THE MULTIPLIERS FOR
FUNCTIONS WITH FOURIER TRANSFORMS IN \( L_p \)

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1. Introduction.

In a previous paper [12] the author, together with T. S. Liu and J. K. Wang, began the study of certain subspaces of the group algebra \( L_1(G) \) of a locally compact Abelian group \( G \). These subspaces \( A_p(G) \), \( 1 \leq p < \infty \), were defined to be the spaces of all \( f \in L_1(G) \) whose Fourier transforms \( \hat{f} \) belong to \( L_p(\hat{G}) \), where \( \hat{G} \) denotes the dual group of the locally compact Abelian group \( G \) and \( L_p(\hat{G}) \) is the space of complex valued functions on \( \hat{G} \) whose \( p \)th powers are integrable with respect to Haar measure on \( \hat{G} \). If for each \( p \), \( 1 \leq p < \infty \), we set

\[
\|f\|_p^p = \|f\|_1 + \|\hat{f}\|_p, \quad f \in A_p(G),
\]

where

\[
\|f\|_1 = \int_G |f(t)| \, dt, \quad \|\hat{f}\|_p = \left( \int_{\hat{G}} |\hat{f}(\gamma)|^p \, d\gamma \right)^{1/p},
\]

and \( dt \) and \( d\gamma \) denote integration with respect to Haar measures on \( G \) and \( \hat{G} \) respectively, then \( A_p(G) \) is a Banach algebra with the indicated norm and the usual convolution product. It is then possible to study the relationship between various Banach algebra properties of \( L_1(G) \) and \( A_p(G) \), \( 1 \leq p < \infty \), as was done in [12]. Since the appearance of [12] a number of writers have further investigated the algebras \( A_p(G) \) or their generalizations [6], [8], [9], [10], [11], [13], [14], [15], [16], [19]. In particular, we asserted in [12] that if \( G \) is a noncompact locally compact Abelian group, then the multipliers for the algebras \( A_p(G) \), \( 1 \leq p < \infty \), correspond precisely to the Fourier–Stieltjes transforms of bounded regular Borel measures on \( G \), that is, the multipliers for \( A_p(G) \) are the same as those for \( L_1(G) \) [17, p. 73]. The proof of this assertion given in [12] is defective, but a correct proof has subsequently been given in [6]. However, as indicated in [12], when \( G \) is compact there in general exist multipliers for \( A_p(G) \) different from those defined by Fourier–Stieltjes

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transforms of measures. The main purpose of this article is to investigate more fully the nature of the multipliers for $A_p(G)$, $1 \leq p < \infty$, when $G$ is a compact Abelian group.

Let us first recall that a multiplier for $A_p(G)$ is a bounded linear operator $T$ on $A_p(G)$ which commutes with translation, that is, $T(\tau_s f) = \tau_s (Tf)$ for each $f \in A_p(G)$ and $s \in G$, where $\tau_s f(t) = f(ts^{-1})$. Clearly the translation operators $\tau_s$ are themselves multipliers of norm one. Since $A_p(G)$ is semi-simple [12] to each multiplier $T$ there corresponds, a unique bounded continuous function $\varphi$ on $\hat{G}$ such that $(Tf) \hat{=} = \varphi \hat{f}$ for each $f \in A_p(G)$ [18, p. 1135]. It is well known that these two descriptions of a multiplier are equivalent and so we shall interchange them at will.

When $G$ is compact and $1 \leq p \leq 2$ it is elementary to prove that every bounded function $\varphi$ on $\hat{G}$ defines a multiplier for $A_p(G)$. But if $p > 2$ the subspace of $A_p(G)$ whose Fourier transforms are invariant under multiplication by all bounded functions is the space $A_2(G)$, and this implies that not every bounded function defines a multiplier for $A_p(G)$. In this case, however, we shall show that the space of multipliers for $A_p(G)$ is linearly isomorphic to a proper subspace of the space of pseudo-measures on $G$ [1], [2], [3], and that this subspace properly contains the space of all bounded regular Borel measures on $G$. Finally, for each $p > 2$ we shall norm the linear space $A_1(G)$ in such a way that its completion is a Banach space of continuous functions such that there exists a continuous linear isomorphism from the space of multipliers for $A_p(G)$ onto the dual space of this Banach space.

