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SOME THEOREMS ON DIRECT LIMITS OF EXPANDING SEQUENCES OF MANIFOLDS

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If X is an infinite dimensional smooth manifold, it is often possible to find an expanding sequence $X_1 \subset X_2 \subset \ldots \subset X_n \subset \ldots$ of smooth closed submanifolds of X, such that the direct limit $X_\infty = \varinjlim_n X_n$ is homotopy equivalent to X through the natural map $X_\infty \to X$. Filtrations like that have already turned out to be very useful. Mukherjea showed in [17] that every smooth separable Fredholm manifold with smooth partitions of unity admitted such filtrations and used them to define a certain cohomology theory on Fredholm manifolds. In the fundamental theorem [5] on smooth embedding of a smooth separable Hilbert manifold as an open subset of Hilbert space Eells and Elworthy use also such filtrations in their proof.

These results are the primary motivation for the author to study expanding systems in this paper. Of course expanding systems have always played a role in topology. They enter e.g. in the theory of CW-complexes (skeleta filtrations) and in the construction of classifying spaces.

We outline now the content of the paper. In section 1 we collect some well-known results from the theory of function spaces which we will use without proofs. Section 2 contains the main definitions, expanding systems, and homotopy direct limits of such systems. These definitions are modeled after the situation described in the very beginning of the introduction. In section 3 we give examples of our definitions.

Sections 4 and 5 contain the main results. In section 4 the topology of the limit space X_∞ for an expanding system $X_1 \subset X_2 \subset \ldots \subset X_n \subset \ldots$ of topological spaces is studied. In theorem 4.5 we show that a smooth manifold structure on each of the spaces X_n will induce a smooth structure on X_∞ provided each space in the system is open in the next. If each space is closed in the next the situation is more difficult. Now a smooth structure on each X_n is no longer carried on to the limit. Theorem 4.8 shows however that if each X_n is a finite dimensional

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locally flat topological submanifold of X_{n+1} and the dimension of the manifolds is say strictly increasing, then X_∞ is a topological manifold modeled on R^∞, the real vector space of finitely nonzero real sequences $(x_n)_{n\geqq 1}$ topologized with the finite topology.

Since R^∞ is not metrizable a topological manifold modeled on R^∞ can never be metrizable. The direct limit of an expanding sequence of manifolds is therefore usually not metrizable, expecially not an ANR (absolute neighbourhood retract) for the class of metrizable spaces. This gain in interest if we compare with corollary 6.4 below. We show here that the direct limit of an expanding sequence of ANR's, each closed in the next, has the homotopy type of an ANR. Up to homotopy type the non-metrizability of the limits of manifolds will therefore not cause trouble.

The theorems in section 5 deal with the behaviour of a homotopy direct limit under mapping space constructions. Let the topological space X be a homotopy direct limit of the expanding system of topological spaces $X_1 \subset X_2 \subset \ldots \subset X_n \subset \ldots$. This means that X is homotopy equivalent to X_∞ through a homotopy equivalence $X_\infty \to X$ induced from given continuous maps $X_n \to X$. For a fixed compact space M consider now the covariant functor $C^0(M,\cdot)$ which to any topological space X associates the space $C^0(M,X)$ of all continuous maps from M into X topologized with the compact open topology. It seems then natural to ask whether the functor $C^0(M,\cdot)$ preserves the homotopy direct limit X in the sense that the induced map

$$\varinjlim_n C^0(M,X_n) \to C^0(M,X)$$

is again a homotopy equivalence. We answer this question in the affirmative under comparatively mild restrictions on the spaces involved. If each X_n is open in X_{n+1}, we need e.g. only to know that X_n is a T_1-space for all n. When each X_n is closed in X_{n+1}, we need however all spaces involved to be ANR's in order to make the proof go. Finally in section 5 we give similar theorems for the functors $C^r(M,\cdot)$ for all r, $0 \leqq r \leqq \infty$, in case M is a compact smooth manifold and the homotopy direct limits are smooth.

Sections 6, 7 and 8 are devoted to the proofs of the theorems in section 5. Section 6 contains however also proofs of some folklore theorems about ANR's, for example corollary 6.4.

Finally in section 9 we describe shortly how we can obtain theorems like those in section 5 for other mapping space functors than just the classical $C^r(M,\cdot)$ for M a compact smooth manifold and $0 \leqq r \leqq \infty$.

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1. Preliminaries.

In some of the main theorems we shall be dealing with spaces of continuous maps and, if smooth manifolds are involved, also with spaces of differentiable maps. In this section we list some well-known results (theorems 1.1 and 1.2) from the theory of such spaces. At the same time we fix our notation.

Manifolds will always be Hausdorff and modeled on LCTVS's (locally convex topological vector spaces). Often a LCTVS will be metrizable or even a Fréchet space (complete metrizable LCTVS). For a manifold modeled on a metrizable LCTVS it is equivalent to be metrizable and paracompact, see e.g. Palais [19, theorem 1]. Smooth manifolds will always be modeled on Banach spaces. Often a model Banach space E will be assumed to admit smooth partitions of unity. We say then that E is C^∞-smooth. A metrizable (paracompact) smooth manifold modeled on a C^∞-smooth Banach space allows smooth partions of unity. Therefore the usual constructions in differential topology work for such manifolds, notably the constructions of sprays and tubular neighbourhoods, see Lang [14].

We shall use extensively that a metrizable manifold is an ANR (absolute neighbourhood retract), see Palais [19, theorem 5]. We use the definition of an ANR given in this paper. Especially ANR's will always be metrizable, but not necessarily separable.

If M and X are topological spaces, $C^0(M,X)$ shall denote the space of continuous maps from M into X equipped with the compact open topology.

If M and X are smooth manifolds, we let $C^r(M,X)$ for each r, $0 \leqq r \leqq \infty$, denote the space of differentiable maps from M into X of class C^r equipped with the C^r-topology. In this context M will always be compact and may have a boundary.

If X and Y are topological spaces, we call as usual a continuous map $f: X \to Y$ for an *embedding* if f is a homeomorphism of X onto $f(X)$ considered with the subspace topology in Y. If furthermore $f(X)$ is open or closed in Y, then we call f an *open*, respectively a *closed embedding.*

For X and Y smooth manifolds we have of course smooth counterparts to these notions.

The results we shall use from the theory of spaces of continuous maps are listed in the following theorem.

THEOREM 1.1. *Let M, X and Y be arbitrary topological spaces.*

i) *If X is T_1, Hausdorff or regular, then $C^0(M,X)$ is T_1, Hausdorff or regular.*

ii) *If M is compact metrizable, then $C^0(M,X)$ is an* ANR *if and only if X is.*

iii) *If $f: X \to Y$ is a continuous map, then $f_*: C^0(M,X) \to C^0(M,Y)$, the map induced by composition of maps, is continuous.*

iv) *If $f: X \to Y$ is an embedding, then $f_*: C^0(M,X) \to C^0(M,Y)$ is an embedding. If f is an open or closed embedding, then f_* is an open respectively a closed embedding.*

For the proofs of these facts see e.g. Kelley [11, Chapt. 7] and Hu [9, Chapt. VI, § 2, especially theorem 2.4].

In the differentiable context we need the following results. They can be proved using the general construction principle for manifold structures on spaces of maps formulated by Eells [3, § 6]. This construction works, since, as already remarked, we will have sprays and associated tubular neighbourhoods under the assumptions listed in the theorem. See also Eliasson [6] and Krikorian [12].

THEOREM 1.2. *Let M be a smooth compact manifold, and let X and Y be smooth metrizable manifolds modeled on C^∞-smooth Banach spaces.*

i) *For $0 \leqq r < \infty$, $C^r(M,X)$ can be given the structure of a smooth metrizable manifold.*

ii) *$C^\infty(M,X)$ can be given the structure of a metrizable manifold modeled on Fréchet spaces.*

iii) *$C^r(M,X)$ is an* ANR *for all $0 \leqq r \leqq \infty$ (consequence of* i) *and* ii)).

iv) *A smooth map $f: X \to Y$ induces a smooth map $f_*: C^r(M,X) \to C^r(M,Y)$ for $0 \leqq r < \infty$ by composition of maps. For $r = \infty$, the map f_* is continuous.*

v) *If $f: X \to Y$ is a smooth embedding, then $f_*: C^r(M,X) \to C^r(M,Y)$ is a smooth embedding for $0 \leqq r < \infty$ (continuous for $r = \infty$). If f is an open or closed embedding, then f_* is an open respectively a closed embedding.*

2. Expanding systems and homotopy direct limits.

This section contains the definitions of the main objects for our investigations. The reader will find a strong resemblance with the definitions given in the appendix in Milnor [15]. Our definition of a homotopy direct limit is however slightly more general.

Definition 2.1. An *expanding system* of topological spaces $(\boldsymbol{X},\boldsymbol{f},n_0) = \{X_n, f_{n,n+1}\}_{n \geqq n_0}$ is a system of topological spaces X_n and embeddings $f_{n,n+1}: X_n \to X_{n+1}$ indexed over the integers $n \geqq n_0$.

If all the embeddings $f_{n,n+1}$ are open, then we call $(\boldsymbol{X},\boldsymbol{f},n_0)$ an *open expanding system*. Similarly, if all $f_{n,n+1}$ are closed embeddings, then we call $(\boldsymbol{X},\boldsymbol{f},n_0)$ a *closed expanding system*.

If all the topological spaces X_n are smooth manifolds, and all the embeddings $f_{n,n+1}$ are smooth embeddings, then we call $(\boldsymbol{X},\boldsymbol{f},n_0)$ a *smooth expanding system*.

Remark 2.2. If $(\boldsymbol{X},\boldsymbol{f},n_0)$ is a smooth open expanding system, then we can obviously assume that all the manifolds X_n are modeled on the same Banach space E without loss of generality. We will always make this assumption in the following.

The terms in definition 1.1 will occur so often in this paper, that we will abbreviate expanding system, open expanding system and closed expanding system to ES, OES and CES respectively.

If $(\boldsymbol{X},\boldsymbol{f},n_0)$ is an arbitrary ES, then we define its *limit space* $\boldsymbol{X}_\infty$ as the direct limit of the system $\{X_n, f_{n,n+1}\}_{n \geqq n_0}$, so

$$\boldsymbol{X}_\infty = \varinjlim_n \{X_n, f_{n,n+1}\}_{n \geqq n_0} .$$

As usual the direct limit $\boldsymbol{X}_\infty$ is the identification space obtained from the disjoint union $\bigvee_{n=n_0}^{\infty} X_n$ of the spaces X_n by identifying $x_n \in X_n$ with $f_{n,n+1}(x_n) \in X_{n+1}$. If $f_{n,\infty}: X_n \to \boldsymbol{X}_\infty$ denotes the composition of the inclusion of X_n into $\bigvee_{n=n_0}^{\infty} X_n$ followed by the projection of this space onto $\boldsymbol{X}_\infty$, then the topology on $\boldsymbol{X}_\infty$ can be described as the largest (finest) topology making all the maps $f_{n,\infty}$ continuous.

