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A NEW CHARACTERIZATION OF BESICOVITCH ALMOST PERIODIC FUNCTIONS

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1. Introduction and history.

The set of Besicovitch almost periodic functions, $\{B^p\text{-AP}\}$, may be defined as the closure of the Bohr almost periodic functions via the Besicovitch norm $\|\cdot\|_{B(p)}$, where

$$\|f\|_{B(p)} = \overline{\lim_{T\to\infty}} \left[\frac{1}{2T}\int_{-T}^{T} |f(x)|^p \, dx\right]^{1/p},$$

and $p \in [1,\infty)$ is a fixed parameter. Several authors have given structural characterizations which assure that a function, known to be in $L_p(-T,T)$ for all $T>0$, is also in $\{B^p\text{-AP}\}$. The first was by Bohr and Besicovitch [2]:

A set E of real numbers is called satisfactorily uniform if there exists $L>0$ such that the ratio of maximum number of elements of E included in an interval of length L to the minimum number is less than 2. Then $f \in \{B^p\text{-AP}\}$ if and only if for every $\varepsilon>0$ the set of $\|\cdot\|_{B(p)}$-ε-translation numbers,

$$B^p E(\varepsilon,f) = \{u \in R:\ \|f_u - f\|_{B(p)} < \varepsilon\},$$

contains a satisfactorily uniform subset,

$$\ldots u_{-2} < u_{-1} < u_0 < u_1 < u_2 < \ldots,$$

such that

$$\bar{M}_x \bar{M}_i \left[\frac{1}{c}\int_{x}^{x+c} |f_{u_i}(t) - f(t)|^p \, dt\right] < \varepsilon^p$$

whenever $c>0$. Here $f_u(x) = f(x+u)$,

$$\bar{M}_x[g(x)] = \overline{\lim_{T\to\infty}} \frac{1}{2T}\int_{-T}^{T} g(x) \, dx,$$

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and

$$\bar{M}_i[g(i)] = \overline{\lim_{k\to\infty}} \frac{1}{2k+1} \sum_{i=-k}^{k} g(i) .$$

In the case $p=1$ Besicovitch [1] has shown: $f \in \{\text{B-AP}\}$ if and only if for every $\varepsilon > 0$ there is a satisfactorily uniform set of numbers

$$\dots u_{-2} < u_{-1} < u_0 = 0 < u_1 < u_2 < \dots$$

such that

$$\bar{M}_x \bar{M}_i \left[\int_x^{x+1} |f_{u_i}(t) - f(t)|\, dt \right] < \varepsilon .$$

Incidentally, it is easy to see that a structural characterization of $\{\text{B-AP}\}$ may be turned into a structural characterization of $\{\text{B}^p\text{-AP}\}$ by adjoining the condition

(1) $$\lim_{n\to\infty} \|f - f_n\|_{B(p)} = 0 ,$$

where

$$f_n(x) = f(x) \quad \text{if } |f(x)| < n , \qquad = nf(x)/|f(x)| \quad \text{otherwise} .$$

Alternatively, one may adjoin the condition

(1)′ $$\bar{M}[|f|^p \chi_E] \to 0 \quad \text{as } \bar{\mu}(E) \to 0 .$$

Here E is a measurable set, f is a measurable function, χ_E is the characteristic function of E, and we define $\bar{\mu}(E) = \bar{M}[\chi_E]$.

E. Følner [5] has given the following characterization of $\{\text{B}^p\text{-AP}\}$:

$f \in \{\text{B}^p\text{-AP}\}$ if and only if f satisfies either (1) or (1)′ and

(2) for every $\varepsilon > 0$ there exists a relatively dense set $T = T(\varepsilon)$ and a set $E = E(\varepsilon)$ such that $\bar{\mu}(E) > 1 - \varepsilon$ and $|f(x+t) - f(x)| < \varepsilon$ whenever $t \in T$, and $x, x+t \in E$.

(Actually Følner did not require that $f \in L_p(-T, T)$ for all $T > 0$ but only that f be measurable. His characterization then has the third condition that $\bar{\mu}(\{x\colon |f(x)| = \infty\}) = 0$.) Finally, R. Doss [4] has proven that

$f \in \{\text{B-AP}\}$ if and only if

(a) $\|f\|_{B(1)} < \infty$ and $\lim_{u\to 0} \|f_u - f\|_{B(1)} = 0$;

(b) f is $\|\cdot\|_{B(1)}$-normal, that is, from any sequence b_n can be extracted a subsequence c_n such that

$$\lim_{m,n\to\infty} \|f_{c_n} - f_{c_m}\|_{B(1)} = 0;$$

(c) for any real λ,

$$(\mathrm{c}\lambda)\ \lim_{L\to\infty}\overline{M}_x\left|\frac{1}{L}\int_x^{x+L} f(t)e^{i\lambda t}\,dt-\frac{1}{L}\int_0^L f(t)e^{i\lambda t}\,dt\right| = 0\,.$$

The conditions (cλ) form an infinity of independent conditions.

Two other very interesting characterizations of $\{B^p\text{-AP}\}$ are in the literature, one by R. Doss [3] and one by A. S. Kovanko [10]. These involve certain functions $f^{(a)}$ of period a, where a runs through the real numbers. We shall not restate them here.

