K$ is connected. Since $\nu$ is carried on $bK$ and $\mu = \Psi^*(\Phi^*(\mu)) = \Psi^*(\nu)$, the result follows.

3. Conditions under which $R(X)$ is a Dirichlet algebra.

We now turn to the question: When is $R(X)$ a Dirichlet algebra? We will not try to give a complete discussion of this problem, but concentrate on three special results. We need the following lemma about splitting of orthogonal measures, due to Bishop (see [1, lemma 6]):

3.1. LEMMA. Let $F$ be compact and $\mu \in R(F)^\perp$. Then for almost all $x_0 \in R$ with respect to Lebesgue measure there exists a measure $\beta$ on $L = \{z \in F; \Re z = x_0\}$ such that
\[ \int_{F_1} h \, d\mu = - \int_{\overline{F_2}} h \, d\mu = \int_L h \, d\beta \quad \text{for all } h \in R(F) , \]
where $F_1 = \{z \in F; \Re z \leq x_0\}$, $F_2 = \{z \in F; x_0 \leq \Re z\}$. Further $|\mu|(L) = 0$, and if we define
\[ \mu_1 = \mu / F_1 - \beta, \quad \mu_2 = \mu / F_2 + \beta, \]

then \( \mu = \mu_1 + \mu_2 \) and \( \mu_i \in R(\mathbb{F}^1_i), \ i = 1, 2. \)

We do not give the proof of lemma 3.1, since the proof of lemma 6 in [1] applies with only minor changes.

If \( S \) is a compact plane set, \( F \) is a closed subset of \( S \), and \( B \) is a closed subspace of \( C(S) \) including constants and separating points on \( S \), we say that \( F \) is an interpolation set for \( B \) if \( B / F = C(F) \). We say that \( F \) is a peak set for \( B \) if there exists a function \( f \in B \) such that \( f = 1 \) on \( F \), \( |f| < 1 \) off \( F \). A point \( x \in S \) is called a peak point for \( B \) if \( \{x\} \) is a peak set for \( B \). If \( F \) is both an interpolation set and a peak set for \( B \), we call \( F \) a peak interpolation set for \( B \). We mention the following result, known as the (Bishop–) Rudin–Carleson theorem: If \( B \) is an algebra, then \( F \) is a peak interpolation set for \( B \) if and only if every measure orthogonal to \( B \) vanishes on \( F \). (See [4], [8, p. 284] and [13, p. 429].)

3.2. Lemma. Let \( S \) be a compact set and let \( F \) be a closed subset of \( bS \). Then \( F \) is a peak interpolation set for \( R(S) / bS \) if and only if \( F \) is a peak interpolation set for \( R(S) \).

Proof. Suppose \( F \) is a peak interpolation set for \( R(S) / bS \). Choose \( f \in R(S) \) such that \( f = 1 \) on \( F \) and \( |f| < 1 \) on \( bS \setminus F \). Suppose there exists \( x_0 \in S^\circ \) such that \( |f(x_0)| = 1 \). By the maximum modulus principle we then have \( f = 1 \) on the closure of the component \( V \) of \( S^\circ \) which contains \( x_0 \). Therefore \( bV \subset F \). But since \( R(S) / F = C(F) \) and \( h(z) = (z - x_0)^{-1} \in C(F) \) there exists \( g \in R(S) \) such that \( g(z) = (z - x_0)^{-1} \) or \( g(z) (z - x_0) = 1 \) on \( bV \) and so

\[ g(z)(z - x_0) = 1 \quad \text{for all} \quad z \in V, \]

which is impossible for \( z = x_0 \). This contradiction proves that \( F \) is a peak set for \( R(S) \), and we are done.

3.3. Theorem. Suppose that for all \( x \in X \) there exists an open neighbourhood \( W \) of \( x \) such that \( R(\overline{W} \cap X) \) is a Dirichlet algebra. Then \( R(X) \) is a Dirichlet algebra.

