ON MODELS WITH UNDEFINABLE ELEMENTS

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The following problem was posed by C. Ryll-Nardzewski (cf. [3]): Is there a complete theory $T$ formulated in a first order language with only finitely many non-logical symbols and which has the following properties:

(i) $T$ has a model $\mathfrak{A}$ every element of which is first order definable in $\mathfrak{A}$; hence $\mathfrak{A}$ is a prime model of $T$.

(ii) For every set $\mathcal{X}$ of non-principal (dual) prime ideals of $F_1(T)$ — the Boolean algebra of formulas with $v_0$ as only free variable taken modulo equivalence in $T$ — there is a model $\mathfrak{B}$ of $T$ (which can be taken to be an elementary extension of $\mathfrak{A}$) such that the non-principal prime ideals of $F_1(T)$ which are realized in $\mathfrak{B}$ are exactly those in $\mathcal{X}$; furthermore those prime ideals are realized by exactly one element each.

This problem is an extension of an earlier problem which had been solved in [2]. We give a partial answer to the extended problem in the following

**Theorem.** There is a complete extension $T$ of the first order theory of linear orderings such that: (i) holds; and there is a set $\mathcal{Y}$ of non-principal prime ideals of $F_1(T)$, $\mathcal{Y}$ being of the power of the continuum, such that (ii) holds for subsets of $\mathcal{Y}$.

The theory $T$ will be described as the elementary theory of a particular model $\mathfrak{A}$ which we are now going to describe. Let $Q$ be the set of rational numbers; $\langle r_n : n \in \omega \rangle$ be an enumeration (without repetition) of $Q$; $\langle t_n : n \in \omega \rangle$ a family of positive irrational numbers which are linearly independent over the rationals. For $n \in \omega$ put

$$B_n = \{r_n - t_n \cdot (i + 1)^{-1} : i \in \omega \}.$$

Note: (i) the sets $B_n$ are pairwise disjoint; (ii) each $B_n$ has order type $\omega$; (iii) $\sup B_n = r_n$; (iv) for any real numbers $x$ and $y$, if $x < y$ then there are arbitrarily large $n \in \omega$ such that for some $z \in B_n$, $x < z < y$.

Put $Q' = Q \times \{0\}$, and for $n \in \omega$, $B'_n = B_n \times \{1, \ldots, n+2\}$; finally put

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$A = Q' \cup \bigcup \{B_n' : n \in \omega\}$ and $\mathcal{A} = \langle A, \prec \rangle$, where $\prec$ is the lexicographical ordering. Note:

(1.1) $\mathcal{A}$ is a linearly ordered system.

(1.2) The subsystem of $\mathcal{A}$ determined by $Q'$ is isomorphic to the ordered system of the rationals.

(1.3) The sets $B_n'$ are pairwise disjoint.

(1.4) Each $B_n'$ has order type $\omega$.

(1.5) For each $n \in \omega$, $\sup B_n' = \langle r_n, 0 \rangle$.

(1.6) For any $x, y \in A$, if $x \prec y$ and if the interval from $x$ to $y$ is not finite, then there are arbitrarily large $n$ such that for some $z \in B_n'$, $x \prec z \prec y$.

By definition $\mathcal{A}$ is a model of $T$. Hence in order to establish part (i) of the theorem it suffices to show that every element of $A$ is first order definable in $\mathcal{A}$. This is seen as follows:

(2.1) For each $n \in \omega$, the set $B_n'$ is first order definable in $\mathcal{A}$, viz. by the property of belonging to a maximal discrete subset of power $n + 2$.

(2.2) For each $n \in \omega$, each element of $B_n'$ is first order definable in $\mathcal{A}$; this follows from (2.1) and (1.4).

(2.3) For each $n \in \omega$, $\langle r_n, 0 \rangle$ is first order definable in $\mathcal{A}$; this follows from (2.1) and (1.5).

In order to show part (ii) of the theorem we shall proceed as follows: Let $I$ be a set of irrational numbers. Proceed as in the construction of $\mathcal{A}$ except for taking $Q' = (Q \cup I) \times \{0\}$. Call the resulting model $\mathcal{B}$. We shall show:

1° $\mathcal{B}$ is a model of $T$.