The whole system is shown in the following diagram:

$$\begin{tikzcd}
\vdots \arrow[d] & \\
X_n \arrow[dd, "f_{n,n+1}"'] \arrow[dr, "f_{n,\infty}"] & \\
& \boldsymbol{X}_\infty \\
X_{n+1} \arrow[ur, "f_{n+1,\infty}"'] \arrow[d] & \\
\vdots &
\end{tikzcd}$$

We remark that a subset of X_∞ is open or closed in the direct limit topology on X_∞ if and only if all the preimages of the set in the spaces X_n are open respectively closed.

For $n<m$ we put for convenience $f_{n,m}=f_{m-1,m}\circ\ldots\circ f_{n,n+1}$. Let also $f_{n,n}$ be the identity map on X_n.

We note the following lemma:

LEMMA 2.3. *Let* (X,f,n_0) *be an* ES.

i) *The map* $f_{n,\infty}\colon X_n\to X_\infty$ *is an embedding for all* $n\geqq n_0$.

ii) *If* (X,f,n_0) *is an* OES *or a* CES, *then all the maps* $f_{n,\infty}$ *are open respectively closed embeddings.*

PROOF. The map $f_{n,\infty}$ is clearly continuous and injective. Therefore it will be sufficient to prove that $f_{n,\infty}$ is also an open map onto its image in order to finish the proof of i). For this purpose let U_n be an open set in X_n. Since $f_{n,n+1}$ is an embedding, there exists an open set U_{n+1} in X_{n+1} such that

$$f_{n,n+1}(U_n)=f_{n,n+1}(X_n)\cap U_{n+1}\,.$$

Go on and choose sets $\{U_{n+k}\}_{k\geqq 1}$ such that U_{n+k} is open in X_{n+k} and

$$f_{n+k,n+k+1}(U_{n+k})=f_{n+k,n+k+1}(X_{n+k})\cap U_{n+k+1}\quad\text{for all } k\geqq 0\,.$$

Now $U_\infty=\bigcup_{k=0}^{\infty}f_{n+k,\infty}(U_{n+k})$ will be open in X_∞, and clearly

$$f_{n,\infty}(U_n)=f_{n,\infty}(X_n)\cap U_\infty\,.$$

This shows that $f_{n,\infty}(U_n)$ is open in $f_{n,\infty}(X_n)$. As remarked this finishes the proof of i).

ii) is even easier and is left to the reader.

The n_0 in the definition of an ES (X,f,n_0) is of course not important. What matters is the limit space X_∞, and it is clear that if we take the same spaces X_n and embeddings $f_{n,n+1}$ but only from $m_0\geqq n_0$ and upwards, then (X,f,m_0) gives the same limit space as (X,f,n_0).

We shall often use the following straightforward but important lemma:

LEMMA 2.4. *Let* (X,f,n_0) *be an* ES *of* T_1*-spaces, and let* $g\colon M\to X_\infty$ *be a continuous map such that* $g(M)$ *is a compact subset of* X_∞.

Then there exists an $n\geqq n_0$ *and a continuous map* $g_n\colon M\to X_n$ *such that the following diagram commutes:*

$$\begin{tikzcd} & X_n \arrow[d, "f_{n,\infty}"] \\ M \arrow[ur, "g_n"] \arrow[r, "g"'] & X_\infty \end{tikzcd}$$

PROOF. Observe that $X_\infty = \bigcup_{n=n_0}^{\infty} f_{n,\infty}(X_n)$. Assume that $g(M)$ is not contained in any of the subspaces $f_{n,\infty}(X_n)$ of X_∞. We can then find an infinite subspace $S = \{x_{n_k}\}_{k=1}^{\infty}$ of X_∞ such that

$$x_{n_k} \in (X_\infty \setminus f_{n_k,\infty}(X_{n_k})) \cap g(M) \quad \text{for all } 1 \leqq k \leqq \infty .$$

Since an arbitrary subset of S by construction has at most a finite number of points in common with any $f_{n,\infty}(X_n)$, it is a closed subspace. It is here we need that the spaces X_n are T_1-spaces. S is therefore a closed discrete subspace of $g(M)$. But since $g(M)$ is compact, it cannot contain any infinite closed discrete subspace.

This shows that there exists an n such that $g(M) \subset f_{n,\infty}(X_n)$. Since $f_{n,\infty}$ is an embedding of X_n into X_∞ by lemma 2.3, there exists therefore a unique continuous map $g_n \colon M \to X_n$ such that $g = f_{n,\infty} \circ g_n$.

Finally in this section we give the definition of a homotopy direct limit.

DEFINITION 2.5 (continuous case). Let (X, f, n_0) be an ES of topological spaces, let X be a topological space and let $(g, n_0) = \{g_n\}_{n \geqq n_0}$ be a system of continuous maps $g_n \colon X_n \to X$.

Then X is called a *homotopy direct limit* of (X, f, n_0) with respect to (g, n_0), if:

i) The following diagram commutes:

$$\begin{tikzcd} \vdots \arrow[d] & \\ X_n \arrow[d, "f_{n,n+1}"'] \arrow[dr, "g_n"] & \\ X_{n+1} \arrow[d] \arrow[r, "g_{n+1}"'] & X \\ \vdots & \end{tikzcd}$$

ii) The induced continuous map $g_\infty \colon X_\infty \to X$ is a homotopy equivalence.

DEFINITION 2.5 (smooth case). If $(\boldsymbol{X},\boldsymbol{f},n_0)$ is a smooth ES, X is a smooth manifold and $(\boldsymbol{g},n_0)=\{g_n\}_{n\geqq n_0}$ is a system of smooth maps $g_n: X_n \to X$, then we call X a *smooth homotopy direct limit* of $(\boldsymbol{X},\boldsymbol{f},n_0)$ with respect to $(\boldsymbol{g},n_0)$ provided the conditions i) and ii) above are still satisfied.

In the following we will abbreviate homotpy direct limit to HDL.

Examples of expanding systems and homotopy direct limits are given in the next section.

3. Some examples.

The examples in this section are aimed to show that there are many sources for interesting expanding systems and homotopy direct limits of such systems.

EXAMPLE 3.1. Let X be a topological space with the homotopy type of a CW-complex K, and let $g: K \to X$ be a homotopy equivalence.

If we now take K_n to be the n-skeleton of K and $f_{n,n+1}: K_n \to K_{n+1}$ to be the obvious inclusion, then we get a CES $(\boldsymbol{K},\boldsymbol{f},0)$. By definition of the topology on K we get at once that $\boldsymbol{K}_\infty = K$.

Now let $g_n: K_n \to X$ be the composition of the inclusion $K_n \subset K$ followed by g. Then X is a HDL of $(\boldsymbol{K},\boldsymbol{f},0)$ with respect to the system of continuous maps $(\boldsymbol{g},0)$, since of course $\boldsymbol{g}_\infty = g$.

EXAMPLE 3.2. Let E be a separable Hilbert space with basis $(e_i)_{i\geqq 1}$. For each n put $E_n = \operatorname{span}\{e_1,\ldots,e_n\}$.

If $G_k(\cdot)$ denotes the Grassmann manifold of k-planes in a linear space, then for all $n \geqq k$ we have natural inclusions

$$f_{n,n+1}: G_k(E_n) \to G_k(E_{n+1}) \quad \text{and} \quad g_n: G_k(E_n) \to G_k(E)$$

$G_k(E)$ is a classifying space for k-dimensional real vector bundles. In the usual notation $G_k(E) = BO(k)$. It is well known that $BO(k)$ is a HDL of the CES

$$(\boldsymbol{X},\boldsymbol{f},k) = \{G_k(E_n),f_{n,n+1}\}_{n\geqq k}$$

with respect to the system of continuous maps $(\boldsymbol{g},k)$. That is, the limit map

$$\boldsymbol{g}_\infty: \varinjlim_n G_k(E_n) \to BO(k)$$

is a homotopy equivalence.

EXAMPLE 3.3. Let E be a LCTVS, and let $E_1 \subset E_2 \subset \ldots \subset E_n \subset \ldots$ be an increasing sequence of finite dimensional linear subspaces whose union $\bigcup_{n=1}^{\infty} E_n$ is dense in E. If X is a subset of E, put $X_n = X \cap E_n$. Let also

$$f_{n,n+1}\colon X_n \to X_{n+1} \quad \text{and} \quad g_n\colon X_n \to X$$

be the obvious inclusions. Assume now, that X is an open subset of E. Then a theorem of Palais [19, corollary to theorem 17] can be restated as follows:

If E is metrizable or, more generally, if X is paracompact, then X is a HDL *of the* CES $(\boldsymbol{X},\boldsymbol{f},1)$ *with respect to the system of continuous maps* $(\boldsymbol{g},1)$.

EXAMPLE 3.4. See Eells [4, § 8] for details in this important example. Let X be a smooth, separable Fredholm manifold modeled on a C^∞-smooth Banach space E.

Observe that, by a theorem of Elworthy [7], every smooth separable Hilbert manifold can be given a Fredholm structure. Elworthy's theorem is actually more general than that.

A theorem of Mukherjea [17] can now be stated as follows:

There exists a smooth CES $(\boldsymbol{X},\boldsymbol{f},1)$ *such that each X_n is a compact submanifold of X of dimension n, say with inclusion $g_n\colon X_n \to X$, and each $f_{n,n+1}$ is an inclusion of X_n as a submanifold of X_{n+1}.*

Furthermore, this system can be chosen such that $\bigcup_{n=1}^{\infty} X_n$ is dense in X, and such that X is a smooth HDL *of* $(\boldsymbol{X},\boldsymbol{f},1)$ *with respect to the system of smooth embeddings* $(\boldsymbol{g},1)$.

This finishes example 3.4.

The examples 3.1–3.4 dealt only with CES's. These systems will however probably also be the most interesting, since it is in connection with example 3.4 the theorems we will prove are most naturally asked for. From a mathematical point of view CES's are also much more difficult to handle than OES's.

Finally we give now an example of an OES. There are of course much simpler examples than the one, we give here. Take e.g. an increasing family of open balls in euclidean space.

EXAMPLE 3.5. Let E be an infinite dimensional Banach space, and let $L(E)$ denote the Banach space of continuous linear operators on E. Consider then for each $n \geqq 0$ the subset $L(E;n)$ of $L(E)$ consisting of those operators for which the cokernel has dimension $\leqq n$. This is then an open subset of $L(E)$, and $L(E;0)$ is just the space of all surjective operators on E. Obviously we have now an OES

$$L(E;0) \subset L(E;1) \subset \ldots \subset L(E;n) \subset \ldots .$$

The limit space of this OES is precisely the open subset of $L(E)$ consisting of the operators with finite dimensional cokernel, the right Fredholm operators on E.