Throughout the paper $M_p$ will denote the Banach space of multipliers for $A_p(G)$, $1 \leq p < \infty$, and $\|T\|_p$ will denote the norm of the multiplier $T$ as an operator on $A_p(G)$ [18]. $C(G)$ is the space of complex valued (bounded) continuous functions on the compact group $G$, and $L_p(G)$, $1 \leq p < \infty$, the space of complex valued functions on the compact group $G$ whose $p$th powers are integrable with respect to Haar measure on $G$. If $S \subset L_1(G)$, then $\hat{S}$ will denote the set of Fourier transforms of elements of $S$. It will always be assumed that the Haar measures on $G$ and $\hat{G}$ are chosen in such a way that the Fourier inversion formula is valid. General results from harmonic analysis which are used in the body of the paper can all be found in [17].

Remarks. a) We collect here several facts about the spaces $A_p(G)$ which we shall use in the sequel, generally without explicit reference. If $G$ is compact it follows from [12] that $A_2(G) = L_2(G)$ as linear spaces, and applications of the Hausdorff–Young Theorem [20, p. 190] reveal that for $1 < p' < 2$, $1/p' + 1/p = 1$, we have
\( A_1(G) \subseteq A_p(G) \subseteq L_p(G) \subseteq L_2(G) = A_2(G) \subseteq L_p(G) \subseteq A_p(G) \subseteq L_1(G) \).

In general, if \( q < p \), then \( A_q(G) \) is a norm dense ideal in \( A_p(G) \) [12]. It is also easy to establish that as linear spaces \( M_p \subseteq M_q \) when \( q < p \). Finally, for compact \( G \), it is easily seen by means of the Fourier inversion formula that

\[
A_1(G) = \hat{L}_1(\hat{G}) \subseteq C(G).
\]

Moreover \( A_1(G) \) is supremum norm dense in \( C(G) \) and \( \hat{A}_1(G) \) is norm dense in \( L_p(\hat{G}) \), \( 1 \leq p < \infty \).

b) If \( G \) is finite, then \( \hat{G} \) is finite, and it is evident that \( L_1(G) = A_p(G) \), \( 1 \leq p < \infty \). In this case nothing remains to be examined. However, this is the only instance in which the spaces \( A_p(G) \) and \( L_1(G) \) are identical when \( G \) is compact. More generally the following easily established theorem is valid.

**Theorem 1.** Let \( G \) be a locally compact Abelian group. Then the following are equivalent.

(i) \( G \) is nondiscrete.

(ii) \( A_p(G) = L_1(G) \), \( 1 \leq p < \infty \).

(iii) \( A_p(G) = A_q(G) \), \( 1 \leq p, q < \infty \), \( p \neq q \).

Thus we shall restrict our attention to **infinite** compact Abelian groups.

2. Spaces invariant under multiplication by bounded functions.

It is well known that every bounded measurable function \( \varphi \) on \( \hat{G} \) defines a multiplier for \( L_2(G) \) [4, p. 496], and hence, when \( G \) is compact, for \( A_2(G) \). Moreover an elementary argument shows that, if \( \varphi \) defines a multiplier for \( A_2(G) \), then it does so also for \( A_p(G) \), \( 1 \leq p < 2 \). Thus we obtain the following theorem.

**Theorem 2.** Let \( G \) be an infinite compact Abelian group. Then every \( \varphi \in C(\hat{G}) \) defines a multiplier for \( A_p(G) \), \( 1 \leq p \leq 2 \).