Proof. Choose open sets \( W_1, \ldots, W_n \) such that \( R(\overline{W}_i \cap X) \) is a Dirichlet algebra, \( 1 \leq i \leq n \), and \( X \subset \bigcup_{i=1}^n W_i \). Choose \( \delta > 0 \) such that for every closed rectangle \( R \) of diameter less than \( \delta \) there exists \( W_i \) with \( R \cap X \subset W_i \). Let \( \mu \) be a measure on \( bX \) which is orthogonal to \( R(X) \). Using lemma 3.1 a finite number of times we can write
\[ \mu = \sum_{i=1}^{N} \mu_i, \quad X = \bigcup_{i=1}^{N} X_i, \]

where \( X_i = X \cap R_i \), \( R_i \) is a closed rectangle of diameter less that \( \delta \), and \( \mu_i \) is a measure on \( bX = (R_i \cap bX) \cup (X \cap bR_i) \)

orthogonal to \( R(X_i) \). Moreover, lemma 3.1 tells us that

\[ \mu_i/R_i \cap bX = \mu/R_i \cap bX. \]

Let \( i \) be fixed, \( 1 \leq i \leq N \). Choose \( W = W_j \) such that \( X_i \subseteq W \) and put \( Y = Y_i = \overline{W} \cap X \). For \( x \in Y^\circ \) we let \( \lambda_x \) denote the harmonic measure for \( x \) with respect to \( Y^\circ \). Let \( S_1, S_2, \ldots \) be the components of \( Y^\circ \), choose \( y_i \in S_i \) for all \( i \), and define \( \lambda = \sum_i 2^{-i} \lambda_{y_i} \). Let \( K = K_X \), and \( \Phi, \Psi, \Psi^* \) be as in 2.1.

The idea of the proof is to construct a function \( \Phi^* \in L^\infty(\mu) \) which satisfies:

(i) \( \Phi^*(t) \in bK \) for a.a. \( t \) with respect to \( \mu \),

(ii) \( \int (\Phi^*)^k d\mu = 0 \) for \( k = 0, 1, 2, \ldots \),

(iii) \( \Psi^*(\Phi^*(t)) = t \) a.e. \( \mu \).

When (i), (ii) and (iii) are established, it is easy to prove that if \( \mu \) is real, \( \mu \) must be the zero measure. For if \( \mu \) is real, so is the measure \( \nu = \Phi^*(\mu) \). \( \nu \) is carried on \( bK \), by (i), and it is orthogonal to the polynomials, by (ii).

It is well known and easy to see that \( R(K) \) is a Dirichlet algebra. Hence \( \nu = 0 \), and (iii) gives that

\[ \mu = \Psi^*(\Phi^*(\mu)) = \Psi^*(\nu) = 0. \]

To construct such a function \( \Phi^* \) we first prove this local result: There exists a function \( \Phi_{x^*} \in H^\infty(\lambda) \) such that

(I) \( \Phi_{x^*}(t) \in bK \) for a.a. \( t \) on \( bX_i \cap bY \) with respect to \( \mu_i \)

(II) \( \int_{bX_i \cap bY} (\Phi_{x^*})^k d\mu_i = -\int_{bX_i \cap Y^\circ} \Phi^k d\mu_i, \quad k = 0, 1, 2, \ldots \)

(III) \( \Psi^*(\Phi_{x^*}(t)) = t \) a.e. \( \mu_i \) on \( bX_i \cap bY \).

When (I), (II) and (III) are established, the construction of \( \Phi^* \) goes as follows: For each \( i \) between 1 and \( N \) we construct one such function \( \Phi_{x^*} = \Phi_{y^*} \), and since \( \mu \) has no mass on \( \bigcup_{i=1}^{N} bR_i \), by lemma 3.1, we can define \( \Phi^* \in L^\infty(\mu) \) by

\[ \Phi^*(t) = \Phi_{y^*}(t) \quad \text{for} \quad t \in R_i \cap bX = bX_i \cap bY_i. \]
Then of course \( \Phi^*(t) \in bK \) a.e. \( \mu \), and since \( \mu = \sum_{i=1}^{N} \mu_i \), (II) gives that

\[
\int_{bX} (\Phi^*)^k \, d\mu = \sum_{i} \int_{R_i \cap bX} (\Phi^*)^k \, d\mu_i \\
= \sum_{i} \int_{bX_i \cap bY_i} (\Phi^*_{i})^k \, d\mu_i \\
= - \sum_{i} \int_{bX_i \cap X^*} \Phi^k \, d\mu_i = - \int_{\bar{X}^*} \Phi^k \, d\mu = 0, \quad k = 0, 1, 2, \ldots.
\]

Here we have used the fact that \( bX_i \cap Y^* = bX_i \cap X^* \) and that \( \mu \) is a measure on \( bX \). Moreover, since (III) is valid, it is easy to see that \( \Phi^* \) also satisfies (iii).