2° For each $i \in I$, the prime ideal defined by $\langle i, 0 \rangle$ in $\mathcal{B}$ is non-principal.

3° For each $i, j \in I$, if $i \neq j$ then the prime ideals defined by $\langle i, 0 \rangle$ and by $\langle j, 0 \rangle$ are distinct.

Part (ii) of the theorem will then be proved by taking $I$ the set of all irrational numbers and $Y$ the set of all non-principal prime ideals realized in the corresponding $\mathcal{B}$. Note that these are not all non-principal prime ideals of $F_1(T)$; in fact there are continuum many others. Those have the property that if they are realized in a model, then they are realized by infinitely many elements.

Let us recall Fraisse's relation $\equiv_n$, $n \in \omega$. Let $\mathcal{A}$ and $\mathcal{B}$ be as above. For $m \in \omega$, $a \in ^mA$ and $b \in ^mB$, put
\[ a \equiv_0 b \] iff \[
\{ \langle a_i, b_i \rangle : i \in m \} \text{ establishes an isomorphism between } \mathcal{U} \upharpoonright \{a_0, \ldots, a_{m-1}\} \text{ and } \mathcal{B} \upharpoonright \{b_0, \ldots, b_{m-1}\};
\]

\[ a \equiv_{n+1} b \] iff \[
\text{for each } x \in A \text{ there is a } y \in B \text{ such that } \langle x \rangle \equiv_n b \langle y \rangle \text{ and for each } y \in B \text{ there is a } x \in A \text{ such that } a \langle x \rangle \equiv_n b \langle y \rangle.
\]

The two important properties of this relation which we need are (e.g. see [1]):

(3.1) If for each \( n \in \omega, \mathcal{O} \equiv_n \mathcal{O} \), then \( \mathcal{U} \equiv \mathcal{B} \).

(3.2) For each formula \( \varphi \) in \( F_1(T) \) there is an \( n \in \omega \) (which is given by the quantifier depth of \( \varphi \)) such that for any \( x, y \in A \), if \( \langle x \rangle \equiv_n \langle y \rangle \), then \( x \) satisfies \( \varphi \) in \( \mathcal{U} \) if and only if \( y \) satisfies \( \varphi \) in \( \mathcal{U} \).

For the formulation and proof of the next lemma it is convenient to expand the language of \( T \) by the following defined symbols:

(i) \( Sv_0v_1 = v_0 < v_1 \land \neg \exists v_2 [ v_0 < v_2 \land v_2 < v_1 ] \),

(ii) \( Snv_0v_1 = \exists v_2 \cdots \exists v_n [ Sv_0v_2 \land Sv_2v_3 \land \cdots \land Sv_nv_1 ] \), \( n = 2, 3, \ldots \),

(iii) \( L_nv_0 = \exists v_1S^nv_0v_1 \), \( n = 1, 2, \ldots \),

(iv) \( R_nv_0 = \exists v_1S^nv_0v_1 \), \( n = 1, 2, \ldots \),

(v) \( Dv_0v_1 = v_0 < v_1 \land R_1v_0 \land L_1v_1 \land v_2 [ v_0 < v_2 \land v_2 < v_1 \rightarrow L_1v_2 \land R_1v_2 ] \).

(The intuitive meaning of the latter is that the interval from \( v_0 \) to \( v_1 \) is discrete; in \( \mathcal{U} \) or \( \mathcal{B} \) this implies that the interval is finite.) For \( k \in \omega \), put \( \mathcal{U}_k = (A, S^i, L^i, R^i, D^i) \) and define \( \mathcal{B}_k \) similarly.

**Lemma.** Let \( n, m \in \omega \), \( a \in \mathcal{N}A \), \( b \in \mathcal{N}B \), \( k = 3^n \). Assume

(i) \( \{ \langle a_i, b_i \rangle : i \in \omega \} \) establishes an isomorphism between \( A_k \upharpoonright \{a_0, \ldots, a_{m-1}\} \) and \( B_k \upharpoonright \{b_0, \ldots, b_{m-1}\} \); say, the \( a_i \)'s are in increasing order.