Since, as noted previously, every multiplier for \( A_p(G) \) corresponds to a bounded continuous function, the preceding theorem shows that the multipliers for \( A_p(G) \), \( 1 \leq p \leq 2 \), correspond precisely to \( C(\hat{G}) \) when \( G \) is compact.

The situation for \( p > 2 \) and compact \( G \) is quite different. In this case we set

\[
(A_p(G))_0 = \{ f \mid f \in A_p(G), \ \varphi \hat{f} \in \hat{A}_p(G), \ \varphi \in C(\hat{G}) \},
\]

and prove the following result.
Theorem 3. Let $G$ be an infinite compact Abelian group and $p > 2$. Then $(A_p(G))_0 = A_2(G) = L_2(G)$.

Proof. Let $f \in (A_p(G))_0$. Then $f \in L_1(G)$ and $q^f \in \hat{A}_p(G) \subseteq \hat{L}_1(G)$ for each $q \in C(\hat{G})$. However, the set of such functions in $L_1(G)$ is precisely $L_2(G)$ [7, p. 244], and hence $(A_p(G))_0 \subseteq L_2(G)$. Conversely, if $f \in L_2(G) = A_2(G)$, then by the preceding result $q^f \in \hat{A}_2(G)$ for each $q \in C(\hat{G})$. But $\hat{A}_2(G) \subseteq \hat{A}_p(G)$ and so $A_2(G) \subseteq (A_p(G))_0$.

Therefore $(A_p(G))_0 = A_2(G) = L_2(G)$.

Corollary. Let $G$ be an infinite compact Abelian group and $p > 2$. Then not every function $q \in C(\hat{G})$ defines a multiplier for $A_p(G)$.

3. Multipliers, measures and pseudomeasures.

As indicated previously, when $G$ is compact the linear space $A_1(G) = \hat{L}_1(\hat{G})$. This linear space is a Banach space when equipped with the norm $\|f\|_A = \|\hat{f}\|_1$, $f \in \hat{L}_1(\hat{G})$, and its dual space is the space of pseudomeasures on $G$, which we denote by $P(G)$ [3, p. 259]. The next result asserts that for $p > 2$ the space of multipliers $M_p$ can be identified with a subspace of $P(G)$. Notation and results on pseudomeasures used below can all be found in [3].

Theorem 4. Let $G$ be an infinite compact Abelian group and $p > 2$. Then there exists a continuous linear injective mapping from $M_p$ into $P(G)$.

Proof. Let $T \in M_p$. Since $C(G) \subseteq L_2(G) = A_2(G) \subseteq A_p(G)$ we see that $T$ defines a linear mapping from $C(G)$ to $L_1(G)$ which commutes with translation. Furthermore, suppose $f_n, f \in C(G)$ and $\lim_n \|f_n - f\|_\infty = 0$, where $\|\cdot\|_\infty$ denotes the usual supremum norm in $C(G)$. Because $G$ is compact it follows that $\lim_n \|f_n - f\|_1 = \lim_n \|f_n - f\|_2 = 0$, and hence by the Plancherel Theorem $\lim_n \|f_n - f\|^2 = 0$. But $p > 2$ implies that $T \in M_2$, and so $\lim_n \|Tf_n - Tf\|^2 = 0$, from which we conclude that $\lim_n \|Tf_n - Tf\|_1 = 0$. Thus the mapping defined by $T$ is continuous.