It remains to prove the existence of \( \Phi^*_Y \).

Since \( \Phi/Y^* \in H^\infty (Y^*) \) and \( R(Y) \) is a Dirichlet algebra, there exists by lemma 2.3 a function \( \Phi^*_Y \in H^\infty (\lambda) \) such that

\[
\hat{\Phi}^*_Y (z) = \Phi (z) \quad \text{for all } z \in Y^*.
\]

Since \( \lambda_z \) is a multiplicative measure with respect to \( H^\infty (\lambda) \), we get

\[
\int (\Phi^*_Y)^k \, d\lambda_z = \Phi^k (z) \quad \text{for all } z \in Y^*.
\]

Let \( E \subset bX_i \cap bY \) be closed and suppose \( \lambda (E) = 0 \). Since \( R(Y) \) is a Dirichlet algebra, all measures on \( bY \) orthogonal to \( R(Y) \) vanishes on \( E \), by the Wermer–Glicksberg theorem ([17]) and the fact that \( R(Y) \) has no nonzero completely singular orthogonal measures. Hence by the Rudin–Carleson theorem \( E \) is a peak interpolation set for \( R(Y)/bY \) and by lemma 3.2 \( E \) is a peak interpolation set for \( R(Y) \). Again by the Rudin–Carleson theorem we get that \( |\mu_i| (E) = 0 \), since

\[
\mu_i \in R(X_i)^{1} \subset R(Y)^{1}.
\]

Hence

(1) \[
\mu_i/bX_i \cap bY \ll \lambda/bX_i \cap bY.
\]

Define the measure \( \sigma_i \) on \( bY \) to be the sweep of \( \mu_i \) to \( bY \), that is

\[
\int f \, d\sigma_i = \int \hat{f} \, d\mu_i \quad \text{for } f \in C(bY),
\]

where as before \( \hat{f} \) denotes the harmonic extension of \( f \) in \( Y^* \), and we set \( \hat{f} (x) = f (x) \) for \( x \in bY \). Then \( \sigma_i \in R(Y)^{1} \), \( \sigma_i \ll \lambda \) and therefore
(2) \[ (\Phi_Y^*)^k \, d\sigma_i = 0, \quad k = 0, 1, 2, \ldots, \]

since \( \Phi_Y^* \in H^\infty(\lambda) \). Choose a sequence \( \{f_n\}_{n=1}^\infty \subset C(bY) \) such that \( \|f_n\|_\infty \leq \|\Phi_Y^*\|_\infty \) and \( f_n(t) \to \Phi_Y^*(t) \) a.e. \( \lambda \). Then by the dominated convergence
\[ (f_n^k)^*(z) \to \Phi^k(z) \text{ for } z \in Y^c, \]

\( k = 0, 1, 2, \ldots, \) and (1) and (2) gives
\[
0 = \int_{bY} (\Phi_Y^*)^k \, d\sigma_i = \lim_n \int_{bY} f_n^k \, d\sigma_i \\
= \lim_n \int_{bX_i} (f_n^k)^* \, d\mu_i \\
= \lim_n \int_{bX_i \cap Y^c} (f_n^k)^* \, d\mu_i + \lim_n \int_{bX_i \cap bY} f_n^k \, d\mu_i \\
= \int_{bX_i \cap Y^c} \Phi^k \, d\mu_i + \int_{bX_i \cap bY} (\Phi_Y^*)^k \, d\mu_i, \quad k = 0, 1, 2, \ldots,
\]

which is (II).

To verify (I) and (III) define \( \Psi_0 : K \to X \) by
\[
\Psi_0(t) = \Psi^*(t) \quad \text{for } t \in bK, \\
= \Psi(t) \quad \text{for } t \in K^c.
\]

We want to show that
\[
\Psi_0(\Phi_Y^*(t)) = t \quad \text{a.e. } \lambda.
\]

Let \( G = \Phi(Y^c) \). We assert that
\[
\Phi_Y^*(t) \in \tilde{G} \quad \text{for a.a. } t \text{ with respect to } \lambda.
\]

Suppose \( a \notin \tilde{G} \). Then \( (\Phi/Y^c-a)^{-1} \in H^\infty(Y^c) \), and by lemma 2.3 there exists \( \beta^* \in H^\infty(\lambda) \) such that
\[
\tilde{\beta}^* = (\Phi/Y^c-a)^{-1} \quad \text{and} \quad \|\beta^*\|_\infty = \|(\Phi(Y^c-a)^{-1}\|.
\]