(ii) For each \( z \in B_0 \cup \ldots \cup B_{k-1} \) the following conditions hold: (a) if \( z \preceq a_0 \) or \( z \preceq b_0 \), then \( a_0 = b_0 \); (b) if \( a_m-1 \preceq z \) or \( b_m-1 \preceq z \), then \( a_m-1 = b_m-1 \); (c) if, for \( i = 0, \ldots, m-2 \), \( a_i \preceq z \preceq a_{i+1} \) or \( b_i \preceq z \preceq b_{i+1} \), then \( a_i = b_i \) and \( a_{i+1} = b_{i+1} \). Under these conditions \( a \equiv_n b \).

The proof is by induction on \( n \). We shall only treat a typical case. Assume the lemma holds for \( n \) (and all \( m \)). Given \( a \) and \( b \) satisfying conditions (i) and (ii) with \( k = 3^{n+1} \) and \( x \in A \) with \( a_i < x < a_{i+1} \). (Other cases are: \( x < a_0 \); \( a_{m-1} < x \); \( x = a_i \); and the cases with the roles of \( x \) and \( y \) interchanged.) We shall find a \( y \in B \) such that \( a \langle x \rangle \) and \( b \langle y \rangle \) satisfy conditions (i) and (ii) with \( k = 3^n \). There are various possibilities:

(I) \( x \in B_p \), for some \( p < k \), or more generally, \( a_i = b_i \) and \( a_{i+1} = b_{i+1} \). In this case take \( y = x \).
(II) $x \in B_p'$, for some $p \geq k$. Again various possibilities have to be distinguished:

(IIa) $\langle a_i, x \rangle \in D^M$. If there are at most $3^n$ elements between $a_i$ and $x$, say $h$ of them, take as $y$ the $(h+1)$st element to the right of $b_i$; if there are more than $3^n$ elements between $a_i$ and $x$ take as $y$ the $3^n$th element to the right of $b_i$.

(IIb) $\langle x, a_{i+1} \rangle \in D^M$ and the previous case does not apply. Proceed similarly.

(IIc) Neither (IIa) nor (IIb) holds though (II) holds. Let $x$ be the $h$th term of its discrete component. If $h \leq 3^n$ take as $y$ the $h$th term of a discrete component between $b_i$ and $b_{i+1}$, using (1.6); if $p-h \leq 3^n$ proceed similarly; in the remaining case take as $y$ the $\lfloor p/2 \rfloor$th term of a discrete component between $b_i$ and $b_{i+1}$, again using (1.6).

(III) $x \in Q'$. Take as $y$ any element between $b_i$ and $b_{i+1}$ belonging to the $Q'$ (of $B$).

**Corollary.** (i) $\mathfrak{A} \equiv \mathfrak{B}$, hence $\mathfrak{B}$ is a model of $T$.

(ii) Let $y \in B$, $y = \langle i, 0 \rangle$, where $i \in I$. Then the prime ideal of $F_1(T)$ which is defined by $y$ in $\mathfrak{B}$ is non-principal.

**Proof.** Part (i) follows from the lemma and (3.1). For part (ii), let $\varphi$ be an element of $F_1(T)$ satisfied by $y$ in $\mathfrak{B}$. Let $n$ be the number obtained for $\varphi$ from (3.2). Put $k = 3^n$ and let $x \in Q'$ such that for no $z \in B'_0 \cup \ldots \cup B'_{k-1}, z$ is between $x$ and $y$. Let $\chi$ be a formula with a single free variable which defines according to (2.3) $x$ in $\mathfrak{A}$ and hence in $\mathfrak{B}$. Then by the lemma and (3.2) $y$ satisfies $\varphi \wedge \neg \chi$ in $\mathfrak{B}$ while $\varphi \wedge \neg \chi$ is not equivalent in $T$ with $\varphi$.

Finally let $i, j \in I$ and $i \neq j$, say $i < j$. Let $r \in Q$ with $i < r < j$. By (2.3), $\langle r, 0 \rangle$ is definable in $\mathfrak{A}$ and hence in $\mathfrak{B}$, say by the formula $\chi$. Then $\langle i, 0 \rangle$ satisfies $\exists v_1[v_0 < v_1 \wedge \chi(v_1)]$ in $\mathfrak{B}$ while $\langle j, 0 \rangle$ does not. Hence the prime ideal defined by those elements are distinct.

This concludes the proof of the theorem. We may remark that every element of $B$ is definable in $B$ by formulas of $L_{\omega_1\omega}$.

**Literature**