In this paper we show that $f \in \{\text{B-AP}\}$ if and only if

(A1) f is $\|\cdot\|_{B(1)}$-normal, and

(B) for all but a countable set of $\varepsilon > 0$ it is the case that

$$\overline{M}_x\overline{M}_w[|f(w+x)-f(x)|\ \chi_{BE(\varepsilon,f)}(w)] \leqq \varepsilon\bar{\mu}(BE(\varepsilon,f))\,.$$

(A1) may be replaced by the equivalent condition

(A2) for every $\varepsilon > 0$, the set $BE(\varepsilon,f)$ is relatively dense and open.

The requirement in (A2) that $BE(\varepsilon,f)$ be open may be weakened to require only that $BE(\varepsilon,f)$ be of positive measure or of second category. However some sort of "width" requirement on $BE(\varepsilon,f)$ is necessary. Examples illustrating this and other points are discussed in the last section.

2. The main theorem.

We begin with a few notational remarks, additional to those made above. We denote by R the set of real numbers and by $\alpha(\mathsf{R})$ the set of (continuous) Bohr almost periodic functions on R while μ denotes Lebesgue measure on R. If f is a measurable function on R, $\|f\|_\infty$ is its essential supremum and $\|f\|$ is its Besicovitch 1-norm:

$$\|f\| = \overline{\lim_{T\to\infty}}\,\frac{1}{2T}\int_{-T}^{T}|f|\,d\mu\,.$$

Also

$$BE(\varepsilon,f) = \{x \in R\colon\ \|f_x - f\| < \varepsilon\}\,,$$
$$E(\varepsilon,f) = \{x \in R\colon\ \|f_x - f\|_\infty < \varepsilon\}\,.$$

$L_{1,\mathrm{loc}}(\mathsf{R})$ denotes the set of all complex-valued functions f on R such that $f \in L_1(-T,T)$ for all $T > 0$. Notice that $\|f_a\| = \|f\|$ for all $f \in L_{1,\mathrm{loc}}(\mathsf{R})$ and all $a \in \mathsf{R}$ even when one side is ∞. Indeed

$$\|f_a\| = \overline{\lim_{T\to\infty}} \left[\frac{1}{2T}\int_{-T+a}^{T+a} |f|\, d\mu\right]$$

$$\leqq \overline{\lim_{T\to\infty}} \left[\frac{2(T+|a|)}{2T}\,\frac{1}{2(T+|a|)}\int_{-(T+|a|)}^{T+|a|} |f|\, d\mu\right] = \|f\|\,,$$

which includes the opposite inequality. By a δ-mesh in a metric space is meant a finite set of points of the space such that every point of the space is within δ of one of the points of the finite set. Finally, we introduce two conditions for a function $f \in L_{1,\,\mathrm{loc}}(\mathsf{R})$:

(A3) for every $\varepsilon > 0$, the set $BE(\varepsilon,f)$ is relatively dense and either of positive measure or of second category;

(A4) for every $\varepsilon > 0$ there exists a finite set $w_1,\ldots,w_n \in \mathsf{R}$ such that $\mathsf{R} = \bigcup_{i=1}^{n} [w_i + BE(\varepsilon,f)]$.

2.1. Proposition. *For a function* $f \in L_{1,\,\mathrm{loc}}(\mathsf{R})$ *the conditions* (A1), (A2), (A3), (A4) *are equivalent.*

Proof. Suppose f satisfies (A1) and take $\varepsilon > 0$. As $(\{f_a\colon a \in \mathsf{R}\}, \|\cdot\|)$ is conditionally compact it is totally bounded so it contains an ε-mesh, say $f_{u_1},\ldots,f_{u_n}$. If $u \in \mathsf{R}$, then for some i

$$\|f_{u-u_i} - f\| = \|f_u - f_{u_i}\| < \varepsilon\,,$$

so $u \in u_i + BE(\varepsilon,f)$. Thus $\mathsf{R} = \bigcup_{i=1}^{n}[u_i + BE(\varepsilon,f)]$. ε being arbitrary, f satisfies (A4).

Suppose f satisfies (A4). For each $\varepsilon > 0$, the set $BE(\varepsilon,f)$ is clearly measurable and it follows from (A4) that it is of positive measure and of second category. To see that each $BE(\varepsilon,f)$ is relatively dense take $\varepsilon > 0$ and

$$L > 2\sup\{|w_i|\colon\ 1 \leqq i \leqq n\}\,,$$

where the w_i are as in (A4). We claim that $BE(\varepsilon,f)$ meets every interval of the form $(a, a+L)$, $a \in \mathsf{R}$. Indeed if $a \in \mathsf{R}$, then $a + \frac{1}{2}L = w_i + b$ for some $b \in BE(\varepsilon,f)$. Hence

$$b = a + \tfrac{1}{2}L - w_i \in (a, a+L)\,.$$

Consequently $BE(\varepsilon,f)$ is relatively dense, whence (A4) implies (A3).