Since \( \Phi_Y^* - a \in H^\infty(\lambda) \) we then have
\[
\int \beta^*(\Phi_Y^*-a) \, d\lambda_z = (\Phi(z)-a)^{-1}(\Phi(z)-a) = 1
\]

for all \( z \in Y^c \). Hence by injectivity \( \beta^* = (\Phi_Y^*-a)^{-1} \) and so
\[
\|(\Phi_Y^*-a)^{-1}\|_\infty = \|(\Phi/Y^c-a)^{-1}\|.
\]
Since this is valid for all \( a \notin \bar{G} \), we must have
\[
\Phi_Y^*(t) \in \bar{G} \subset K \quad \text{for a.a. } t \text{ with respect to } \lambda.
\]
Now choose \( z \in Y^\circ \) and put \( x = \Phi(z) \in K^\circ \). Since
\[
\int (\Phi_Y^*)^k \, d\lambda_z = \Phi^k(z) = x^k, \quad k = 0, 1, 2, \ldots ,
\]
the measure \( \beta_x = \Phi_Y^*(\lambda_z) \) is a representing measure on \( \bar{G} \subset K \) for \( x \) with respect to the polynomials. Choose \( k \) such that \( x \in \Lambda_k \). Since the function which is 1 on \( \Lambda_k \) and 0 on \( K \setminus \Lambda_k \) belongs to \( R(K) = P(K) \) (this follows, for instance, by lemma 3.1), \( \beta_x \) must be a measure on \( \Lambda_k \). Let \( \epsilon_x \) be the Poisson measure for \( x \) with respect to \( \Lambda_k \). Then \( \epsilon_x - \beta_x \in R(\Lambda_k)^\perp \).

Let \( F \) be a closed subset of \( b\Lambda_k \) such that \( d\theta_k(F) = 0 \). Then, by the F. and M. Riesz theorem, lemma 3.2 and the Rudin–Carleson theorem we get that \( |\epsilon_x - \beta_x|(F) = 0 \). Since \( \epsilon_x(F) = 0 \), \( \beta_x(F) = 0 \) and so \( \beta_x/b\Lambda_k \ll d\theta_k \).

Now let \( \{P_n\} \) be a sequence of polynomials which is uniformly bounded on \( \Lambda_k \) and converges to \( \Psi^* \) a.e. \( d\theta_k \) on \( b\Lambda_k \) [17, lemma 5]. Then \( P_n(z) \to \Psi(z) \) for all \( z \in \Lambda_k \) and we have
\[
\int \Psi_0(\Phi_Y^*(t)) \, d\lambda_x(t) = \int \Psi_0(t) \, d\beta_x(t)
\]
\[
= \lim_n \int P_n(t) \, d\beta_x(t)
\]
\[
= \lim_n P_n(x) = \Psi(x) = z = \int t \, d\lambda_x(t).
\]
Since this is valid for all \( z \in Y^\circ \), lemma 2.3 gives that
\[
\Psi_0(\Phi_Y^*(t)) = t \text{ a.e. } \lambda.
\]
Using the fact that \( \Psi_0(K^\circ) = X^\circ \), it follows that
\[
\Psi_0(\Phi_Y^*(t)) = \Psi^*(\Phi_Y^*(t)) = t \text{ a.e. } \lambda \text{ on } bX_i \cap bY = R_i \cap bX,
\]
and since
\[
\mu_i/bX_i \cap bY \ll \lambda/bX_i \cap bY
\]
by (1), we have (I) and (III).

This completes the proof.

3.4. Corollary. Suppose that the diameters of the components of \( C \setminus X \) are bounded away from zero. Then \( R(X) \) is a Dirichlet algebra.
Proof. This is an immediate consequence of theorem 3.3 and Walsh’ theorem, which states that $R(Y)$ is a Dirichlet algebra whenever $C \setminus Y$ is connected [5, lemma 3].

We say that $R(X)$ is pointwise boundedly dense (p.b.d.) in $H^\infty(X^\circ)$ if every function in $H^\infty(X^\circ)$ can be approximated pointwise in $X^\circ$ by a bounded sequence of functions in $R(X)$. From lemma 1.2 in [7] and theorem VIII.11.1 in [9] we have the following lemma ($\lambda$ is as defined before lemma 2.3).