4. Topology of the limit space for an expanding system.

This section will mainly deal with the question, whether the limit space for an ES of manifolds can be given the structure of a manifold. The main results are theorem 4.5 and theorem 4.8. Only theorem 4.5 is needed in later sections.

The topology on the limit space X_∞ for an ES (X,f,n_0) is a quotient topology, a fact, which keeps one from expecting too much of it. For a general ES we have however the following result. Recall that a topological space is a Lindelöf space if every open covering of it contains a countable sub-covering.

Proposition 4.1. *Let* (X,f,n_0) *be an* ES *of topological spaces.*
(i) *If all the spaces* X_n *are* T_1*-spaces, then* X_∞ *is a* T_1*-space.*
(ii) *If all the spaces* X_n *are locally compact Hausdorff spaces, then* X_∞ *is a Hausdorff space.*
(iii) *If all the spaces* X_n *are Lindelöf spaces, then* X_∞ *is a Lindelöf space.*

Proof. We leave the proofs of (i) and (iii) to the reader and concentrate on the proof of (ii).

By lemma 2.3 we can identify the spaces X_n with the subspaces $f_{n,\infty}(X_n)$ of X_∞. We can therefore assume that all the maps $f_{n,n+1}$ are inclusion maps.

Let now $x,y \in X_\infty$ with $x \neq y$ be given. We have to find disjoint open neighbourhoods of these points in X_∞. Pick $n_1 \geqq n_0$ such that $x,y \in X_{n_1}$. Since X_{n_1} is locally compact and Hausdorff, we can find, in X_{n_1}, open neighbourhoods U_{n_1} and V_{n_1} of x and y respectively such that the closures $\overline{U}_{n_1}$ and $\overline{V}_{n_1}$ in X_{n_1} are compact and disjoint. Since $\overline{U}_{n_1}$ and $\overline{V}_{n_1}$ are also compact and disjoint subsets of the Hausdorff space X_{n_1+1}, they can be separated by open sets in X_{n_1+1}. Using the local compactness of X_{n_1+1} it is then easy to find open sets U_{n_1+1} and V_{n_1+1} in X_{n_1+1} which extend U_{n_1} and V_{n_1} and have disjoint compact closures $\overline{U}_{n_1+1}$ and $\overline{V}_{n_1+1}$. This indicates how we can construct expanding sequences $\{U_n\}_{n\geqq n_1}$ and $\{V_n\}_{n\geqq n_1}$, where U_n and V_n are disjoint open neighbourhoods of x and y in X_n for each $n \geqq n_1$. Then $U_\infty = \bigcup_{n\geqq n_1}^\infty U_n$ and $V_\infty = \bigcup_{n\geqq n_1}^\infty V_n$ will be the wanted disjoint open neighbourhoods of x and y in X_∞.

We are particularly interested in the Lindelöf property, since it is known that a regular Lindelöf space is paracompact. For a connected, locally compact space the converse statement is also true. See Kelley [11]. We remark that compact and paracompact include the Hausdorff axiom in this paper.

For an OES we can strengthen proposition 4.1 (ii) as follows. The proof is trivial.

PROPOSITION 4.2. *If* $(\boldsymbol{X},\boldsymbol{f},n_0)$ *is an* OES *of Hausdorff spaces, then* $\boldsymbol{X}_\infty$ *is a Hausdorff space.*

In general the Hausdorff property is not carried on to the limit in a CES. We have however this result:

PROPOSITION 4.3. *Let* $(\boldsymbol{X},\boldsymbol{f},n_0)$ *be a* CES *of topological spaces.*

(i) *If all the spaces* X_n *are normal, then* $\boldsymbol{X}_\infty$ *is normal.*

(ii) *If all the spaces* X_n *are regular Lindelöf spaces, then* $\boldsymbol{X}_\infty$ *is a regular Lindelöf space.*

PROOF. Assume for a moment that (i) is proved. Then (ii) follows in this way. Each X_n is a regular Lindelöf space, hence paracompact and therefore normal. By (i), $\boldsymbol{X}_\infty$ is therefore also normal, in particular regular. That $\boldsymbol{X}_\infty$ is Lindelöf, follows by proposition 4.1.

We turn then to the proof of (i). By proposition 4.1 we get immediately that $\boldsymbol{X}_\infty$ is a T_1-space. It will therefore be sufficient to show that Urysohn's lemma holds in $\boldsymbol{X}_\infty$. For that purpose, let A and B be closed, non-empty, disjoint subsets of $\boldsymbol{X}_\infty$. We have to find a continuous function $h\colon \boldsymbol{X}_\infty \to [0,1]$ such that $h|A\equiv 0$ and $h|B\equiv 1$. To do this we proceed as follows:

Choose a sufficiently high $n_1 \geqq n_0$ such that both $f^{-1}_{n_1,\infty}(A)$ and $f^{-1}_{n_1,\infty}(B)$ are non-empty. Using the normality of X_{n_1}, we can now find a continuous function $h_{n_1}\colon X_{n_1} \to [0,1]$ such that

$$h_{n_1}|f^{-1}_{n_1,\infty}(A) \equiv 0 \quad \text{and} \quad h_{n_1}|f^{-1}_{n_1,\infty}(B) \equiv 1 .$$

Consider then the closed subset

$$f_{n_1,n_1+1}(X_{n_1}) \cup f^{-1}_{n_1+1,\infty}(A) \cup f^{-1}_{n_1+1,\infty}(B)$$

of X_{n_1+1}. Using Tietze's extension theorem, it follows now immediately that there exists a continuous function $h_{n_1+1}\colon X_{n_1+1} \to [0,1]$ such that

$$h_{n_1} = h_{n_1+1}\circ f_{n_1,n_1+1}, \quad h_{n_1+1}|f^{-1}_{n_1+1,\infty}(A) \equiv 0, \quad h_{n_1+1}|f^{-1}_{n_1+1,\infty}(B) \equiv 1 .$$

This indicates how we can construct a family $\{h_n\}_{n\geqq n_1}$ of continuous functions $h_n\colon X_n \to [0,1]$ such that

$$h_n = h_{n+1} \circ f_{n,n+1}, \quad h_n | f_{n,\infty}^{-1}(A) \equiv 0, \quad h_n | f_{n,\infty}^{-1}(B) \equiv 1 .$$

But then we get an induced continuous map $h = h_\infty : X_\infty \to [0,1]$, which by construction will satisfy the conditions $h|A \equiv 0$ and $h|B \equiv 1$.

As already noted this finishes the proof.

Theorem 4.8 below will produce lots of examples of CES's where all the spaces in the systems are metrizable manifolds but the limit spaces are not metrizable (see remark 4.10). We will now give an example of an OES of smooth metrizable manifolds, where the limit space is a smooth manifold which is Hausdorff but not regular. The statements in the propositions 4.1–4.3 seem therefore to be about the best one can hope for in general.

Example 4.4. We follow closely an example given by Palais [20].

Let E be a separable Hilbert space of infinite dimension with inner product $(\cdot,\cdot)$, and let $e_0 \in E$ be a unit vector in E. Let E^+ denote the closed half-space $\{e \in E \mid (e,e_0) \geqq 0\}$, $E^{+\circ}$ its interior, and ∂E^+ its boundary. Let $S = \{s_n\}_{n \geqq 1}$ be a countable dense subset of ∂E^+, and put $X_n = E^{+\circ} \cup \{s_1, \ldots, s_n\}$ for each $n \geqq 1$. Give X_n the induced topology from E and let $f_{n,n+1} : X_n \to X_{n+1}$ be the obvious inclusion. It follows easily that $(\boldsymbol{X}, \boldsymbol{f}, 1)$ is an OES and that the limit space X_∞ is Hausdorff. X_∞ is however not regular. To see this, observe that $E^{+\circ} \cup \{s\}$ is an open subset of X_∞ for each $s \in S$, and that every open neighbourhood of $s \in X_\infty$ in X_∞ will have other points from S in its closure since S is dense in ∂E^+.

Assertion. Each X_n is a metrizable smooth manifold modeled on E, and X_∞ is a smooth manifold modeled on E, which is Hausdorff but not regular.

Proof. The manifold X_n is a subspace of the metrizable space E and is therefore itself metrizable. The purely topological statements in the assertion are therefore now all proved.

To show that X_n and X_∞ are smooth manifolds, it will suffice to define a homeomorphism $\theta_0 : E^{+\circ} \cup \{0\} \to E$ which restricts to a smooth diffeomorphism of $E^{+\circ}$ onto $E \setminus \{0\}$. For then we can define smooth atlasses

$$\{\theta_s\}_{s \in \{s_1, \ldots, s_n\}} \quad \text{and} \quad \{\theta_s\}_{s \in S}$$

on X_n and X_∞ respectively, where $\theta_s : E^{+\circ} \cup \{s\} \to E$ is defined by $\theta_s(x) = \theta_0(x - s)$. To define θ_0 we recall that, by a theorem of Bessaga [1], E is diffeomorphic to its own unit sphere $\Sigma = \{e \in E \mid \|e\| = 1\}$. Now ∂E^+ is linearly diffeomorphic to E and hence diffeomorphic to Σ. On the other hand ∂E^+ is also diffeomorphic to $D = \{v \in \partial E^+ \mid \|v\| < 1\}$, which by stereographic projection from $-e_0$ is mapped diffeomorphically onto $\Sigma \cap E^{+\circ}$. Altogether there exists therefore a diffeomorphism $g\colon \Sigma \cap E^{+\circ} \to \Sigma$. Now define

$$\theta_0\colon E^{+\circ} \cup \{0\} \to E$$

by

$$\theta_0(x) = \|x\|\, g(x/\|x\|) \quad \text{for } x \neq 0$$

and $\theta_0(0)=0$. Then θ_0 is a bijection with inverse given by

$$\theta_0^{-1}(x) = \|x\|\, g^{-1}(x/\|x\|) \quad \text{for } x \neq 0$$

and $\theta_0^{-1}(0)=0$. Clearly θ_0 and θ_0^{-1} are continuous, and since $x \to \|x\|$ is a smooth map of $E \setminus \{0\}$ into the reals, it follows, that θ_0 maps $E^{+\circ}$ diffeomorphically onto $E \setminus \{0\}$.

This finishes the proof of the assertion and ends the example.

We start now our investigation of the limit space for an ES of manifolds.

For an OES one could have hoped for nothing better than

THEOREM 4.5. *Let $(\boldsymbol{X},\boldsymbol{f},n_0)$ be a smooth* OES *with the Banach space E as the model for all the manifolds X_n.*

Then $\boldsymbol{X}_\infty$ has a unique structure as a smooth manifold modeled on E, such that all the maps $f_{n,\infty}\colon X_n \to \boldsymbol{X}_\infty$ are smooth open embeddings.