Assuming f satisfies (A3) we show f satisfies (A2) by showing that each $BE(\varepsilon,f)$ is open. Now

$$BE(\varepsilon,f) \supset \{x - y\colon\ x, y \in BE(\tfrac{1}{2}\varepsilon,f)\}\,.$$

It follows from the fact that $BE(\frac{1}{2}\varepsilon,f)$ is Borel and of positive measure or from the fact that $BE(\frac{1}{2}\varepsilon,f)$ is of second category that the right side, above, contains a neighborhood of 0 (cf. [7, 61.3], [9, Chap. 6, problem P]). Thus each $BE(\varepsilon,f)$ contains a neighborhood of 0. Now for $\varepsilon>0$ take $b\in BE(\varepsilon,f)$. Take $\varepsilon_1>0$ such that

$$\|f_b-f\|+\varepsilon_1 < \varepsilon .$$

Let $\delta>0$ be such that $(-\delta,\delta)\subset BE(\varepsilon_1,f)$. Then, if $|b-c|<\delta$ we have

$$\begin{aligned}\|f_c-f\| = \|f_{-c}-f\| &= \|f_{b-c}-f_b\|\\ &\leqq \|f_{b-c}-f\| + \|f-f_b\|\\ &< \varepsilon_1 + \|f_b-f\| < \varepsilon ,\end{aligned}$$

whence $c\in BE(\varepsilon,f)$. It follows that $BE(\varepsilon,f)$ is open and, ε being arbitrary, (A3) implies (A2).

Suppose f satisfies (A2) and take $\varepsilon>0$. As $0\in BE(\varepsilon,f)$ and $BE(\varepsilon,f)$ is open, there exists $\delta>0$ such that $(-\delta,\delta)\subset BE(\varepsilon,f)$. Take $L>0$ such that every interval of length L meets $BE(\varepsilon,f)$. As

$$BE(2\varepsilon,f) \supset BE(\varepsilon,f)+BE(\varepsilon,f) ,$$

every interval of length L contains an interval of length δ all of whose points are in $BE(2\varepsilon,f)$. Let n be an integer larger than $2L/\delta$. Then

$$\bigcup_{i=-n}^{n}[\tfrac{1}{2}i\delta+BE(2\varepsilon,f)] = \mathsf{R} .$$

Consequently $\{f_{\frac{1}{2}i\delta}\colon -n\leqq i\leqq n\}$ is a 2ε-mesh in $(\{f_a\colon a\in\mathsf{R}\}, \|\cdot\|)$. As $\varepsilon>0$ is arbitrary, $(\{f_a\colon a\in\mathsf{R}\}, \|\cdot\|)$ is totally bounded and (A1) follows. This proves the proposition.

2.2. LEMMA. *If $f\in\{$B-AP$\}$, then f satisfies* (A3). *Hence f also satisfies* (A1), (A2) *and* (A4).

PROOF. This is well-known and follows from the fact that if $g\in\alpha(\mathsf{R})$ then

$$\|f_u-f\| \leqq \|f_u-g_u\| + \|g_u-g\| + \|g-f\| \leqq 2\|f-g\| + \|g_u-g\|_\infty .$$

Thus if $\|f-g\|<\frac{1}{3}\varepsilon$ we get that $E(\frac{1}{3}\varepsilon,g)\subset BE(\varepsilon,f)$. The set $E(\frac{1}{3}\varepsilon,g)$ is relatively dense and contains a neighborhood of 0, as g is uniformly continuous. Since ε is arbitrary, f satisfies (A3).

2.3 LEMMA. *Let $f\in L_{1,\mathrm{loc}}(\mathsf{R})$ satisfy* (A2). *For $x\in\mathsf{R}$ define $h(x)=\|f_x-f\|$. Then $h\in\alpha(\mathsf{R})$ and for every $\varepsilon>0$,*

$$E(\varepsilon,h) = BE(\varepsilon,f) .$$

PROOF. By (A2), the set $B(1,f)$ contains a neighborhood of 0, so $h(x)$ is finite for x near 0. As

$$h(nx) \leqq \|f_{nx}-f_{(n-1)x}\| + \cdots + \|f_x-f\| = nh(x)\ ,$$

it follows that $h(x)$ is finite for all $x \in \mathsf{R}$. Take $\varepsilon > 0$. From

$$\|h_u-h\|_\infty = \sup_{x\in\mathsf{R}}\big|\|f_{x+u}-f\|-\|f_x-f\|\big| \leqq \sup_{x\in\mathsf{R}}\|f_{x+u}-f_x\| = \|f_u-f\|$$

it follows that $BE(\varepsilon,f)\subset E(\varepsilon,h)$. On the other hand, if $u\in E(\varepsilon,h)$, then

$$\sup_{x\in\mathsf{R}}\big|\|f_{x+u}-f\|-\|f_x-f\|\big| < \varepsilon\ .$$

Letting $x=0$ gives $\|f_u-f\|<\varepsilon$, so $u\in BE(\varepsilon,f)$. Thus $E(\varepsilon,h)\subset BE(\varepsilon,h)$. As $\varepsilon>0$ is arbitrary, we have $E(\varepsilon,h)=BE(\varepsilon,f)$ for all $\varepsilon>0$. That $h\in\alpha(\mathsf{R})$ now follows from the fact that f satisfies (A2).

2.4 NOTATION. We let $\overline{\mathsf{R}}$ denote the Bohr compactification of R and consider R as a dense subset of $\overline{\mathsf{R}}$. For $f\in\alpha(\mathsf{R})$ we let $\bar{f}$ denote its continuous extension to $\overline{\mathsf{R}}$. Letting $C(\overline{\mathsf{R}})$ denote the set of continuous complex valued functions on $\overline{\mathsf{R}}$, we get that $f\to\bar{f}$ is a vector space isomorphism from $\alpha(\mathsf{R})$ onto $C(\overline{\mathsf{R}})$. See, for example, [9, pp. 247–249]. If $A\subset\overline{\mathsf{R}}$, we let A^{C} denote its closure. If $\bar{h}\in C(\overline{\mathsf{R}})$, we define

$$E(\varepsilon,\bar{h}) = \{x\in\overline{\mathsf{R}}\colon \|\bar{h}_x-\bar{h}\|_\infty<\varepsilon\}\ ,$$

for each $\varepsilon>0$. Finally, we let ν denote Haar measure on $\overline{\mathsf{R}}$.