Consequently there exists a unique pseudomeasure $\nu \in P(G)$ such that $Tf = \nu * f$ for each $f \in C(G)$ [3, p. 260]. Define $\alpha(T) = \nu$. Clearly $\alpha$ is a linear mapping from $M_p$ into $P(G)$. That $\alpha$ is injective follows immediately from the denseness of $A_1(G)$ in $A_p(G)$. Moreover, since for each $\gamma \in \hat{G}$ the Fourier transform of the continuous character $(\cdot, \gamma)$ is the characteristic function of the set $\{\gamma\}$, and $T(\cdot, \gamma) = \varphi(\gamma)(\cdot, \gamma)$ where $\varphi \in C(\hat{G})$ is the unique function for which $(Tf)^\sim = \varphi \hat{f}$ for each $f \in A_p(G)$, we see that...
\[ |\hat{v}(\gamma)| = |||\hat{v}(\cdot, \gamma)|||_\infty \leq ||v \ast (\cdot, \gamma)||_1 = ||T(\cdot, \gamma)||_1 = |\varphi(\gamma)| \leq ||\varphi||_\infty \leq ||T||_P . \]

The validity of the last inequality follows from [18, p. 1135] and the semisimplicity of \( A_p(G) \). But the Fourier transform for pseudomeasures is an isometry [3, p. 260], and hence

\[ ||\alpha(T)||_P = ||v||_P = ||\hat{v}||_\infty \leq ||T||_P , \]

where \( ||\cdot||_P \) denotes the norm in \( P(G) \).

Therefore \( \alpha \) is a continuous linear injective mapping from \( M_p \) into \( P(G) \).

**Remarks.**

a) The norms \( ||\cdot||_A \) and \( ||\cdot||^1 \) on \( \hat{L}_1(\hat{G}) \) are obviously equivalent.

b) The mapping \( \alpha \) from \( M_p \) to \( P(G) \) is not surjective. As if it were, then since the Fourier transform of pseudomeasures maps \( P(G) \) onto \( C(\hat{G}) \) [3, p. 260], as \( G \) is compact, the composition of the Fourier transform with \( \alpha \) would produce a surjective mapping from \( M_p \) onto \( C(\hat{G}) \). That is, every function in \( C(\hat{G}) \) would define a multiplier for \( A_p(G) \), thereby contradicting the Corollary to Theorem 3.

c) On the other hand, the subspace \( \alpha(M_p) \) of \( P(G) \) properly contains the space \( M(G) \) consisting of all bounded regular Borel measures on \( G \). Thus, when \( p > 2 \) there exist multipliers for \( A_p(G) \) which are not defined by the Fourier–Stieltjes transform of any bounded regular Borel measure on \( G \). To see this, given \( p > 2 \), set \( m = p/2 \), \( n = m/(m-1) \), and choose \( r \) such that \( 0 < r < 2 \) and \( rn > 2 \). Let \( E \subset \hat{G} \) be any infinite Sidon set [17, p. 120] and choose \( \varphi \in C(\hat{G}) \) such that

\begin{enumerate}
  \item \( \varphi(\gamma) = 0 \), \( \gamma \notin E \),
  \item \( \sum_\gamma |\varphi(\gamma)|^2 = \infty \),
  \item \( \sum_\gamma |\varphi(\gamma)|^m < \infty \).
\end{enumerate}

It is easy to see that such choices can always be made. An application of Hölder's inequality shows that \( \varphi \hat{f} \in L_2(\hat{G}) \subset L_p(\hat{G}) \) for each \( f \in A_p(G) \), and so \( \varphi \) defines a multiplier for \( A_p(G) \). However, \( \varphi \neq \hat{\mu} \) for any \( \mu \in M(G) \) because \( \varphi \) is a Fourier–Stieltjes transform if and only if \( \sum_\gamma |\varphi(\gamma)|^2 < \infty \) [2, p. 841].

4. The space \( B_p(G) \).

In the preceding section we saw for \( p > 2 \) that \( M_p \) is linearly isomorphic to a proper subspace of the continuous linear functionals on a certain Banach space of continuous functions. However, it is not imme-
dially obvious whether $M_p$ can be considered as a dual space of such a Banach space. The development of this section will show that this is indeed possible. We begin by defining the normed spaces of continuous functions $B_p(G)$, and shall ultimately prove that there exists a continuous linear isomorphism from $M_p$ onto $B_p'(G)$, the dual space of $B_p(G)$. Furthermore we shall show that the completion $\hat{B}_p(G)$ of $B_p(G)$ can be considered as a Banach space of continuous functions.