Furthermore, if the smooth manifold X is a smooth HDL *of $(\boldsymbol{X},\boldsymbol{f},n_0)$ with respect to the system of smooth maps $(\boldsymbol{g},n_0)$, then the induced map $\boldsymbol{g}_\infty\colon \boldsymbol{X}_\infty \to X$ is smooth.*

PROOF. We use again that $\boldsymbol{X}_\infty = \bigcup_{n=n_0}^{\infty} f_{n,\infty}(X_n)$. Since $f_{n,\infty}$ is a homeomorphism of X_n onto $f_{n,\infty}(X_n)$, there is a unique smooth structure on $f_{n,\infty}(X_n)$ making $f_{n,\infty}$ a smooth diffeomorphism. Consider now $f_{n,\infty}(X_n)$ with this smooth structure for all $n \geqq n_0$. Observe then that $f_{n,\infty}(X_n)$ is a smooth open submanifold of $f_{n+1,\infty}(X_{n+1})$ for each $n \geqq n_0$ since $f_{n,n+1}$ is a smooth open embedding. When we now use that the subsets $f_{n,\infty}(X_n)$ are open subsets of $\boldsymbol{X}_\infty$, it follows immediately, that there exists a unique smooth structure on $\boldsymbol{X}_\infty$ which induces the above smooth structures on the subspaces $f_{n,\infty}(X_n)$. This proves clearly existence and uniqueness of the wanted smooth structure on $\boldsymbol{X}_\infty$.

The second part of the theorem is obvious, since $\boldsymbol{g}_\infty \circ f_{n,\infty} = g_n$ is smooth for all $n \geqq n_0$.

Results like theorem 4.5 for a CES of smooth manifolds are much more delicate and far more than one can hope for. The main reason is that the direct limit of topological vector spaces (TVS) is not always a

TVS. Before we state our result for a CES of manifolds, we investigate therefore this problem a little.

Let $E_{n_0} \subset E_{n_0+1} \subset \ldots \subset E_n \subset \ldots$ be an increasing sequence of Fréchet spaces E_n such that E_n is a subspace of E_{n+1} in the sense of TVS's for all $n \geqq n_0$. Put $E_\infty = \bigcup_{n=n_0}^\infty E_n$. Then E_∞ has a natural real vector space structure, and it is known that it can be given the structure of a Hausdorff LCTVS by taking as neighbourhoods of 0 convex sets which intersect each E_n in an open neighbourhood of $0 \in E_n$. With this LCTVS structure E_∞ is a socalled LF-space. See e.g. Treves [25] for the result just mentioned. The topology in this locally convex structure on E_∞ is usually different from the direct limit topology (the weak topology) with respect to the topological spaces E_n. We have however this result:

LEMMA 4.6. *Let $E_{n_0} \subset E_{n_0+1} \subset \ldots \subset E_n \subset \ldots$ be an increasing sequence of finite dimensional vector spaces with their canonical Hausdorff* TVS *structures. The inclusions are inclusions as linear subspaces.*

Then the locally convex topology and the direct limit topology on $E_\infty = \bigcup_{n=n_0}^\infty E_n$ with respect to the subspaces E_n coincide.

In particular, E_∞ will therefore be a LCTVS *in the direct limit topology with respect to the subspaces E_n.*

PROOF. The locally convex topology is always smaller than the direct limit topology, so it is in the proof of the converse statement we need the vector spaces E_n to be finite dimensional. To prove that an open set in the direct limit topology is also open in the locally convex topology, it will be sufficient to prove the following. If $U \subset E_\infty$ is an arbitrary subset of E_∞ such that $U \cap E_n$ is open in E_n for each $n \geqq n_0$, and $x \in U$, then there exists a convex neighbourhood K of x in E_∞ such that $K \cap E_n$ is open in E_n for each $n \geqq n_0$ and $x \in K \subset U$. On the other hand this statement is easily proved using the local compactness of the finite dimensional vector spaces E_n. One merely starts in the space $E_{n(x)}$ with the lowest index $n \geqq n_0$ such that $x \in E_n$, and then builds a K with the wanted properties step by step.

If E is an arbitrary vector space, the finite topology on E is the direct limit topology on E with respect to the directed set of finite dimensional subspaces of E considered with their unique Hausdorff TVS topologies. A subset $U \subset E$ is therefore open (closed) in the finite topology if and only if $U \cap F$ is open (closed) in F for every finite dimensional subspace F of E. When the subspaces E_n of E_∞ are finite dimensional as in lemma 4.6, it is obvious, that the finite topology on E_∞ coincides with the direct limit topology with respect to the subspaces E_n and thus also with the locally convex topology. Under the assumptions in lemma 4.6, E_∞ will therefore be a LCTVS in the finite topology. In general it is known that a vector space E is a TVS in its finite topology if and only if E is at most countable dimensional. See Palais [19] and the reference there to a paper by Kakutani and Klee [10]. There is however a slight mistake in Palais's argument for his lemma 6.10. (The convex neighbourhood $N(x^\circ, \{\varepsilon_i\})$ is not necessarily contained in U.) This rather trivial mistake is corrected by our lemma 4.6.

We have a canonical countable dimensional LCTVS in its finite topology, denoted R^∞. It is the vector space of real sequences $(x_n)_{n\geqq 1}$ such that $x_n \neq 0$ for at most a finite number of indexes.

If E is an arbitrary countable dimensional TVS in its finite topology, then it is isomorphic to R^∞ as topological vector spaces, since it is obviously isomorphic to R^∞ as vector spaces, and the topologies cause no trouble in this case, because we have the finite topology on both spaces.

The following lemma is now easily proved.

LEMMA 4.7. *Let $E_{n_0} \subset E_{n_0+1} \subset \ldots \subset E_n \subset \ldots$ be an increasing sequence of finite dimensional vector spaces with their canonical Hausdorff* TVS *structures. The inclusions are inclusions as linear subspaces. Assume also that the dimension of the spaces E_n is unbounded.*

Then $E_\infty = \bigcup_{n=n_0}^\infty E_n$ is a LCTVS *in the direct limit topology with respect to the subspaces E_n. Furthermore E_∞ is isomorphic to R^∞ as topological vector spaces.*

The result we have for a CES of manifolds is strongest when formulated entirely in the topological context. We recall therefore a few definitions. Let E be a TVS and let F and G be closed linear subspaces of E which split E into $E = F \times G$. A subset X of a topological manifold Y modeled on E is then called a topological submanifold of Y modeled on F if for each $x \in X$ there exists a coordinate chart (U, θ) on Y centered at x $(\theta(x) = 0)$ such that $\theta(U) = E$ and $\theta(U \cap X) = F$. If X and Y are topological manifolds modeled on the TVS's F and E respectively, then an embedding $f: X \to Y$ is called a locally flat embedding if $f(X)$ is a topological submanifold of Y modeled on F. Observe that a smooth embedding is always locally flat in finite dimensions.

We can now state and prove

THEOREM 4.8. *Let (X, f, n_0) be a* CES *of finite dimensional topological manifolds X_n, where all the maps $f_{n,n+1}$ are locally flat embeddings. Assume also that the dimension of the manifolds X_n is unbounded.*

Then X_∞ is a topological manifold modeled on R^∞.

PROOF. Let E_n be a finite dimensional model for X_n. We can assume that E_n is a linear subspace of E_{n+1} for each $n \geqq n_0$ such that we have an increasing sequence

$$E_{n_0} \subset E_{n_0+1} \subset \ldots \subset E_n \subset \ldots$$

as in lemma 4.7. By this lemma we know already that $E_\infty = \bigcup_{n=n_0}^\infty E_n$ with the direct limit topology with respect to the subspaces E_n is isomorphic to R^∞ as topological vector spaces. Since we know from proposition 4.1 (ii) that X_∞ is Hausdorff, it will therefore be sufficient to prove the following assertion in order to finish the proof of the theorem.

Assertion. Each point of X_∞ has an open neighbourhood homeomorphic to an open subset of E_∞.

In order to prove this assertion we proceed as follows: By lemma 2.3 we can identify the spaces X_n with closed subspaces of X_∞. We can therefore assume that the CES is an expanding sequence

$$X_{n_0} \subset X_{n_0+1} \subset \ldots \subset X_n \subset \ldots \subset X_\infty$$

with $X_\infty = \bigcup_{n=n_0}^\infty X_n$ and each space closed in the next. Since the maps $f_{n,n+1}$ are locally flat embeddings, X_n is a topological submanifold of X_{n+1} for each $n \geqq n_0$. Now let $x \in X_\infty$ be an arbitrary point in X_∞, and let $n(x)$ be the smallest index $n \geqq n_0$ such that $x \in X_n$. Choose a coordinate chart $(U_{n(x)}, \theta_{n(x)})$ on $X_{n(x)}$ centred at x such that $\theta_{n(x)}(U_{n(x)}) = E_{n(x)}$. Using a result of Lacher [13, theorem 2.2] one proves easily that there exists a coordinate chart $(U_{n(x)+1}, \theta_{n(x)+1})$ on $X_{n(x)+1}$ centered at x such that

$$\theta_{n(x)+1}(U_{n(x)+1}) = E_{n(x)+1}, \quad U_{n(x)} = X_{n(x)} \cap U_{n(x)+1}, \quad \theta_{n(x)} = \theta_{n(x)+1} \,|\, U_{n(x)} .$$

It is obvious that we can continue this extension procedure ending up with charts (U_n, θ_n) on X_n centered at x for all $n \geqq n(x)$ such that $\theta_n(U_n) = E_n$ and such that (U_{n+1}, θ_{n+1}) restricts to (U_n, θ_n). Put then $U_\infty = \bigcup_{n=n(x)}^\infty U_n$ and define $\theta \colon U_\infty \to E_\infty$ by

$$\theta \,|\, U_n = \theta_n \quad \text{for all } n \geqq n(x) .$$

By definition of the topologies on X_∞ and E_∞ it follows now immediately that $\theta \colon U_\infty \to E_\infty$ is a homeomorphism from the open neighbourhood U_∞ of $x \in X_\infty$ onto E_∞.

This proves the assertion and therefore the theorem.

EXAMPLE 4.9. Let as usual S^n and RP^n denote the n-sphere and the n-dimensional real projective space respectively.

We have the following closed expanding systems using the standard inclusions

$$S^1 \subset S^2 \subset \ldots \subset S^n \subset \ldots,$$

$$RP^1 \subset RP^2 \subset \ldots \subset RP^n \subset \ldots.$$

Consider the limit spaces

$$S^\infty = \varinjlim_n S^n \quad \text{and} \quad RP^\infty = \varinjlim_n RP^n .$$

By theorem 4.8 both S^∞ and RP^∞ are topological manifolds modeled on R^∞.

This statement is of course also valid for the limit spaces in examples 3.2, 3.3 and 3.4.

REMARK 4.10. A topological manifold modeled on R^∞ can never be metrizable since R^∞ itself is not metrizable.

The manifolds X_∞ modeled on R^∞ obtained from theorem 4.8 are therefore never metrizable but nearly always paracompact by proposition 4.3. Remember that a regular Lindelöf space is paracompact, and that these conditions actually are equivalent for locally compact, connected spaces.