2.5 LEMMA. *Let $\bar{h}\in C(\overline{\mathsf{R}})$. Then for all but a countable set of $\varepsilon>0$ we have*

$$\nu(E(\varepsilon,\bar{h})) = \nu(E(\varepsilon,\bar{h})^{\mathrm{C}})\ .$$

PROOF. Take $\varepsilon>0$ and let $A_\varepsilon=\{x\in\overline{\mathsf{R}}\colon \|\bar{h}_x-\bar{h}\|_\infty=\varepsilon\}$. As $\bar{h}$ is uniformly continuous, $\|\bar{h}_x-\bar{h}\|_\infty$ is a continuous function of $x\in\mathsf{R}$. Thus

$$A_\varepsilon \supset E(\varepsilon,\bar{h})^{\mathrm{C}} \sim E(\varepsilon,\bar{h})\ .$$

The sets $\{A_\varepsilon\}_{\varepsilon>0}$ are pairwise disjoint. If $\nu(A_\varepsilon)>0$ for uncountably many ε, we would have that $\nu(\overline{\mathsf{R}})=\infty$, contrary to the compactness of $\overline{\mathsf{R}}$. Hence for all but a countable set of $\varepsilon>0$ we have $\nu(A_\varepsilon)=0$. Further $\nu(E(\varepsilon,\bar{h})^{\mathrm{C}})=\nu(E(\varepsilon,\bar{h}))$ for such ε.

2.6 NOTATION. For $h\in\alpha(\mathsf{R})$ we define

$$T(h) = \{\eta>0\colon \nu(E(\eta,\bar{h})^{\mathrm{C}})=\nu(E(\eta,\bar{h}))\}\ .$$

By 2.5, the set $T(h)$ contains all but a countable set of the positive numbers.

2.7 **Lemma.** *If* $h \in \alpha(\mathsf{R})$ *and* $\eta \in T(h)$, *then* $\chi_{E(\eta,h)} \in \{\text{B-AP}\}$.

Proof. Observe that $E(\eta,h) = E(\eta,\bar{h}) \cap \mathsf{R}$ and $E(\eta,\bar{h})$ is open in $\bar{\mathsf{R}}$. Take $\bar{a}_n \in C(\bar{\mathsf{R}})$ such that $\bar{a}_n \uparrow \chi_{E(\eta,\bar{h})}$ and $\bar{a}_n(x) \geqq 0$ for all $x \in \bar{\mathsf{R}}$. Then

$$0 \leqq \int_{\bar{\mathsf{R}}} (\chi_{E(\eta,\bar{h})} - \bar{a}_n)\, d\nu \to 0 \quad \text{as } n \to \infty .$$

As $\eta \in T(h)$, we may apply Theorem 26.17 of [8] to conclude that

$$\bar{M}|\chi_{E(\eta,h)} - a_n| = M[\chi_{E(\eta,h)} - a_n] \to 0 \quad \text{as } n \to \infty .$$

The lemma follows.

2.8 **Theorem.** *If* $f \in \{\text{B-AP}\}$, *then* f *satisfies* (B).

Proof. Take $f \in \{\text{B-AP}\}$ and define $h(x) = \|f_x - f\|$ for all $x \in \mathsf{R}$. By 2.2, 2.3 and 2.5 it suffices to show that for all ε in $T(h)$,

$$(1) \qquad \bar{M}_x \bar{M}_w[|f(w+x) - f(x)|\,\chi_{BE(\varepsilon,f)}(w)] \leqq \varepsilon \bar{\mu}(BE(\varepsilon,f)) .$$

Take any ε in $T(h)$. Take $f_n \in \alpha(\mathsf{R})$ (and not identially 0) such that $\|f - f_n\| \to 0$ as $n \to \infty$. By 2.2, 2.3 and 2.7, $\chi_{BE(\varepsilon,f)} \in \{\text{B-AP}\}$. Take $b_n \in \alpha(\mathsf{R})$ such that

$$(2) \qquad \|\chi_{BE(\varepsilon,f)} - b_n\| \leqq 1/(n\|f_n\|_\infty), \quad n = 1, 2, \dots .$$

We shall show that

$$(3) \qquad \bar{M}_x \bar{M}_w[|f(w+x) - f(x)|\,\chi_{BE(\varepsilon,f)}(w)] \\ = \lim_{n\to\infty} M_x M_w[|f_n(w+x) - f_n(x)|\,b_n(w)] .$$