Consider a fixed $p > 2$. For $T \in M_p$ we shall denote by $q$ the unique function in $C(\hat{G})$ such that $(Tf)^\wedge = q \hat{f}$ for each $f \in A_p(G)$. If $T \in M_p$, then for each $f \in \hat{L}_1(\hat{G})$ we set

$$\beta(T)(f) = \int_{\hat{G}} (Tf)^\wedge(\gamma) \, d\gamma = \int_{\hat{G}} q(\gamma) \hat{f}(\gamma) \, d\gamma.$$ 

For $f \in \hat{L}_1(\hat{G})$ we define

$$\|f\|_B = \sup \{ |\beta(T)(f)| \mid T \in M_p, \|T\|_p \leq 1 \}.$$ 

These definitions make sense as $M_p \subset M_1$.

It is routine to verify that $\|\cdot\|_B$ is a norm on $\hat{L}_1(\hat{G})$, and the normed linear space so obtained will be denoted by $B_p(G)$. Moreover, from the preceding definitions it is apparent that each $\beta(T)$ defines a continuous linear functional on $B_p(G)$. Thus $\beta$ defines a mapping from $M_p$ into $B_p'(G)$. It is not difficult to show that $\beta$ is a continuous linear injective mapping from $M_p$ to $B_p'(G)$. For example, if $f \in B_p(G)$, then

$$|\beta(T)(f)| = \left| \int_{\hat{G}} (Tf)^\wedge(\gamma) \, d\gamma \right| = \left| \|T\|_p \int_{\hat{G}} (Tf)^\wedge(\gamma)/\|T\|_p \, d\gamma \right|$$

$$= \|T\|_p |\beta(T/\|T\|_p)(f)| \leq \|T\|_p \|f\|_B.$$ 

Hence $\|\beta(T)\| \leq \|T\|_p$, where $\|\cdot\|$ denotes the norm in $B_p'(G)$.

The theorem we wish to establish is the following one.

**Theorem 5.** Let $G$ be an infinite compact Abelian group and $p > 2$. Then $\beta$ is a continuous linear bijective mapping from $M_p$ to $B_p'(G)$.

In light of the preceding discussion we need only prove that $\beta$ is surjective. Before turning to the proof proper of this fact we shall establish several technical lemmas.

**Lemma 1.** Let $G$ be an infinite compact Abelian group, $p > 2$ and $f, g \in B_p(G)$. 
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(i) If $1 \leq r \leq \infty$ and $1/r + 1/r' = 1$, then $\|f*g\|_B \leq \|\hat{f}\|_r \|\hat{g}\|_{r'}$.

(ii) $\|f*g\|_B \leq \|f\|_P \|g\|_\infty$.

Proof. Clearly $f*g \in B_p(G)$ as $\hat{L}_1(\hat{G}) = A_1(\hat{G})$ is an algebra under convolution. Using [18, p. 1135] we see that for each $T \in M_p$, 

$$|\beta(T)(f*g)| = \left| \int_{\hat{G}} q_T(\gamma) \hat{g}(\gamma) \ d\gamma \right| \leq \|q_T\|_r \|\hat{g}\|_{r'} \leq \|T\|_P \|\hat{f}\|_r \|\hat{g}\|_{r'}.$$ 

The application of Hölder’s inequality is justified since $L_1(\hat{G}) \subset L_q(\hat{G})$, $1 \leq q \leq \infty$. Hence $\|f*g\|_B \leq \|\hat{f}\|_r \|\hat{g}\|_{r'}$.