REMARK 4.11. In a recent paper Henderson and West [8] have obtained a theorem like our theorem 4.8. They work however with metrizable topologies in the following sense:

Let $X_1 \subset X_2 \subset \ldots \subset X_n \subset \ldots$ be a sequence of metric spaces such that all inclusions are isometries. In our terminology a *metric expanding system.* Let X_∞^{metric} denote the direct limit of this sequence in the category of metric spaces and isometries. X_∞^{metric} is then just the union $\bigcup_{n=1}^\infty X_n$ with the unique metric topology such that all inclusions $X_n \subset X_\infty^{\text{metric}}$ are isometries. Call X_∞^{metric} the metric direct limit of the sequence. Let X_∞^{weak} denote the usual direct limit of the sequence. The identity map (as sets) $X_\infty^{\text{weak}} \to X_\infty^{\text{metric}}$ is clearly continuous. When dealing with manifolds it is usually a homotopy equivalence.

If $l_2^f(\aleph_0)$ denotes the metrizable LCTVS of finitely non-zero real sequences $(x_n)_{n\geq 1}$ with its standard pre-Hilbert structure, then the result of Henderson and West can be formulated as follows:

If $M^1 \subset M^2 \subset \ldots \subset M^n \subset \ldots$ is a sequence of metrizable manifolds ($\dim M^n = n$) *without boundary, each bicollared in the next, then the manifolds may be metrized, so that it is a metric expanding system whose metric direct limit is an $l_2^f(\aleph_0)$-manifold of the same homotopy type as the usual direct limit.*

As Henderson and West remark, it is not all choices of metrics making the sequence a metric expanding system which give the metric direct limit the structure of an $l_2^f(\aleph_0)$-manifold.

This theorem together with example 4.9 shows e.g. that S^∞ and RP^∞ are topological manifolds in both their direct limit topologies and their metric direct limit topologies inherited from the defining (metric) sequences.

5. Induced expanding systems and induced homotopy direct limits of spaces of maps.

In this section we state our results concerning the behaviour of the covariant functors $C^0(M,\cdot)$ and $C^r(M,\cdot)$ when they are applied to expanding systems and homotopy direct limits. The proofs of the theorems stated will be given in sections 7 and 8.

Let $(\boldsymbol{X},\boldsymbol{f},n_0)$ be an ES of topological spaces, and let M be an arbitrary topological space. Then we get an induced system

$$\ldots \to C^0(M,X_n) \xrightarrow{(f_{n,n+1})_*} C^0(M,X_{n+1}) \to \ldots$$

starting at $n=n_0$. Let us denote this system by $(C^0(M,\boldsymbol{X}),\boldsymbol{f}_*,n_0)$ and call it the *induced system.*

From theorem 1.1 we get immediately

LEMMA 5.1. *An induced system of an* ES *is itself an* ES. *If* $(\boldsymbol{X},\boldsymbol{f},n_0)$ *is an* OES *or a* CES, *then* $(C^0(M,\boldsymbol{X}),\boldsymbol{f}_*,n_0)$ *is an* OES *or a* CES.

Similarly, if $(\boldsymbol{X},\boldsymbol{f},n_0)$ is a smooth ES, and M is an arbitrary smooth compact manifold, we get for each $0 \leqq r \leqq \infty$ an induced system

$$\ldots \to C^r(M,X_n) \xrightarrow{(f_{n,n+1})_*} C^r(M,X_{n+1}) \to \ldots$$

starting at $n=n_0$. We denote this system by $(C^r(M,\boldsymbol{X}),\boldsymbol{f}_*,n_0)$ and call it again the *induced system.*

Using theorem 1.2 we get

LEMMA 5.2. *Let* $(\boldsymbol{X},\boldsymbol{f},n_0)$ *be a smooth* ES, *and let* M *be a smooth compact manifold.*

Then for all $0 \leqq r \leqq \infty$ *the induced system* $(C^r(M,\boldsymbol{X}),\boldsymbol{f}_*,n_0)$ *is an* ES, *smooth for* $0 \leqq r < \infty$.

If $(\boldsymbol{X},\boldsymbol{f},n_0)$ *is a smooth* OES *or* CES, *then* $(C^r(M,\boldsymbol{X}),\boldsymbol{f}_*,n_0)$ *is an* OES *or a* CES, *again smooth for* $0 \leqq r < \infty$.

Assume now, that X is a HDL of the ES $(\boldsymbol{X},\boldsymbol{f},n_0)$ with respect to the system of continuous maps $(\boldsymbol{g},n_0)$.

For an arbitrary topological space M we get then an induced diagram

$$\begin{tikzcd}
\vdots \arrow[d] & & \\
C^0(M,X_n) \arrow[dd, "(f_{n,n+1})_*"'] \arrow[dr, "f^0_{n,\infty}"] \arrow[drr, "(g_n)_*"] & & \\
& C^0(M,X)_\infty \arrow[r, "g_{*\infty}"] & C^0(M,X) \\
C^0(M,X_{n+1}) \arrow[ur, "f^0_{n+1,\infty}"] \arrow[urr, "(g_{n+1})_*"'] \arrow[d] & & \\
\vdots & &
\end{tikzcd}$$

In this diagram $C^0(M,X)_\infty$ is the limit space of the induced system $(C^0(M,\boldsymbol{X}),\boldsymbol{f}_*,n_0)$, $f^0{}_{n,\infty}$ is the inclusion of $C^0(M,X_n)$ into the limit space and $g_{*\infty}$ is the limit map for the *induced system* of continuous maps $(\boldsymbol{g}_*,n_0)=\{(g_n)_*\}_{n\geqq n_0}$.

Dealing with smooth HDL we get of course similar diagrams for all $0\leqq r\leqq\infty$.

The main theorems in the continuous context can now be formulated. We have a theorem for each of the cases OES and CES. For a CES the theorem is not so general, but general enough to cover many interesting examples.

THEOREM 5.3. *Let $(\boldsymbol{X},\boldsymbol{f},n_0)$ be an* OES *of T_1-spaces and let M be a compact space. Furthermore, let X be a* HDL *of $(\boldsymbol{X},\boldsymbol{f},n_0)$ with respect to the system $(\boldsymbol{g},n_0)$ of continuous maps.*

Then $C^0(M,X)$ is a HDL *of the induced* OES *$(C^0(M,\boldsymbol{X}),\boldsymbol{f}_*,n_0)$ with respect to the induced system $(\boldsymbol{g}_*,n_0)$ of continuous maps.*

THEOREM 5.4. *Let $(\boldsymbol{X},\boldsymbol{f},n_0)$ be a* CES *of* ANR's, *and let M be compact metrizable. Furthermore, let X be an* ANR *which is a* HDL *of $(\boldsymbol{X},\boldsymbol{f},n_0)$ with respect to the system $(\boldsymbol{g},n_0)$ of continuous maps.*

Then $C^0(M,X)$ is a HDL *of the induced* CES *$(C^0(M,\boldsymbol{X}),\boldsymbol{f}_*,n_0)$ with respect to the induced system $(\boldsymbol{g}_*,n_0)$ of continuous maps.*

In the differentiable context we have analogous results. There is nearly no difference between OES and CES in this case.

THEOREM 5.5. *Let $(\boldsymbol{X},\boldsymbol{f},n_0)$ be a smooth* OES *or* CES, *and let X be a smooth manifold which is a smooth* HDL *of $(\boldsymbol{X},\boldsymbol{f},n_0)$ with respect to the system $(\boldsymbol{g},n_0)$ of smooth maps. Assume furthermore, that all the manifolds are metrizable and modeled on C^∞-smooth Banach spaces. For an* OES *assume also that X_∞ is metrizable.*

Then for every smooth compact manifold M and all $0\leqq r\leqq\infty$, $C^r(M,X)$

is a HDL *of the induced* OES *respectively* CES $(C^r(M,\boldsymbol{X}),\boldsymbol{f}_*,n_0)$ *with respect to the induced system* $(\boldsymbol{g}_*,n_0)$ *of continuous maps.*

For $0 \leqq r < \infty$ *the induced* HDL *is actually smooth.*

REMARK 5.6. Example 4.4 shows that the metrizability of $\boldsymbol{X}_\infty$ for the smooth OES $(\boldsymbol{X},\boldsymbol{f},n_0)$ is not a consequence of the metrizability of the manifolds X_n.

6. Homotopy theory of closed expanding systems.

The material in this section is probably well known. In the literature there seems, however, not to be a completely adequate treatment along the lines we need. This is the excuse for the details we give here.

DEFINITION 6.1. Let $(\boldsymbol{X},\boldsymbol{f},n_0)$ and $(\boldsymbol{Y},\boldsymbol{g},n_0)$ be expanding systems of topological spaces. A *map between expanding systems* $\boldsymbol{h}: (\boldsymbol{X},\boldsymbol{f},n_0) \to (\boldsymbol{Y},\boldsymbol{g},n_0)$ is a system of continuous maps $h_n: X_n \to Y_n$ making the following diagram commutative

$$\begin{array}{ccccccc}
\cdots \to & X_n & \xrightarrow{f_{n,n+1}} & X_{n+1} & \to \cdots \\
 & \downarrow{\scriptstyle h_n} & & \downarrow{\scriptstyle h_{n+1}} & \\
\cdots \to & Y_n & \xrightarrow[g_{n,n+1}]{} & Y_{n+1} & \to \cdots
\end{array}$$

When composition of maps is defined in the obvious way, it is clear that we get a category consisting of ES's starting at $n = n_0$ as objects and the maps in definition 6.1 as morphisms.

Call a map $\boldsymbol{h}: (\boldsymbol{X},\boldsymbol{f},n_0) \to (\boldsymbol{Y},\boldsymbol{g},n_0)$ for a *homotopy equivalence between* ES's if each h_n is a homotopy equivalence in the usual sense.

A map $\boldsymbol{h}: (\boldsymbol{X},\boldsymbol{f},n_0) \to (\boldsymbol{Y},\boldsymbol{g},n_0)$ induces in the usual way a continuous *limit map* $\boldsymbol{h}_\infty: \boldsymbol{X}_\infty \to \boldsymbol{Y}_\infty$.

Theorems 6.2 and 6.3 below are the main theorems in this section. The author owes the basic idea in their proofs to lectures by Tammo tom Dieck (Aarhus, fall 1968). See also the Springer lecture notes by Puppe et al. [24]. If one takes into account lemma 6.7 below, then the theorems will also be immediate consequences of the corresponding theorems in the appendix in Milnor [15]. We give however nearly all details here.

THEOREM 6.2. *Let* $(\boldsymbol{X},\boldsymbol{f},n_0)$ *and* $(\boldsymbol{Y},\boldsymbol{g},n_0)$ *be* CES's *of topological spaces, such that all the maps* $f_{n,n+1}$ *and* $g_{n,n+1}$ *are cofibrations.*

Then a homotopy equivalence $\boldsymbol{h}\colon (\boldsymbol{X},\boldsymbol{f},n_0) \to (\boldsymbol{Y},\boldsymbol{g},n_0)$ *between* ES's *induces an ordinary homotopy equivalence* $\boldsymbol{h}_\infty\colon \boldsymbol{X}_\infty \to \boldsymbol{Y}_\infty$.