For any fixed $x \in \mathsf{R}$, we have $|f(w+x) - f(x)| \in \{\text{B-AP}\}$ whence also

$$|f(w+x) - f(x)|\chi_{BE(\varepsilon,f)}(w) \in \{\text{B-AP}\}$$

(cf. [6, page 7]). Thus $M_w[|f(w+x) - f(x)|\,\chi_{BE(\varepsilon,f)}(w)]$ exists for each $x \in \mathsf{R}$. Also

$$\begin{aligned} &\bar{M}_x|M_w[|f(w+x) - f(x)|\,\chi_{BE(\varepsilon,f)}(w)] - M_w[|f_n(w+x) - f_n(x)|\,b_n(w)]| \\ &\quad \leqq \bar{M}_x|M[(|f_x - f(x)| - |f_{nx} - f_n(x)|)\chi_{BE(\varepsilon,f)}]| \\ &\qquad + \bar{M}_x|M(|f_{nx} - f_n(x)|\,\chi_{BE(\varepsilon,f)}) - M(|f_{nx} - f_n(x)|\,b_n)| \\ &\quad \leqq \bar{M}_x|M|f_x - f(x) - f_{nx} + f_n(x)|| + \bar{M}_x|M[|f_{nx} - f_n(x)|(\chi_{BE(\varepsilon,f)} - b_n)]| \\ &\quad \leqq \bar{M}_x[M|f_x - f_{nx}| + M|f(x) - f_n(x)|] + 2\|f_n\|_\infty\|\chi_{BE(\varepsilon,f)} - b_n\| \\ &\quad \leqq 2\|f - f_n\| + 2/n \to 0 \quad \text{as } n \to \infty, \quad \text{by (2)} . \end{aligned}$$

Thus, as a function of x,

$$M_w[|f(w+x) - f(x)|\,\chi_{BE(\varepsilon,f)}(w)] \in \{\text{B-AP}\}$$

and (3) holds. We now show that

$$\text{(4)}\qquad \bar{M}_w \bar{M}_x[|f(w+x)-f(x)|\,\chi_{BE(\varepsilon,f)}(w)] = \lim_{n\to\infty} M_w M_x[|f_n(w+x)-f_n(x)|\,b_n(w)]\;.$$

For fixed $w \in \mathsf{R}$,

$$|f(w+x)-f(x)|\,\chi_{BE(\varepsilon,f)}(w) \in \{\text{B-AP}\}\;,$$

so $M_x[|f(w+x)-f(x)|\chi_{BE(\varepsilon,f)}(w)]$ exists for each $w \in \mathsf{R}$. Arguing as before,

$$\begin{aligned}
&\bar{M}_w|M(|f_w-f|\,\chi_{BE(\varepsilon,f)}(w)) - M(|f_{nw}-f_n|\,b_n(w))| \\
&\quad \leqq \bar{M}_w|M[(|f_w-f|-|f_{nw}-f_n|)\chi_{BE(\varepsilon,f)}(w)]| \\
&\qquad + \bar{M}_w|M(|f_{nw}-f_n|\,\chi_{BE(\varepsilon,f)}(w)) - M(|f_{nw}-f_n|\,b_n(w))| \\
&\quad \leqq \bar{M}_w|M|f_w-f-f_{nw}+f_n|| + \bar{M}_w|M[|f_{nw}-f_n|\,(\chi_{BE(\varepsilon,f)}(w)-b_n(w))]| \\
&\quad \leqq \bar{M}_w|M|f_w-f_{nw}| + M|f-f_n|| + 2\,\|f_n\|_\infty \|\chi_{BE(\varepsilon,f)}-b_n\| \\
&\quad \leqq 2\|f-f_n\| + 2/n \to 0 \quad \text{as } n\to\infty, \text{ by (2)}\;.
\end{aligned}$$

Thus, as a function of w,

$$M_x[|f(w+x)-f(x)|\chi_{BE(\varepsilon,f)}(w)] \in \{\text{B-AP}\}$$

and (4) holds. As $f_n, b_n \in \alpha(\mathsf{R})$,

$$M_x M_w[|f_n(w+x)-f_n(x)|\,b_n(w)] = M_w M_x[|f_n(w+x)-f_n(x)|\,b_n(w)]$$

for all $n = 1,2,\ldots$. Applying this to (3) and (4) gives

$$\begin{aligned}
\bar{M}_x\bar{M}_w[|f(w+x)-f(x)|\,\chi_{BE(\varepsilon,f)}(w)] &= \bar{M}_w\bar{M}_x[|f(w+x)-f(x)|\,\chi_{BE(\varepsilon,f)}(w)] \\
&= \bar{M}_w[\|f_w-f\|\,\chi_{BE(\varepsilon,f)}(w)] \\
&\leqq \varepsilon\bar{\mu}(BE(\varepsilon,f))\;.
\end{aligned}$$

This proves (1), from which the theorem follows.

2.9 LEMMA. *Let $f \in L_{1,\text{loc}}(\mathsf{R})$ and suppose that for some $\varepsilon > 0$*

$$\text{(1)}\qquad \bar{M}_x\bar{M}_w\,|f(w+x)-f(x)|\,\chi_{BE(\varepsilon,f)}(w)] < \infty\;.$$

Suppose $0 < L_i \uparrow \infty$ as $i \to \infty$ and let U be an open set in R. Then there exists $u \in U$ and a subsequence $\{L_i'\}$ of $\{L_i\}$ such that

$$\lim_{i\to\infty} \frac{1}{2L_i'} \int_{-L_i'}^{L_i'} |f_u|\,\chi_{BE(\varepsilon,f)}\,d\mu$$

exists and is finite.