3.5. Lemma. Suppose $R(bX) = C(bX)$ and $R(X)$ is p.b.d. in $H^\infty(X^\circ)$. Then the map $f \rightarrow \tilde{f}$ is an isometric isomorphism from $H^\infty(\lambda + |\mu|)$ onto $H^\infty(X^\circ)$, for every measure $\mu$ on $bX$ orthogonal to $R(X)$.

When we have this fact, an argument similar to the one in the proof of theorem 3.3 gives the next result.

3.6. Theorem. Suppose $R(bX) = C(bX)$ and $R(X)$ is p.b.d. in $H^\infty(X^\circ)$. Then $R(X)$ is a Dirichlet algebra.

Proof. We use the same notation as in theorem 3.3. Let $\mu$ be a measure on $bX$ orthogonal to $R(X)$. Then by lemma 3.5 there exists $\phi^* \in H^\infty(\lambda + |\mu|)$ such that $\tilde{\phi}^* = \phi$, and using the same technique as in the proof of theorem 3.3 we get that

$$\phi^*(t) \in \overline{\phi(X^\circ)} = K \text{ a.e. } \lambda + |\mu|.$$ 

Choose $z \in X^\circ$ and let $x = \phi(z)$. Then if $\nu_x$ denotes the poissonmeasure for $x$, we have that $\nu^*(\nu_x)$ is a representing measure for $x$ with respect to the functions which are continuous on $X$ and harmonic on $X^\circ$. Hence by theorem 5.3 in [12], $\nu^*(\nu_x) = \lambda_z$. This gives that

$$x = \phi(z) = \int \phi^* d\lambda_z = \int \phi^* \circ \nu^* d\nu_x \text{ for all } x \in K^\circ ,$$

and so

$$\phi^*(\nu^*(t)) = t \text{ a.e. } d\theta .$$

Therefore

$$\nu_x = \phi^*(\nu^*(\nu_x)) = \phi^*(\lambda_z) \text{ for all } x \in K^\circ .$$

If $\nu_0$ is defined as in the proof of theorem 3.3, then $\nu_0 \circ \phi^* \in L^\infty(\lambda)$, and for $z \in X^\circ$ we have

$$\int \nu_0 \circ \phi^* d\lambda_z = \int \nu_0 d\nu_x = \nu(x) = z$$

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Hence \( \int \Psi_0 \circ \Phi^* \circ \Psi^* \, d\theta = \Psi(x) \) and therefore

\[
\Psi_0 \circ \Phi^* \circ \Psi^* (t) = \Psi^* (t) \quad \text{a.e. } d\theta,
\]

that is,

\[
(2) \quad \Psi_0 \circ \Phi^* (t) = t \quad \text{a.e. } \lambda.
\]

Choose a sequence \( \{P_n\} \) of polynomials such that \( \|P_n\|_K \leq M < \infty \) and

\[
P_n(z) \to \Psi_0 (z) \quad \text{for all } z \in K^c ,
\]

\[
P_n(t) \to \Psi_0 (t) \quad \text{a.e. } d\theta \text{ on } bK .
\]

Then \( \{P_n \circ \Phi^* \} \) is a bounded sequence of functions in \( H^\infty (\lambda + |\mu|) \). Therefore there exists a function \( F \in H^\infty (\lambda + |\mu|) \) and a subnet \( \{P_i \circ \Phi^* \} \) converging to \( F \) in the weak* topology in \( L^\infty (\lambda + |\mu|) \). Using (1) we get for \( z \in X^c \):

\[
\tilde{F} (z) = \int F \, d\lambda_z = \lim_i \int P_i \circ \Phi^* \, d\lambda_z
\]

\[
= \lim_i \int P_i \, d\theta_z = \int \Psi^* \, d\theta_x = \Psi (x) = z .
\]

Hence by lemma 3.4

\[
(3) \quad F (t) = t \quad \text{a.e. } \lambda + |\mu| .
\]

Let \( E \) denote the convex hull of \( \{P_i \circ \Phi^* \} \). Then since

\[
\int g \, P_i \circ \Phi^* \, d(\lambda + |\mu|) \to \int g \, F \, d(\lambda + |\mu|)
\]

for all \( g \in L^2 (\lambda + |\mu|) \) and the unit ball in \( L^\infty (\lambda + |\mu|) \) is metrizable, there exists a sequence \( \{Q_n\} \) of polynomials such that

\[
Q_n \circ \Phi^* \in E \quad \text{and} \quad Q_n \circ \Phi^* \to F
\]

in \( L^2 (\lambda + |\mu|) \). We can assume that

\[
Q_n \circ \Phi^* (t) \to F (t) \quad \text{a.e. } \lambda + |\mu| ,
\]

and since \( \|P_m \circ \Phi^*\| \leq M \) for all \( m \),

\[
\|Q_n \circ \Phi^*\| \leq M \quad \text{for all } n .
\]