THEOREM 6.3. *Let* $(\boldsymbol{X},\boldsymbol{f},n_0)$ *be a* CES *of topological spaces, such that all the maps* $f_{n,n+1}$ *are cofibrations.*

Then, if each X_n *has the homotopy type of a CW-complex,* $\boldsymbol{X}_\infty$ *has too.*

Before entering in the proofs of these theorems we mention a corollary, which we shall use in the proofs of the theorems in section 5.

COROLLARY 6.4. *Let* $(\boldsymbol{X},\boldsymbol{f},n_0)$ *be a* CES *of* ANR*'s. Then* $\boldsymbol{X}_\infty$ *has the homotopy type of an* ANR.

Notice, that homotopy type is the most we can hope for by remark 4.10.

Corollary 6.4 is an easy consequence of theorem 6.3 and lemmas 6.5 and 6.6 below. Lemma 6.6 has some interest in itself. As pointed out to me by J. Eells, separability of ANR's is not needed in the statement of this lemma.

LEMMA 6.5. *Let A and X be* ANR*'s, and let* $f\colon A \to X$ *be a closed embedding.*

Then f is a cofibration.

PROOF. See e.g. Palais [19, theorem 7].

LEMMA 6.6. *It is equivalent for a topological space to have the homotopy type of an* ANR *and of a CW-complex.*

PROOF. Assume first, that X is an ANR. Then by an extension of Hanner's result in the separable case Palais proves [19, theorem 14] that X is dominated by a simplicial complex. Then by a theorem of Milnor [16, theorem 2] X has the homotopy type of a CW-complex.

Next assume, that X is a CW-complex. Then by the above mentioned theorem of Milnor, X has the homotopy type of a simplicial complex with the metric topology. But a simplicial complex with the metric topology is an ANR, see Hu [9, theorem 11.3, p. 106].

We begin now the proof of theorems 6.2 and 6.3. For that purpose we introduce the *iterated mapping cylinder* (or telescope) of an ES $(\boldsymbol{X},\boldsymbol{f},n_0)$, denote it by $Z(\boldsymbol{X},\boldsymbol{f},n_0)$, and define it as the quotient space of the disjoint union $\bigvee_{n=n_0}^{\infty}(X_n \times [n,n+1])$ modulo the relations $(x_n, n+1) \sim (f_{n,n+1}(x_n), n+1)$ for all $x_n \in X_n$ and all $n \geqq n_0$.

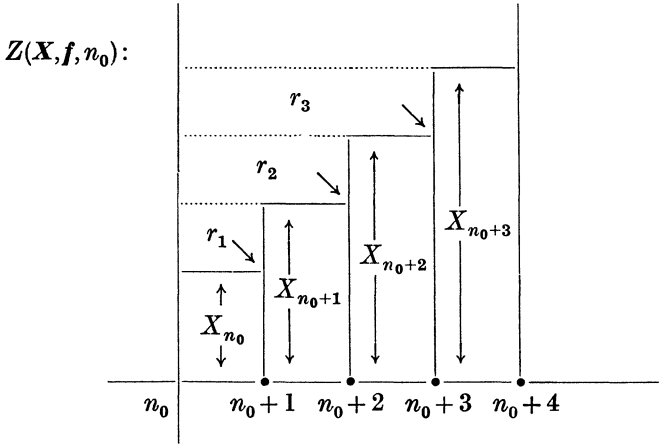

fig. 6.1

The projections $X_n \times [n, n+1] \to X_n$ induce a canonical projection

$$p(X, f, n_0) \colon Z(X, f, n_0) \to X_\infty$$

LEMMA 6.7. *Let (X, f, n_0) be a CES of topological spaces such that all the maps $f_{n,n+1}$ are cofibrations.*

Then the canonical projection $p(X, f, n_0)$ is a homotopy equivalence.

PROOF. By lemma 2.3 we may assume that all the maps $f_{n,n+1}$ are inclusions and that all the spaces X_n are closed subspaces of X_∞. We have then the tower

$$X_{n_0} \subset X_{n_0+1} \subset \ldots \subset X_n \subset \ldots \subset X_\infty .$$

The telescope $Z(X, f, n_0)$ can now be identified with a subspace of $X_\infty \times [n_0, \infty[$, and the canonical projection $p(X, f, n_0)$ is just the composite map

$$Z(X, f, n_0) \subset X_\infty \times [n_0, \infty[\xrightarrow{\text{proj.}} X_\infty .$$

Since the projection is trivially a homotopy equivalence, it will be sufficient to prove, that $Z(X, f, n_0)$ is a strong deformation retract of $X_\infty \times [n_0, \infty[$. To prove this we use a theorem of Puppe [23, Satz 4, p. 87] to construct a strong deformation retraction

$$r_n \colon X_{n+1} \times [n_0, n+1] \to X_n \times [n_0, n+1] \cup X_{n+1} \times \{n+1\}$$

for all $n \geqq n_0$. As observed by Puppe, the existence of such a strong deformation retraction follows just from the fact that the inclusion $X_n \to X_{n+1}$ is a cofibration. Using these strong deformation retractions, it is easy to construct a strong deformation retraction of $X_\infty \times [n_0, \infty[$ onto $Z(\boldsymbol{X},\boldsymbol{f},n_0)$. See fig. 6.1.

Let now $(\boldsymbol{X},\boldsymbol{f},n_0)$ and $(\boldsymbol{Y},\boldsymbol{g},n_0)$ be ES's of topological spaces and let $\boldsymbol{h} = \{h_n\}_{n \geqq n_0}$ and $\boldsymbol{\varphi} = \{\varphi_n\}_{n \geqq n_0}$ be systems of continuous maps $h_n : X_n \to Y_n$ and homotopies $\varphi_n : X_n \times [0,1] \to Y_{n+1}$ such that $(\varphi_n)_0 = g_{n,n+1} \circ h_n$ and $(\varphi_n)_1 = h_{n+1} \circ f_{n,n+1}$ for all $n \geqq n_0$. The following diagram is thus homotopy commutative with the homotopies φ_n as the homotopies in the squares

$$\begin{tikzcd}
\cdots \arrow[r] & X_n \arrow[r, "f_{n,n+1}"] \arrow[d, "h_n"'] \arrow[dr, phantom, "\circlearrowleft\ \varphi_n"] & X_{n+1} \arrow[r] \arrow[d, "h_{n+1}"] & \cdots \\
\cdots \arrow[r] & Y_n \arrow[r, "g_{n,n+1}"'] & Y_{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

To the systems of maps $\boldsymbol{h}$ and $\boldsymbol{\varphi}$ we associate a map

$$Z(\boldsymbol{h},\boldsymbol{\varphi}) : Z(\boldsymbol{X},\boldsymbol{f},n_0) \to Z(\boldsymbol{Y},\boldsymbol{g},n_0)$$

defined by

$$\begin{aligned} Z(\boldsymbol{h},\boldsymbol{\varphi})(x_n, n+t) &= (h_n(x_n), n+2t), && \text{for } 0 \leqq t \leqq \tfrac{1}{2}, \\ &= (\varphi_n(x_n, 2t-1), n+1), && \text{for } \tfrac{1}{2} \leqq t \leqq 1 \end{aligned}$$

for all $x_n \in X_n$ and all $n \geqq n_0$. With these assumptions we have

LEMMA 6.8. *If all the maps h_n are homotopy equivalences, then the map $Z(\boldsymbol{h},\boldsymbol{\varphi})$ is a homotopy equivalence.*

PROOF. The proof is analogous to the proof of Hilfsatz 7, p. 314 in [22]. Puppe proves here the corresponding fact for the ordinary mapping cylinder.

PROOF OF THEOREM 6.2. Define the homotopy φ_n as the constant homotopy

$$(\varphi_n)_t = g_{n,n+1} \circ h_n = h_{n+1} \circ f_{n,n+1}$$

for all $t \in [0,1]$. Then we have the commutative diagram

$$\begin{tikzcd}
Z(\boldsymbol{X},\boldsymbol{f},n_0) \arrow[r, "p(\boldsymbol{X},\boldsymbol{f},n_0)"] \arrow[d, "Z(\boldsymbol{h},\boldsymbol{\varphi})"'] & X_\infty \arrow[d, "h_\infty"] \\
Z(\boldsymbol{Y},\boldsymbol{f},n_0) \arrow[r, "p(\boldsymbol{Y},\boldsymbol{g},n_0)"'] & Y_\infty .
\end{tikzcd}$$

Lemmas 6.7 and 6.8 finish now the proof.

Proof of Theorem 6.3. It is easy to construct a homotopy commutative diagram

$$\begin{tikzcd}
X_{n_0} \arrow[r, "f_{n_0,n_0+1}"] \arrow[d, "h_{n_0}"'] & X_{n_0+1} \arrow[r, "f_{n_0+1,n_0+2}"] \arrow[d, "h_{n_0+1}"'] & X_{n_0+2} \arrow[r] \arrow[d, "h_{n_0+2}"'] & \cdots \\
K_{n_0} \arrow[r, "i_{n_0,n_0+1}"] & K_{n_0+1} \arrow[r, "i_{n_0+1,n_0+2}"] & K_{n_0+2} \arrow[r] & \cdots ,
\end{tikzcd}$$

where each K_n is a CW-complex, each $i_{n,n+1}$ is an inclusion of K_n as a subcomplex of K_{n+1} and each h_n is a homotopy equivalence.

Let $\varphi = \{\varphi_n\}_{n \geq n_0}$ be the system of homotopies in the squares. Consider then the diagram

$$\begin{tikzcd}
Z(X,f,n_0) \arrow[r, "p(X,f,n_0)"] \arrow[d, "Z(h,\varphi)"'] & X_\infty \\
Z(K,i,n_0) \arrow[r, "p(K,i,n_0)"] & K_\infty
\end{tikzcd}$$

All the maps in this diagram are homotopy equivalences by lemmas 6.7 and 6.8. Therefore X_∞ is homotopy equivalent to K_∞, which by construction is a CW-complex.