To prove (ii) we observe that for $T \in M_p$ we have 

$$|\beta(T)(f*g)| = \left| \int_{\hat{G}} [T(f*g)](\gamma) \ d\gamma \right| = \left| \int_{\hat{G}} (Tf)(\gamma) \hat{g}(\gamma^{-1}) \ d\gamma \right|,$n

where $\hat{g}(t) = g(t^{-1})$. However $Tf, g \in A_1(G) \subset L_2(G)$ as $M_p \subset M_1$. Thus we may apply Parseval’s formula to obtain 

$$|\beta(T)(f*g)| = \left| \int_{\hat{G}} Tf(t) \hat{g}(t) \ dt \right| \leq \|Tf\|_1 \|\hat{g}\|_\infty \leq \|Tf\|_P \|\hat{g}\|_\infty \leq \|T\|_P \|f\|_P \|g\|_\infty.$$ 

Therefore $\|f*g\|_B \leq \|f\|_P \|g\|_\infty$.

Lemma 2. Let $G$ be an infinite compact Abelian group and $p > 2$. Suppose $F \in B_p'(G)$, $f \in B_p(G)$ and define $F_f(g) = F(f*g)$ for each $g \in B_p(G)$. Then $F_f$ defines a continuous linear functional on $L_p(\hat{G})$.

Proof. It is evident that $F_f$ defines a linear functional on $\hat{B}_p(G) \subset L_p(\hat{G})$. Moreover from the first portion of Lemma 1 we see for each $g \in B_p(G)$ that 

$$|F_f(g)| = |F(f*g)| \leq \|F\| \|f*g\|_B \leq \|F\| \|\hat{f}\|_{r'} \|\hat{g}\|_P,$$

where $1/p + 1/p' = 1$. Thus $F_f$ is continuous on $\hat{B}_p(G)$ considered as a subspace of $L_p(\hat{G})$. Moreover $\hat{B}_p(G)$ is norm dense in $L_p(\hat{G})$.

Therefore $F_f$ can be uniquely extended to a continuous linear functional on all of $L_p(\hat{G})$. 

Given $F_f$ as in the previous lemma we denote by $\hat{h}$ the unique element of $L_{p'}(\hat{G})$, $1/p + 1/p' = 1$, such that for each $\hat{g} \in \hat{B}_p(G)$

$$F_f(\hat{g}) = \langle \hat{h}, \hat{g} \rangle = \int_{\hat{g}} \hat{h}(\gamma) \hat{g}(\gamma) \, d\gamma.$$

Since $1 < p' < 2$, the Hausdorff–Young theorem [20, p. 190] implies the existence of a unique $h \in L_p(G)$ whose Fourier transform is $\hat{h}$. Thus, given $F \in B_p'(G)$, for each $f \in B_p(G)$ we define $Tf = h$, where $h$ is chosen as above. Clearly $T$ is a linear transformation from the linear subspace $A_1(G) = B_p(G)$ of $A_p(G)$ to $A_p'(G) \subset A_p(G)$.

**Lemma 3.** Let $G$ be an infinite compact Abelian group, $p > 2$ and $F \in B_p'(G)$. If $T$ is defined as above, then $T$ is a continuous linear transformation from the subspace $A_1(G)$ of $A_p(G)$ to $A_p(G)$.

**Proof.** Suppose $f \in A_1(G)$ and $1/p + 1/p' = 1$. Then, since $\hat{B}_p(G) \subset L_{p'}(\hat{G}) \subset L_p(\hat{G})$ and $\hat{B}_p(G)$ is norm dense in $L_p(\hat{G})$, we conclude that

$$\|(Tf)^\sim\|_p = \|\hat{h}\|_p = \|\hat{h}\|_p$$

$$= \sup \{ |\langle \hat{h}, \hat{g} \rangle| \mid \hat{g} \in \hat{B}_p(G), \|\hat{g}\|_{p'} \leq 1 \}$$