Since

\[
F (t) = \Psi_0 (\Phi^* (t)) \quad \text{a.e. } \lambda
\]

by (2) and (3) this gives that, if \( \epsilon_k = \Phi^* (\lambda_k) \), then
\[
\sum_k 2^{-k} \int |\Psi_0 - Q_n| \, d\theta_k = \sum_k 2^{-k} \int |\Psi_0 \circ \Phi^* - Q_n \circ \Phi^*| \, d\lambda_k \\
= \sum_k 2^{-k} \int |F - Q_n \circ \Phi^*| \, d\lambda_k \\
= \int |F - Q_n \circ \Phi^*| \, d\lambda \to 0.
\]

By passing to a subsequence, we may therefore assume that
\[Q_n(t) \to \Psi_0(t) \quad \text{a.e.} \ d\theta \text{ on } bK.\]

Hence
\[Q_n(x) \to \Psi_0(x) \quad \text{for all } x \in K^c.\]

Define \(G(t) = \lim_n Q_n(t)\) for those \(t \in K\) such that \(\lim_n Q_n(t)\) exists. Put
\[T = \{t \in bX ; \lim_n Q_n(\Phi^*(t)) = F(t) = t\}.\]

Then \((\lambda + |\mu|)(bX \setminus T) = 0\) so that
\[G(\Phi^*(t)) = t \quad \text{a.e.} \ \lambda + |\mu|.\]

Since \(G(K^c) \subset X^c\), it follows that
\[\Phi^*(t) \in bK \quad \text{a.e.} \ \lambda + |\mu|.\]

Hence if we define \(\nu = \Phi^*(\mu)\), then \(\nu\) is a measure on \(bK\). Since \(\Phi^* \in H^\infty(\lambda + |\mu|),\)
\[\int (\Phi^*)^k \, d\mu = 0 \quad \text{for } k = 0, 1, 2, \ldots ,\]

and \(\nu\) is orthogonal to the polynomials.

Now suppose \(\mu\) is a real measure. Then \(\nu\) is real, and so \(\nu = 0\). Hence \(\mu = G(\Phi^*(\mu)) = G(\nu) = 0\), and the proof is complete.

If \(Y\) is a compact plane set we define \(H(Y)\) to be the space of real-valued functions harmonic in a neighbourhood of \(Y\), and \(\overline{H}(Y)\) to be the uniform closure of \(H(Y)\) on \(Y\). To get the other main result of this section, we need the following two lemmas from potential theory, which we state without proof. The first is due to Davie [6, lemma 2.1] and the other is due to Carleson [5, lemma 1] and Davie [6, lemma 1.5].

3.7. Lemma. Let \(Y\) be a compact set and suppose \(x_0\) is a peak point for \(\overline{H}(Y)\). Let \(f\) be a superharmonic function defined in a neighbourhood of \(x_0\). Then
\[f(x_0) = \liminf_{x \to x_0} f(x), \quad x \in C \setminus Y.\]
3.8. **Lemma.** Let $Y$ and $E$ be subsets of $\mathbb{C}$ such that $E$ has zero one-dimensional Hausdorff outer measure and $Y \cup E$ is connected. Let $\mu$ be a finite real measure with compact support such that

$$P_{\mu}(z) = \int \log |\zeta - z|^{-1} \, d\mu(\zeta) < \infty$$

for all $z \in Y$. Let $f$ be a real continuous function on an open set $U$ and define

$$g(z) = P_{\mu}(z) - f(z),$$

where

$$P_{\mu}(z) = \int \log |\zeta - z|^{-1} \, d\mu(\zeta),$$

for those $z \in U$ which satisfies $P_{\mu}(z) < \infty$. Suppose $g(z)$ is a constant $\alpha$ for all $z \in Y \cap U$. Let $z_0 \in \overline{Y} \cap U$ and suppose $P_{\mu}(z_0) < \infty$.

Then $g(z_0) = \alpha$.