7. Proof of theorems 5.3 and 5.4.

Let (X,f,n_0) be an ES of topological spaces, and let M be an arbitrary topological space. Consider then the induced ES $(C^0(M,X),f_*,n_0)$. Utilizing the universal property of a direct limit we get an induced continuous map $\theta\colon C^0(M,X)_\infty \to C^0(M,X_\infty)$ as shown in the diagram

$$\begin{tikzcd}
\vdots \arrow[d] & & \\
C^0(M,X_n) \arrow[dr, "f^0_{n,\infty}"] \arrow[drr, "(f_{n,\infty})_*"] \arrow[dd, "(f_{n,n+1})_*"'] & & \\
& C^0(M,X)_\infty \arrow[r, "\theta"] & C^0(M,X_\infty) . \\
C^0(M,X_{n+1}) \arrow[ur, "f^0_{n+1,\infty}"] \arrow[urr, "(f_{n+1,\infty})_*"'] \arrow[d] & & \\
\vdots & &
\end{tikzcd}$$

Lemma 7.1. *Let M be a compact space.*

i) *If (X,f,n_0) is an* OES *of T_1-spaces, then θ is a homeomorphism.*

ii) *If (X,f,n_0) is a* CES *of T_1-spaces, then θ is a weak homotopy equivalence.*

Proof. Assume that $g \in C^0(M,X_\infty)$. By lemma 2.4 there exists then an n and a map $g_n \in C^0(M,X_n)$ such that $g = f_{n,\infty} \circ g_n$ or equivalently $(f_{n,\infty})_*(g_n) = g$. This shows, that θ is surjective. It is also easy to show that θ is a monomorphism. Therefore θ is always a continuous bijection in both cases.

i) Let $O \subset C^0(M,X)_\infty$ be an open subset. Put $O_n = (f^0{}_{n,\infty})^{-1}(O)$. Observe then, that $O = \bigcup_{n=n_0}^\infty f^0{}_{n,\infty}(O_n)$, and therefore, that $\theta(O) = \bigcup_{n=n_0}^\infty (f_{n,\infty})_*(O_n)$. Since $(f_{n,\infty})_*$ is an open embedding by lemma 2.3 and theorem 1.1, it follows, that $\theta(O)$ is open. This shows that θ is an open map, and therefore together with the remarks preceding i), that θ is a homeomorphism.

ii) Let Q be an arbitrary compact space. It will be sufficient to prove the assertions A° and B° below. Assertion A° will prove, that the induced map θ_* in homotopy is surjective in all dimensions. Assertion B° will prove, that θ_* is injective in all dimensions.

- A° For each continuous map $h\colon Q \to C^0(M,X_\infty)$ there exists an n and a continuous map $h_n\colon Q \to C^\circ(M,X_n)$, such that $h = (f_{n,\infty})_* \circ h_n$.
- B° If $h_n\colon Q \to C^0(M,X_n)$ is a continuous map, such that $h = (f_{n,\infty})_* \circ h_n$ is homotopic to a constant map in $C^0(M,X_\infty)$, then there exists a $k \geqq 0$, such that $h_{n+k} = (f_{n,n+k})_* \circ h_n$ is homotopic to a constant map in $C^0(M,X_{n+k})$.

To prove assertion A°, observe that h is continuous if and only if the map $\hat{h}\colon Q \times M \to X_\infty$ is continuous when $\hat{h} = \mathrm{Ev} \circ (h \times 1_M)$, and $\mathrm{Ev}\colon C^0(M,X_\infty) \times M \to X_\infty$ is the usual evaluation map. Since $Q \times M$ is compact, $\hat{h}$ can be factored continuously through X_n for some n by lemma 2.4. This factorization provides us in the obvious way with the needed map $h_n\colon Q \to C^0(M,X_n)$.

Now we prove assertion B°. From the hypothesis in B° it follows by arguments similar to those under A°, that there exist a $k \geqq 0$, a map $c \in C^0(M,X_{n+k})$ and a homotopy $H\colon Q \times [0,1] \to C^0(M,X_{n+k})$ such that

$$(f_{n+k,\infty})_*(H(q,0)) = h(q) \quad \text{and} \quad (f_{n+k,\infty})_*(H(q,1)) = (f_{n+k,\infty})_*(c)$$

for all $q \in Q$. But then it follows that $h_{n+k} = (f_{n,n+k})_* \circ h_n$ is homotopic to the constant map $Q \to C^0(M,X_{n+k})$ with value c under the homotopy H.

This proves lemma 7.1.

LEMMA 7.2. *Let M be a locally compact Hausdorff space, and let f: $X \to Y$ be a homotopy equivalence.*

Then the induced map $f_*: C^\circ(M,X) \to C^0(M,Y)$ is a homotopy equivalence.

PROOF. Let g: $Y \to X$ be a homotopy inverse to f. Then we have homotopies $g \circ f \sim 1_X$ and $f \circ g \sim 1_Y$. From sublemma 7.3 and the functoriallity of the mapping space construction it follows now easily that these homotopies induce homotopies

$$g_* \circ f_* \sim 1_{C^\circ(M,X)} \quad \text{and} \quad f_* \circ g_* \sim 1_{C^\circ(M,Y)} \,.$$

SUBLEMMA 7.3. *Let M be a locally compact Hausdorff space, and let X, Y and T be arbitrary topological spaces. For a map H: $X \times T \to Y$ we define an induced map*

$$H_*: C^0(M,X) \times T \to C^0(M,Y)$$

by

$$(H_*(f,t))(x) = H(f(x),t)$$

for all $f \in C^\circ(M,X)$, $x \in M$ and $t \in T$.

Then, if H is continuous, H_* is continuous.

PROOF. Consider the composite map $\hat{H}_*$ defined by the diagram

$$C^0(M,X) \times T \times M \xrightarrow{\text{twist}} C^0(M,X) \times M \times T \xrightarrow{\mathrm{Ev} \times 1_T} X \times T \xrightarrow{H} Y \,.$$

This map sends (f,t,x) into $H(f(x),t)$. If H is continuous, then $\hat{H}_*$ is continuous, since the evaluation map Ev: $C^0(M,X) \times M \to X$ is continuous. Since $\hat{H}_*$ induces H_* in the obvious way, H_* is therefore also continuous if H is continuous.

Assume now, that the topological space X is a HDL of the ES $(\boldsymbol{X},\boldsymbol{f},n_0)$ with respect to the system of continuous maps $(\boldsymbol{g},n_0)$. If M is an arbitrary topological space, we have then the induced system $(C^0(M,\boldsymbol{X}),\boldsymbol{f}_*,n_0)$ and the induced system of continuous maps $(\boldsymbol{g}_*,n_0)$. Utilizing the following commutative diagram, it is obvious that the limit map of the induced system $\boldsymbol{g}_{*\infty}$ factors as $\boldsymbol{g}_{*\infty} = (\boldsymbol{g}_\infty)_* \circ \theta$.

$$\begin{tikzcd}
\vdots \arrow[d] & & & \\
C^0(M,X_n) \arrow[dd, "(f_{n,n+1})_*"'] \arrow[dr, "f^0_{n,\infty}"] \arrow[drr, "(f_{n,\infty})_*"] \arrow[drrr, "(g_n)_*"] & & & \\
& C^0(M,X)_\infty \arrow[r, "\theta"] & C^0(M,X_\infty) \arrow[r, "(g_\infty)_*"] & C^0(M,X) \,. \\
C^0(M,X_{n+1}) \arrow[d] \arrow[ur, "f^0_{n+1,\infty}"] \arrow[urr, "(f_{n+1,\infty})_*"'] \arrow[urrr, "(g_{n+1})_*"'] & & & \\
\vdots & & &
\end{tikzcd}$$

PROOF OF THEOREM 5.3. We have to show, that the limit map $g_{*\infty}\colon C^0(M,X)_\infty \to C^0(M,X)$ is a homotopy equivalence. This follows however immediately from the factorization $g_{*\infty}=(g_\infty)_* \circ \theta$, lemma 7.1 i) and lemma 7.2.

This proves theorem 5.3.

PROOF OF THEOREM 5.4. From lemma 7.1, ii), lemma 7.2 and the factorization $g_{*\infty}=(g_\infty)_* \circ \theta$ it follows that $g_{*\infty}$ is at least a weak homotopy equivalence. If we can show that the spaces $C^0(M,X)_\infty$ and $C^0(M,X)$ have the homotopy type of ANR's, then it follows from a theorem of Whitehead [26] (see also Palais [19, theorem 15]), that $g_{*\infty}$ is a homotopy equivalence. Under the assumptions listed in the theorem, this follows however immediately from theorem 1.1, ii) and corollary 6.4.

This proves theorem 5.4.

8. Proof of theorem 5.5.

In this section (X,f,n_0) is a smooth ES, and M is a smooth compact manifold. Assume also, that the smooth manifold X is a smooth HDL of (X,f,n_0) with respect to the system (g,n_0) of smooth maps. All manifolds will be metrizable and modeled on C^∞-smooth Banach spaces, such that the spaces of maps we deal with have manifold structures.

Although the statement of theorem 5.5 is the same for an OES and a CES, the proof proceeds a little different in the two cases.

PROOF OF THEOREM 5.5 FOR AN OES. We consider X_∞ with the unique smooth structure from theorem 4.5. We get then a factorization $g_{*\infty}=(g_\infty)_* \circ \theta$ of $g_{*\infty}$ just as in the continuous case, but now for all $0 \leqq r \leqq \infty$:

$$\begin{tikzcd}
\vdots \arrow[d] \\
C^r(M,X_n) \arrow[dr, "f^r_{n,\infty}"] \arrow[drr, "(f_{n,\infty})_*"] \arrow[drrr, "(g_n)_*"] \arrow[dd, "(f_{n,n+1})_*"'] \\
& C^r(M,X)_\infty \arrow[r, "\theta"] & C^r(M,X_\infty) \arrow[r, "(g_\infty)_*"] & C^r(M,X) . \\
C^r(M,X_{n+1}) \arrow[ur, "f^r_{n+1,\infty}"'] \arrow[urr, "(f_{n+1,\infty})_*"'] \arrow[urrr, "(g_{n+1})_*"'] \arrow[d] \\
\vdots
\end{tikzcd}$$

Just as in the proof of lemma 7.1, i) we can show that θ is a homeomorphism.

The proof is therefore finished, if we can prove, that $(g_\infty)_*$ is a homotopy equivalence. For that purpose consider the diagram

$$\begin{tikzcd} C^r(M,X_\infty) \arrow[r, "(g_\infty)_*{}^r"] \arrow[d] & C^r(M,X) \arrow[d] \\ C^0(M,X_\infty) \arrow[r, "(g_\infty)_*{}^0"'] & C^0(M,X)\,. \end{tikzcd}$$

We have labelled the induced maps with r and 0 respectively just to distinguish between them.

The vertical maps are the obvious inclusions. By a theorem of Palais [21, theorem 13.14] stated below these inclusions are known to be homotopy equivalences. Since we know that $(g_\infty)_*{}^0$ is a homotopy equivalence (lemma 7.2), it follows then that $(g_\infty)_*{}^r$ is a homotopy equivalence. As already remarked, this finishes the proof of theorem 5.5 for a smooth OES.

REMARK 8.1. Strictly speaking we use in this paper the slightly generalized version of Palais' theorem ([21, theorem 13.14]) stated below. If X in this theorem is a separable smooth manifold modeled on a separable Hilbert space, then Palais' method of proof described in [21] will go through with minor obvious changes as e.g. substitution of the use of Whitney's embedding theorem in lemma 13.13 with McAlpin's generalization of this theorem to the Hilbert case ([3], § 4).