PROOF. We may as well assume U is bounded. We show first that there exists $u \in U$ and a subsequence $\{L_i''\}$ of $\{L_i\}$ such that

(2) $$\lim_{i\to\infty} \frac{1}{2L_i''} \int_{-L_i''}^{L_i''} |f(u+w)-f(u)|\,\chi_{BE(\varepsilon,f)}(w)\, d\mu(w)$$

exists and is finite. Otherwise

$$\bar{M}_w[|f(u+w)-f(u)|\,\chi_{BE(\varepsilon,f)}(w)] = \infty$$

for all $u \in U$. Take T_0 such that $U \subset (-T_0, T_0)$. Then for every $T \geqq T_0$,

$$\infty = \frac{1}{T}\int_U \bar{M}[|f_x - f(x)|\,\chi_{BE(\varepsilon,f)}]\, d\mu(x) \leqq \frac{1}{T}\int_{-T}^{T} \bar{M}[|f_x - f(x)|\,\chi_{BE(\varepsilon,f)}]\, d\mu(x)\,,$$

contrary to (1).

From the fact that (2) exists and is finite it follows that there exists $N < \infty$ such that

$$\begin{aligned} 0 &\leqq \frac{1}{2L_i''}\int_{-L_i''}^{L_i''} |f_u|\,\chi_{BE(\varepsilon,f)}\, d\mu \\ &\leqq \frac{1}{2L_i''}\int_{-L_i''}^{L_i''} |f(u+w)-f(u)|\,\chi_{BE(\varepsilon,f)}(w)\, d\mu(w) + \\ &\quad + \frac{1}{2L_i''}\int_{-L_i''}^{L_i''} |f(u)|\,\chi_{BE(\varepsilon,f)}(w)\, d\mu(w) \\ &< N \quad \text{for all } i = 1,2,\ldots\,. \end{aligned}$$

Thus we may take $\{L_i'\}$ to be a suitable subsequence of $\{L_i''\}$, proving the lemma.

2.10 Lemma. *If* $h \in \alpha(\mathsf{R})$, *then* $\bar{\mu}(E(\varepsilon,h)) > 0$ *for every* $\varepsilon > 0$.

Proof. Take $\varepsilon > 0$. Take $\delta > 0$ such that $(-\delta,\delta) \subset E(\frac{1}{2}\varepsilon,h)$. Take $L > 0$ such that $E(\frac{1}{2}\varepsilon,h) \cap (kL,(k+1)L) \neq \emptyset$ for all $k = 0, \pm 1, \ldots$. Then

$$\begin{aligned} \bar{\mu}(E(\varepsilon,h)) &\geqq \overline{\lim_{n\to\infty}} \frac{1}{2nL} \sum_{k=-n}^{n-1} \int_{kL}^{(k+1)L} \chi_{E(\varepsilon,h)}\, d\mu \\ &\geqq \overline{\lim_{n\to\infty}} \frac{1}{2nL}(2n\delta) = \delta/L > 0\,. \end{aligned}$$

2.11 Theorem. *Let* $f \in L_{1,\mathrm{loc}}(\mathsf{R})$ *satisfy* (A2) *and* (B). *Then* $f \in$ {B-AP}.

Proof. Take arbitrary $\delta > 0$. For $x \in \mathsf{R}$ define $h(x) = \|f_x - f\|$. By 2.3, $h \in \alpha(\mathsf{R})$ and $E(\varepsilon,h) = BE(\varepsilon,f)$ for all $\varepsilon > 0$. By (B) and 2.5 there exists ε in $T(h)$ such that

(1) $\varepsilon < \delta$ and $\bar{M}_x \bar{M}_w[|f(x+w)-f(x)|\chi_{BE(\varepsilon,f)}(w)] \leqq \varepsilon\bar{\mu}(BE(\varepsilon,f))$.

By 2.10 we have $\bar{\mu}(BE(\varepsilon,f))>0$ so we may define

(2) $$g = \chi_{BE(\varepsilon,f)}/\bar{\mu}(BE(\varepsilon,f)) .$$

By 2.7, $g \in \{\text{B-AP}\}$. Hence

(3) $$\begin{cases} Mg \text{ exists and } Mg=1; \\ 0 \leqq g(x) \leqq \|g\|_\infty < \infty \text{ for all } x \in \mathrm{R} . \end{cases}$$

From (1), (2) we get that

(4) $$\bar{M}_x \bar{M}_w[|f(x+w)-f(x)|g(w)] < \delta .$$

Take $j \in \{1,2,\ldots\}$. By (A4) there exists a finite set $F_j \subset \mathrm{R}$ such that for every $y \in \mathrm{R}$ there is some $v \in F_j$ with $y=w+v$ for some $w \in BE(1/j,f)$. Hence

$$\|f_y-f_v\| = \|f_{w+v}-f_v\| = \|f_w-f\| < 1/j .$$

As $BE(1/j,f)$ contains a neighborhood of 0, there is a neighborhood U_v of each $v \in F_j$ such that the following holds: if $y \in \mathrm{R}$ then there is some $v \in F_j$ such that