We can now prove

3.9. **Theorem.** Suppose $X, X_1, X_2, \ldots$ are compact sets such that

(i) $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$,

(ii) $X = \bigcap_{n=1}^{\infty} X_n$,

(iii) $R(X_n)$ is a Dirichlet algebra for $n = 1, 2, \ldots$.

Then $R(X)$ is a Dirichlet algebra.

**Proof.** First we want to prove that if $x_0 \in bX$ then the pointmass at $x, \lambda_x$, is the only representing measure for $x$ on $bX$ with respect to $\overline{H}(X)$. To prove this, it is useful to establish the following:

(*) If $x_0 \in X$ and $\mu$ is a (positive) representing measure for $x_0$ with respect to $\overline{H}(X)$, then

$$P_{\mu}(z) = \log |z - x_0|^{-1} \quad \text{for all } z \in bX \setminus \{x_0\}.$$ 

**Proof of (\*)**: Since $t \to \log |t - z|^{-1} \in H(x)$ for all $z \in \mathbb{C} \setminus X$, we have that

$$P_{\mu}(z) = \log |z - x_0|^{-1} \quad \text{for all } z \in \mathbb{C} \setminus X,$$

and especially for $z \in \mathbb{C} \setminus X_n$, for all $n$. Since $R(X_n)$ is a Dirichlet algebra, it is easy to see that each point in $bX_n$ is a peak point for $\overline{H}(X_n)$. Since $P_{\mu}(z)$ is superharmonic in $\mathbb{C}$ [16, II.23], lemma 3.7 gives that

$$P_{\mu}(z) = \log |z - x_0|^{-1} \quad \text{for } z \in bX_n, \quad n = 1, 2, \ldots.$$
Hence
\[ P_\mu(z) = \log |z - x_0|^{-1} \quad \text{for } z \in Y \]
where \( Y = \bigcup_{n=1}^{\infty} (C \setminus X_n) \). Since \( R(X_n) \) is a Dirichlet algebra, \( C \setminus X_n \) is connected for all \( n \), and so \( Y \) is connected. Let \( E \) be the empty set, \( U = C \setminus \{x_0\} \) and define
\[ f(z) = \log |z - x_0|^{-1} \quad \text{for } z \in U. \]
Then \( P_\mu(z) < \infty \) on \( Y \) and
\[ g(z) = P_\mu(z) - f(z) = 0 \quad \text{on } Y \cap U. \]
Let \( w \in bX \setminus \{x_0\} \). By lower semicontinuity
\[ P_\mu(w) \leq \liminf_{z \to w} P_\mu(z) \leq \log |w - x_0|^{-1} < \infty, \]
and since \( w \in bX \setminus \{x_0\} \subset Y \cap U \), we can apply lemma 3.8 and get that \( g(w) = 0 \), which proves (*).

Let \( x_0 \in bX \) and choose two representing measures \( \mu_1 \) and \( \mu_2 \) for \( x_0 \) on \( bX \) with respect to \( \overline{H}(X) \). Let \( \mu = \mu_1 - \mu_2 \). Using (*) we get that
\[ P_\mu(z) = 0 \quad \text{for all } z \in bX \setminus \{x_0\}. \]
Moreover, if \( a \in X^\circ \), then (*) gives that
\[ P_{\lambda_a}(z) = \log |z - a|^{-1} \quad \text{for all } z \in bX, \]
where \( \lambda_a \) as before denotes the harmonic measure for \( a \) with respect to \( X^\circ \). Since \( \lambda_a(\{x_0\}) = 0 \) (see [12, lemma 5.2]), \( P_\mu(z) = 0 \) a.e. \( \lambda_a \). Since
\[ \int \left( \int \log |t - z|^{-1} d\lambda_a(z) \right) d|\mu|(t) = \int \log |t - a|^{-1} d|\mu|(t) < \infty, \]
the Fubini theorem gives
\[ 0 = \int P_\mu(z) d\lambda_a(z) = \int \left( \int \log |t - z|^{-1} d\mu(t) \right) d\lambda_a(z) = \int \left( \int \log |t - z|^{-1} d\lambda_a(z) \right) d\mu(t) = \int \log |t - a|^{-1} d\mu(t) = P_\mu(a). \]