THEOREM (Palais, [21, theorem 13.14]). *Let M be a compact smooth manifold and let X be a metrizable smooth manifold modeled on a C^∞-smooth Banach space.*

Then the inclusion map $C^r(M,X) \to C^0(M,X)$ is a homotopy equivalence for all $1 \leqq r \leqq \infty$.

PROOF. A direct but clumsy proof that the map is a weak homotopy equivalence, and therefore a homotopy equivalence since the spaces involved are ANR's, can be given using standard theorems on approximation of continuous maps with differentiable maps. Proofs of the necessary theorems can be adapted from the proofs given in say Munkres [18]. The following more elegant proof is of course essentially Palais' argument in [21].

Without loss of generality we can identify X with a smooth submanifold of a Banach space E via a smooth closed embedding. The proof of this fact is an almost "classical" application of smooth partitions of unity and has been carried through in all details by J.-P. Penot in his thesis (Univ. Paris, 1970). Since any Banach space admits a spray it follows that X has a tubular neighbourhood in E (the proofs in Lang [14] works without changes). There exists therefore an open neighbourhood U of X in E and a smooth strong deformation retraction $\pi\colon U \to X$. From the fundamental theorem of Palais ([19, theorem 16]) it follows now easily that the inclusion map $C^r(M,U) \to C^0(M,U)$ is a homotopy equivalence. Obvious use of the smooth strong deformation retraction π finishes then the proof.

Proof of theorem 5.5 for a CES. In this case X_∞ does not in general carry a smooth structure, so we have to proceed differently.

Consider the following diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
 & C^r(M,X_n) \arrow[r] \arrow[dl, "f^r{}_{n,\infty}"'] \arrow[dd, "(f_{n,n+1})^r{}_*"] & C^0(M,X_n) \arrow[dr, "f^0{}_{n,\infty}"] \arrow[dd, "(f_{n,n+1})^0{}_*"'] & \\
C^r(M,X)_\infty & & & C^0(M,X)_\infty \\
 & C^r(M,X_{n+1}) \arrow[r] \arrow[ul, "f^r{}_{n+1,\infty}"] \arrow[d] & C^0(M,X_{n+1}) \arrow[ur, "f^0{}_{n+1,\infty}"'] \arrow[d] & \\
 & \vdots & \vdots &
\end{tikzcd}$$

The horizontal maps are the obvious inclusions. By the theorem of Palais they are homotopy equivalences. Using theorem 6.2 and lemma 6.5 in connection with theorem 1.2, iii), it follows then, that the limit map $C^r(M,X)_\infty \to C^0(M,X)_\infty$ is a homotopy equivalence.

Consider now finally the commutative diagram

$$\begin{tikzcd}
C^r(M,X)_\infty \arrow[r, "g^r{}_{*\infty}"] \arrow[d] & C^r(M,X) \arrow[d] \\
C^0(M,X)_\infty \arrow[r, "g^0{}_{*\infty}"'] & C^0(M,X)\,.
\end{tikzcd}$$

The right vertical map is again just an inclusion and a homotopy equivalence. The left vertical map is the limit map from before. From theorem 5.4 it follows, that $g^0{}_{*\infty}$ is a homotopy equivalence. Altogether it follows then immediately, that $g^r{}_{*\infty}$ is a homotopy equivalence. This is exactly, what we wanted to prove.

9. Concluding remarks.

REMARK 9.1. In section 8 it became clear that all the work in proving theorem 5.5 was associated with the continuous case. The result for the functor $C^r(M,\cdot)$ when $r \geqq 1$, was derived from the corresponding result for $C^0(M,\cdot)$ just using that the inclusion $C^r(M,X) \to C^0(M,X)$ was a homotopy equivalence. It is therefore clear, that we get a "theorem 5.5" for all mapping space functors with this property. Palais gives in [21] a lot of functors of this sort. As remarked in remark 8.1, Palais' method of proof will go through not only when the range manifold is finite dimensional, but also when it is a separable manifold modeled on a separable Hilbert space. Let us here just mention one of these functors.

When M is a compact Riemannian manifold of dimension n, and X is a smooth separable manifold modeled on a separable Hilbert space, we can define the Sobolev space $L_r{}^p(M,X)$ of differentiable maps of class C^r from M into X all of whoose differentials of order $\leqq r$ are of class L^p. Here $1 \leqq p \leqq \infty$ and $0 \leqq r \leqq \infty$. For $r > n/p$, the space $L_r{}^p(M,X)$ can be given the structure of a smooth manifold. See Eells [3, § 6] for a more detailed definition and references to the literature. Since the inclusion $L_r{}^p(M,X) \to C^0(M,X)$ is a homotopy equivalence for $r > n/p$, we get the following theorem:

THEOREM. *Let* $(\boldsymbol{X},\boldsymbol{f},n_0)$ *be a smooth* OES *or* CES *and let* X *be a smooth manifold, which is a smooth* HDL *of* $(\boldsymbol{X},\boldsymbol{f},n_0)$ *with respect to the system of smooth maps* $(\boldsymbol{g},n_0)$. *Assume furthermore, that all the manifolds are separable and modeled on separable Hilbert spaces.*

Then for every smooth compact Riemannian manifold M *of dimension* n *and all* $r > n/p$, *the space* $L_r{}^p(M,X)$ *is a* HDL *of the induced system* $(L_r{}^p(M,\boldsymbol{X}),\boldsymbol{f}_*,n_0)$ *with respect to the induced system of continuous maps* $(\boldsymbol{g}_*,n_0)$.

For $0 \leqq r < \infty$ *the induced limit is actually smooth.*

REMARK 9.2. Results similar to those in section 5 and remark 9.1 can of course also be obtained for spaces of sections in smooth locally trivial fibre bundles over a fixed compact base manifold. We have chosen to do the whole programme just for the mapping spaces in order to make the presentation a little simpler.

REMARK 9.3. In stead of the space $C^r(M,X)$ for $r \geqq 1$ one could also be interested in the subspace of embeddings $\mathrm{Emb}^r(M,X)$ or the subspace of immersions $\mathrm{Imm}^r(M,X)$ from M into X. A result of Dax [2] indicates that in the highly stable range in which we work, when the dimension of the manifolds X_n tends to infinity, the limit spaces $\mathrm{Emb}^r(M,\boldsymbol{X})_\infty$

and $\mathrm{Imm}^r(M,X)_\infty$ will have the homotopy type of $C^0(M,X)_\infty$ for any smooth CES (X,f,n_0).

Although the author has not carried through all the details, he feels, that the following conjecture is likely to be a theorem.

CONJECTURE. Let (X,f,n_0) be a smooth CES of finite dimensional manifolds of increasing dimension, and let X be an infinite dimensional smooth metrizable manifold modeled on a C^∞-smooth Banach space. Assume also, that X is a smooth HDL of (X,f,n_0) with respect to the system of smooth embeddings (g,n_0).

Then for all compact smooth manifolds M and all $2 \leqq r \leqq \infty$

$$\mathrm{Emb}^r(M,X) \quad \text{and} \quad \mathrm{Imm}^r(M,X)$$

are HDL's of the induced CES's

$$(\mathrm{Emb}^r(M,X),f_*,n_0) \quad \text{and} \quad (\mathrm{Imm}^r(M,X),f_*,n_0)$$

with respect to the induced system of continuous embeddings (g_*,n_0).

For $2 \leqq r < \infty$ the induced limits will actually be smooth.

REFERENCES

1. C. Bessaga, *Every infinite-dimensional Hilbert space is diffeomorphic with its unit sphere*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 14 (1966), 27–31.
2. J. P. Dax, *Généralisation des théorèmes de plongement de Haefliger à des familles d'applications dépendant d'un nombre quelconque de paramètres*, C. R. Acad. Sci. Paris A 264 (1967), 499–502.
3. J. Eells, *A setting for global analysis*, Bull. Amer. Math. Soc. 72 (1966), 751–807.
4. J. Eells, *Fredholm structures*, Nonlinear Functional Analysis, Chicago 1968 (Amer. Math. Soc. Symp. Pure Math. 18, Part 1), Providence, 1970, 62–85.
5. J. Eells and K. D. Elworthy, *Open embeddings of certain Banach manifolds*, Ann. of Math. 91 (1970), 465–485.
6. H. Eliasson, *Geometry of manifolds of maps*, J. Differential Geometry 1 (1967), 169–194.
7. K. D. Elworthy, *Fredholm maps and* GLk(E)*-structures*, Thesis, Oxford University, 1967, Bull. Amer. Math. Soc. 74 (1968), 582–586.
8. D. W. Henderson and J. E. West, *Triangulated infinite-dimensional manifolds*, Bull. Amer. Math. Soc. 76, (1970), 655–660.
9. S.-T. Hu, *Theory of retracts*, Wayne State University Press, Detroit, 1965.
10. S. Kakutani and V. Klee, *The finite topology of a linear space*, Arch. Mat. 14 (1963), 55–58.
11. J. L. Kelley, *General Topology*, Van Nostrand, Princeton, N.J., 1955.
12. N. Krikorian, *Manifolds of maps*, Thesis, Cornell University, 1969.
13. C. Lacher, *Locally flat strings and half-strings*, Proc. Amer. Math. Soc. 18 (1967), 299–304.
14. S. Lang, *Introduction to differentiable manifolds*, Interscience Publ., New York, 1962.

15. J. Milnor, *Morse theory*, (Annals of Math. Studies 51), Princeton University Press, 1963.
16. J. Milnor, *On spaces having the homotopy type of a CW-complex*, Trans. Amer. Math. Soc. 90 (1959), 272–280.
17. K. K. Mukherjea, *Coincidence theorems for infinite dimensional manifolds*, Bull. Amer. Math. Soc. 74 (1968), 493–496.
18. J. R. Munkres, *Elementary differential topology* (Annals of Math. Studies 54), Princeton University Press, Revised Edition 1966.
19. R. S. Palais, *Homotopy theory of infinite dimensional manifolds*, Topology 5 (1966), 1–16.
20. R. S. Palais, *Correction to "Homotopy theory of infinite dimensional manifolds"*. Unpublished Note.
21. R. S. Palais, *Foundations of global non-linear analysis* (Mathematics Lecture Note Series) W. A. Benjamin Inc., New York · Amsterdam, 1968.
22. D. Puppe, *Homotopiemengen und ihre induzierten Abbildungen* I, Math. Z. 69 (1958), 299–344.
23. D. Puppe, *Bemerkungen über die Erweiterung von Homotopien*, Arch. Math. 18 (1967), 81–88.
24. D. Puppe, (with co-authors T. tom Dieck and K. H. Kamps) *Homotopie Theorie* (Lecture Notes in Math. 157), Springer-Verlag, Berlin · Heidelberg · New York, 1970.
25. F. Treves, *Topological vector spaces, Distributions and Kernels* (Pure and Applied Mathematics 25), Academic Press, New York · London, 1967.
26. J. H. C. Whitehead, *Combinatorial homotopy* I, Bull. Amer. Math. Soc. 55 (1949), 213–245.

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