$$= \sup \{ |\langle F_f(\hat{g}) \rangle| \mid \hat{g} \in \hat{B}_p(G), \|\hat{g}\|_{p'} \leq 1 \}$$

$$= \sup \{ |\langle F(f \ast g) \rangle| \mid \hat{g} \in \hat{B}_p(G), \|\hat{g}\|_{p'} \leq 1 \}$$

$$\leq \sup \{ \|F \| \|f \ast g\|_B \mid \hat{g} \in \hat{B}_p(G), \|\hat{g}\|_{p'} \leq 1 \}$$

$$\leq \sup \{ \|F\| \|\hat{f}\|_p \|\hat{g}\|_{p'} \mid \hat{g} \in \hat{B}_p(G), \|\hat{g}\|_{p'} \leq 1 \} \leq \|F\| \|\hat{f}\|_p.$$

The penultimate inequality is due to Lemma 1(i).

Furthermore, using the fact that $B_p(G)$ is supremum norm dense in $C(G)$ and Parseval’s formula we have

$$\|Tf\|_1 = \sup \{ |\langle Tf, g \rangle| \mid g \in C(G), \|g\|_{\infty} \leq 1 \}$$

$$= \sup \{ \int_T \hat{h}(t)g(t) \, dt \mid g \in B_p(G), \|g\|_{\infty} \leq 1 \}$$

$$= \sup \{ \int_T \hat{h}(\gamma)g(\gamma) \, d\gamma \mid g \in B_p(G), \|g\|_{\infty} \leq 1 \}$$

$$= \sup \{ \|\hat{F}_f(\hat{g})\| \mid g \in B_p(G), \|g\|_{\infty} \leq 1 \}$$

$$\leq \sup \{ \|\hat{F}\| \|f\ast g\|_B \mid g \in B_p(G), \|g\|_{\infty} \leq 1 \}$$

$$\leq \sup \{ \|\hat{F}\| \|f\|_p \|g\|_{\infty} \mid g \in B_p(G), \|g\|_{\infty} \leq 1 \} \leq \|\hat{F}\| \|f\|_p.$$

The penultimate inequality is now due to Lemma 1(ii).

Combining these estimates we see at once that for each $f \in A_1(G)$,
\[ \|Tf\|^p \leq 2\|F\|\|f\|^p . \]

Hence \( T \) is continuous from \( A_1(G) \subseteq A_p(G) \) to \( A_p(G) \).

**Proof of Theorem 5.** As mentioned before, we need only show that \( \beta \) is surjective. Given \( F \in B_p(G) \), let \( T \) be the operator defined preceding Lemma 3. In view of Lemma 3 this operator can be uniquely extended to a bounded linear operator on all of \( A_p(G) \), since \( A_1(G) \) is norm dense in \( A_p(G) \). We shall denote this extension by \( T \).

If \( f, g \in A_1(G) \) and \( s \in G \), then

\[
\int_{\hat{G}} [T(\tau_s f)](\gamma) \hat{g}(\gamma) \, d\gamma = F(\tau_s f * g)
= F(f * \tau_s g)
= \int_{\hat{G}} (Tf)(\gamma)(\tau_s g)(\gamma) \, d\gamma
= \int_{\hat{G}} [Tf * \tau_s g](\gamma) \, d\gamma = \int_{\hat{G}} [\tau_s (Tf)](\gamma) \hat{g}(\gamma) \, d\gamma .
\]

Since \( \hat{A}_1(G) \) is norm dense in \( L_p(\hat{G}) \), \( 1/p + 1/p' = 1 \), and \( (Tf)(\gamma) \in L_p(\hat{G}) \) for each \( f \in A_1(G) \), we conclude that \( [T(\tau_s f)](\gamma) = [\tau_s (Tf)](\gamma) \) per each \( f \in A_1(G) \) and \( s \in G \). The semisimplicity of \( A_1(G) \), the continuity of \( T \) and the norm denseness of \( A_1(G) \) in \( A_p(G) \) combine to imply that \( T\tau_s = \tau_s T \) for each \( s \in G \). Thus \( T \in M_p \).