(5) $$\|f_y-f_u\| < 2/j \quad \text{for all } u \in U_v .$$

Let $F=\bigcup_{j=1}^\infty F_j$. We shall apply lemma 2.9. By (1), the hypotheses of 2.9 are satisfied. The role of U in the lemma is to be taken by the sets U_v, $v \in F$. We enumerate the sets U_v, say $U_1, U_2, \ldots$. Take any sequence $\{T_i\}$ such that $0<T_i \uparrow \infty$ as $i \to \infty$. By applying 2.9 inductively and using a diagonal argument we get a subsequence $\{L_i'\}$ of $\{T_i\}$ and a set $\{x_1,x_2,\ldots\} \subset \mathrm{R}$ such that $x_j \in U_j$, $j=1,2,\ldots$, and

$$\lim_{i\to\infty} \frac{1}{2L_i'} \int_{-L_i'}^{L_i'} |f_{x_j}|\chi_{BE(\varepsilon,f)}\,d\mu$$

exists and is finite for all $j=1,2,\ldots$, Let $D=\{x_1,x_2,\ldots\}$. Because of (5) and the way D is formed, we have that for every $\eta>0$ and every $y \in \mathrm{R}$ there exists $x \in D$ such that

(6) $$\|f_y-f_x\| < \eta .$$

If $x \in D$, then (2), (3) and the above considerations show that

$$\begin{aligned} 0 &\leqq \overline{\lim_{i\to\infty}} \left| \frac{1}{2L_i'} \int_{-L_i'}^{L_i'} f_x g\, d\mu \right| \\ &\leqq \|g\|_\infty \left[\overline{\lim_{i\to\infty}} \frac{1}{2L_i'} \int_{-L_i'}^{L_i'} |f_x|\chi_{BE(\varepsilon,f)}\,d\mu \right] < \infty . \end{aligned}$$

As D is countable, we may use a second diagonal argument to get a subsequence $\{L_i\}$ of $\{L_i'\}$ such that

$$\lim_{i\to\infty} \frac{1}{2L_i} \int_{-L_i}^{L_i} f_x g \, d\mu \tag{7}$$

is finite for all $x \in D$.

We now show that (7) exists and is finite for all $x \in \mathsf{R}$. Indeed take any $y \in \mathsf{R}$ and any $\eta > 0$. Take $x \in D$ such that (6) holds. There exists i_0 such that

$$\frac{1}{2L_i} \int_{-L_i}^{L_i} |f_y - f_x| \, d\mu < \eta$$

for all $i \geqq i_0$. Thus, for $i \geqq i_0$

$$\left| \frac{1}{2L_i} \int_{-L_i}^{L_i} f_y g \, d\mu - \frac{1}{2L_i} \int_{-L_i}^{L_i} f_x g \, d\mu \right| \leqq \|g\|_\infty \frac{1}{2L_i} \int_{-L_i}^{L_i} |f_y - f_x| \, d\mu < \eta \|g\|_\infty .$$

Consequently, for large values of i,

$$\frac{1}{2L_i} \int_{-L_i}^{L_i} f_y g \, d\mu$$

is within $2\eta \|g\|_\infty$ of the finite number

$$\lim_{i\to\infty} \frac{1}{2L_i} \int_{-L_i}^{L_i} f_x g \, d\mu .$$

As $\eta > 0$ is arbitrary, we get by the Cauchy criterion that

$$\varphi(y) = \lim_{i\to\infty} \frac{1}{2L_i} \int_{-L_i}^{L_i} f_y g \, d\mu$$

is a finite number for all $y \in \mathsf{R}$. Now for $x, u \in \mathsf{R}$ we have

$$|\varphi(u+x) - \varphi(x)| \leqq \|g\|_\infty \|f_{u+x} - f_x\| \leqq \|g\|_\infty \|f_u - f\| .$$

Thus for every $\eta > 0$,

$$BE(\eta/\|g\|_\infty, f) \subset E(\eta, \varphi) .$$

Using (A2), it follows that $\varphi \in \alpha(\mathsf{R})$.

We now show that $\|\varphi - f\| < \delta$. As $M[g] = 1$, we get for any $x \in \mathsf{R}$ that

$$|\varphi(x)-f(x)| = \lim_{i\to\infty}\left|\frac{1}{2L_i}\int_{-L_i}^{L_i}[f(x+w)-f(x)]g(w)\,d\mu(w)\right|$$
$$\leqq \bar{M}_w[|f(x+w)-f(x)|g(w)] .$$

Hence,

$$\|\varphi-f\| = \bar{M}_x|\varphi(x)-f(x)| \leqq \bar{M}_x\bar{M}_w[|f(x+w)-f(x)|g(w)] < \delta ,$$

by (4). As $\delta > 0$ is arbitrary, and $\varphi \in \alpha(\mathsf{R})$, the theorem follows.

MAIN THEOREM. *Let* $f \in L_{1,\mathrm{loc}}(\mathsf{R})$. *Then* $f \in \{\text{B-AP}\}$ *if and only if* f *satisfies* (A1) *and* (B). *The condition* (A1) *may be replaced by any of the equivalent conditions* (A2), (A3) *or* (A4).

PROOF. This follows from 2.1, 2.2, 2.8 and 2.11.

3. Further comments.

The most natural ways of weakening (A3) and (B) fail to give a characterization of {B-AP}. For example, (A3) alone will not characterize {B-AP}, as is illustrated by the example on page 5 of [5].

By the phrase "$BE(\varepsilon,f)$ has width" we mean that it is either of positive measure or of second category. We have seen that if $BE(\varepsilon,f)$ has width for every $\varepsilon > 0$, then each $BE(\varepsilon,f)$ is in fact open. Condition (B) plus the requirement that each $BE(\varepsilon,f)$ have width fails to characterize {B-AP}, as is illustrated by the function $f(x)=x$: it is certainly not B-AP, while $BE(\varepsilon,f)=(-\varepsilon,\varepsilon)$ and

$$\bar{M}_x\bar{M}_w[|f(x+w)-f(x)|\chi_{BE(\varepsilon,f)}(w)] = 0$$

for all $\varepsilon > 0$.