Hence \( P_\mu(a) = 0 \) for all \( a \in X^\circ \). Since we also have \( P_\mu(z) = 0 \) for \( z \in (C \setminus X) \cup (bX \setminus \{x_0\}) \), it is well known that \( \mu \) must be the zero measure [5, lemma 2]. Therefore \( \mu_1 = \mu_2 \), and so \( \lambda_{x_0} \) is the only representing measure for \( x_0 \) on \( bX \) with respect to \( \overline{H}(X) \).
For \( x \in bX, \ n = 1, 2, \ldots \), define \( \lambda_x^n \) as follows: If \( x \in X_n^\circ \) we let \( \lambda_x^n \) denote the harmonic measure for \( x \) with respect to \( X_n^\circ \) and if \( x \in bX_n \) we let \( \lambda_x^n \) denote the pointmass at \( x \). Then \( \|\lambda_x^n\| = 1 \) for all \( n \), and so by the theorem of Alaoglu the sequence \( \{\lambda_x^n\}_{n=1}^{\infty} \) has at least one cluster-point in the weak* topology on measures. Let \( \sigma \) be one such point. Since the unit ball of \( C(X_1)^* \) is metrizable, we can find a subsequence \( \{\lambda_x^{n_k}\}_{k=1}^{\infty} \) which converges weak* to \( \sigma \). Since \( \lambda_x^{n_k} \) is a measure on \( bX_{n_k} \), it is easy to see that \( \sigma \) must be a measure on \( bX \). Moreover, if \( f \) is harmonic in a neighbourhood \( U \) of \( X \) then \( X_n \subset U \) for \( n \) big enough, so that

\[
\int f \, d\sigma = \lim_k \int f \, d\lambda_x^{n_k} = f(x) .
\]

Taking uniform limits, we get that \( \sigma \) is a representing measure for \( x \) on \( bX \) with respect to \( \bar{H}(X) \). Hence \( \sigma = \lambda_x \), and this proves that \( \lambda_x^n \) converges to \( \lambda_x \) in the weak* topology.

Now suppose \( \mu \in R(X)^\perp \) is real and carried on \( bX \). Let \( \mu_n \) be the sweep of \( \mu \) to \( bX_n \) defined as follows:

\[
\int f \, d\mu_n = \int \tilde{f}^{(n)} \, d\mu \quad \text{for } f \in C(bX_n) ,
\]

where

\[
\tilde{f}^{(n)}(z) = \int f \, d\lambda_x^n \quad \text{for } z \in X_n .
\]

Then it is easy to see that \( \mu_n \in R(X_n)^\perp \). Since \( R(X_n) \) is a Dirichlet algebra, \( \mu_n = 0 \). But if \( f \in C(X_1) \), then

\[
f(x) = \int f \, d\lambda_x = \lim_n \int f \, d\lambda_x^n = \lim_n \tilde{f}^{(n)}(x)
\]

for all \( x \in bX \), and since

\[
|\tilde{f}^{(n)}(x)| \leq \|f\| \quad \text{for all } x \in bX ,
\]

the dominated convergence gives

\[
\int f \, d\mu = \lim_n \int \tilde{f}^{(n)} \, d\mu = \lim_n \int f \, d\mu_n = 0 ,
\]

and so \( \mu \) must be the zero measure.

Let \( U_0 \) (unbounded), \( U_1, U_2, \ldots \) be the components of \( C \setminus X \). In [3] Bishop proved that if \( bX = bU_0 \), then every measure \( \mu \) on \( bX \) orthogonal to \( R(X) \) is represented by its differential \( (2\pi i)^{-1} \hat{\mu}(z) \, dz \). As a consequence of theorem 3.9 and corollary 3.4 we get the following, which by theorem 2.4 generalizes the following result of Bishop.
3.8. Corollary. Suppose that

\[(**)
\mathbf{b} U_j \cap \bigcup_{i=0}^{j-1} \mathbf{b} U_i \text{ is non-empty } \text{ for } j=1,2,\ldots.
\]

Then \(R(X)\) is a Dirichlet algebra.

Proof. Let \(X_n = X \cup \bigcup_{i=n+1}^{\infty} U_i, n=1,2,\ldots\). Then

\[C \setminus X_n = \bigcup_{i=0}^n U_i \quad \text{and} \quad C \setminus X_n^\circ = \bigcup_{i=0}^n \bar{U}_i,\]

which is connected for all \(n\) by \((**)\). Therefore by corollary 3.4, \(R(X_n)\) is a Dirichlet algebra for all \(n\), and since \(X = \bigcap_{n=1}^{\infty} X_n, R(X)\) is a Dirichlet algebra by theorem 3.9.

References


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