Moreover, if \( f, g \in A_1(G) \), then

\[
\beta(T)(f * g) = \int_{\hat{G}} [T(f * g)](\gamma) \, d\gamma
= \int_{\hat{G}} (Tf)(\gamma) \hat{g}(\gamma) \, d\gamma = F_f(\hat{g}) = F(f * g) ,
\]

by the definition of \( T \). But \( \{f * g \mid f, g \in A_1(G)\} \) is norm dense in \( B_p(G) \). Indeed, let \( \{u_s\} \subseteq A_1(G) \) be an approximate identity for \( A_1(G) \). Then in particular we have \( \lim_s \|\hat{f} - \hat{f}_s\|_1 = 0 \) for each \( f \in A_1(G) \). Furthermore

\[
\|f - f * u_s\|_B = \sup \{ |\beta(T) (f - f * u_s)| \mid T \in M_p, \|T\|_p \leq 1 \}
= \sup \{ |\int_{\hat{G}} \varphi(\gamma)(\hat{f}(\gamma) - \hat{f}_s(\gamma)) \, d\gamma| \mid T \in M_p, \|T\|_p \leq 1 \}
\leq \|\hat{f} - \hat{f}_s\|_1 ,
\]

as \( \|\varphi\|_{\infty} \leq \|T\|_p \leq 1 \) by [18, p. 1135]. Hence \( \lim_s \|f - f * u_s\|_B = 0 \), and \( \{f * g \mid f, g \in A_1(G)\} \) is norm dense in \( B_p(G) \).
Therefore $\beta(T) = f$, and $\beta$ is surjective.

The next result shows that the completion $\overline{B}_p(G)$ of $B_p(G)$ can be identified with a space of continuous functions.

**Theorem 6.** Let $G$ be an infinite compact Abelian group. For each $p > 2$ there exists a continuous linear injective mapping $\iota$ of $\overline{B}_p(G)$ onto a subspace of $C(G)$.

**Proof.** From the Fourier inversion formula we see that if $f \in B_p(G)$ then for each $t \in G$,

$$|f(t)| = \left| \int_G (t, \gamma) \hat{f}(\gamma) \, d\gamma \right|$$

$$= \left| \int_G (\tau_{-1}f)(\gamma) \, d\gamma \right|$$

$$= |\beta(\tau_{-1})(f)|$$

$$\leq \sup \{ |\beta(T)(f)| \mid T \in M_p, \|T\|_p \leq 1 \} = \|f\|_B.$$ 

Hence $\|f\|_\infty \leq \|f\|_B$ for each $f \in B_p(G)$.

Considering the elements of $\overline{B}_p(G)$ as Cauchy sequences of elements of $B_p(G)$ it is apparent from the preceding inequality that, if $\{f_n\} \subset B_p(G)$ is a Cauchy sequence in $B_p(G)$, then there exists a unique function $f \in C(G)$ such that $\lim_n \|f_n - f\|_\infty = 0$. Setting $\iota(\{f_n\}) = f$ we obtain a well defined linear mapping from $\overline{B}_p(G)$ onto a subspace of $C(G)$. It follows at once from the previous estimate that $\iota$ is a continuous mapping. The proof that $\iota$ is injective can be taken *mutatis mutandis* from [4, p. 499].

The proof in [4] carried over to the present context also immediately establishes the following corollary.

**Corollary.** Let $G$ be an infinite compact Abelian group. For each $p > 2$ the space of finite linear combinations of the functionals $\{\beta(s) \mid s \in G\}$ is weak* dense in $B_p'(G)$.

**Remark.** It is clear that the development of this section owes a great deal to [4], where a similar characterization of the multipliers for $L_p(G)$ is studied. The work in [4] has also been extended in [5].
REFERENCES


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