If for some $\varepsilon > 0$, the set $BE(\varepsilon,f)$ has no width, then, by the main theorem, $f \notin \{\text{B-AP}\}$. It is conceivable, however, that if f satisfies (B) and each $BE(\varepsilon,f)$ is relatively dense, then each $BE(\varepsilon,f)$ has width, whence $f \in \{\text{B-AP}\}$. In other words it is conceivable that the width requirement can be removed from (A3). We show that this is false by giving a function f such that for every $\varepsilon > 0$ the set $BE(\varepsilon,f)$ is relatively dense but has no width. (Hence $\mu(BE(\varepsilon,f))=0$ so that f satisfies (B) trivially.) The function f will also satisfy

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f\,d\mu \quad \text{exists} . \tag{1}$$

We define f as follows: On $[0,3)$ and on all intervals of the form $[3n,3n+1)$, $n \in \{\pm 1, \pm 2,\dots\}$, we define $f \equiv 0$. Suppose $k=3n+1$ for some $n \in \{\pm 1, \pm 2,\dots\}$. Then on $[k,k+1)$ we define f to be

$$(2)\qquad (2/1)\chi_{(k,\,k+1/2)}+(2^2/2)\chi_{(k+1/2,\,k+1/2+1/4)}+\ldots+ \\ +(2^n/|n|)\chi_{(k+1/2+\ldots+1/2^{|n|-1},\,k+1/2+\ldots+1/2^{|n|})}\,.$$

If $k=3n+2$ for some $n\in\{\pm 1, \pm 2,\ldots\}$, we define f on $[k,k+1)$ to be the negative of (2).

Let $S(n)=1+1/2+\ldots+1/n$ and for $T>0$ let $n(T)$ be the smallest positive integer such that $T\leqq 3n(T)$. For $T>0$ we have

$$\left|\frac{1}{2T}\int_{-T}^{T} f\,d\mu\right| \leqq \frac{1}{2T}[2\,S(n(T))]$$
$$\leqq \left(\frac{n(T)}{3n(T)-3}\right)\left(\frac{S(n(T))}{n(T)}\right)$$
$$\leqq \frac{\log(n(T))+1}{n(T)} \to 0 \quad \text{as } T\to\infty\,.$$

Thus f satisfies (1).

Take any $i\in\{0,\pm 1,\ldots\}$. Due to the cancellations which occur when we subtract the graph of f from the graph of f_{3i}, we get that for $T>0$

$$\frac{1}{2T}\int_{-T}^{T} |f(x+3i)-f(x)|\,d\mu(x)$$
$$\leqq (1/2T)[k_i+4\,|i|\,S(|i|+n(T))]$$
$$\leqq (k_i/2T)+4\,|i|\left(\frac{|i|+n(T)}{6n(T)-6}\right)\frac{S(|i|+n(T))}{(|i|+n(T))}\to 0 \quad \text{as } T\to\infty;$$

here k_i is a constant which covers the behavior of the graph of $f_{3i}-f$ near zero. It follows that $\{0,\pm 3,\pm 6,\ldots\}\subset BE(\varepsilon,f)$ for every $\varepsilon>0$, whence each $BE(\varepsilon,f)$ is relatively dense.

Take $\varepsilon>0$. We shall show that $BE(\varepsilon,f)$ does not contain a neighborhood of 0. Otherwise, there exists $0<\delta<1$ such that

$$(3)\qquad \|f_\delta - f\| < \varepsilon\,.$$

For positive integers n,k with $n>k$, let $S(k,n)=1/k+\ldots+1/n$. If $k>1$ is such that

$$1/2+1/4+\ldots+1/2^{k-1} > \delta$$

and $n>k$, then

$$\frac{1}{6n}\int_{-3n}^{3n} |f_\delta - f|\, d\mu \geqq \frac{1}{6n}\sum_{i=k+1}^{n} S(k,i)$$
$$\geqq \frac{1}{6n}\sum_{i=k+1}^{n} [\log i - \log k]$$
$$\geqq \frac{1}{6n}\left[\int_{k}^{n} (\log x)\, d\mu(x) - (n-k-1)\log k\right]$$
$$= \tfrac{1}{6}[\log n - 1 - \log k + (k+\log k)/n] \to \infty \quad \text{as } n \to \infty .$$

This contradicts (3). Hence $BE(\varepsilon,f)$ does not contain a neighborhood of 0. As $\varepsilon > 0$ is arbitrary, none of the sets $BE(\varepsilon,f)$ have width, as was to be shown.

We do not know if (B) can be replaced by the stronger condition

(B)′ for every $\varepsilon > 0$

$$\bar{M}_x \bar{M}_w[|f(x+w) - f(x)|\, \chi_{BE(\varepsilon,f)}(w)] \leqq \varepsilon \bar{\mu}(BE(\varepsilon,f)) .$$

Nor do we know if $\{B^p\text{-AP}\}$ can be characterized by inserting the parameter p into (Ai) and (B), $1 \leqq i \leqq 4$, $1 < p < \infty$.

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