ON THE BOHR TOPOLOGY IN AMENABLE TOPOLOGICAL GROUPS

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Introduction.

In [1] E. M. Alfsen and P. Holm have characterized the Bohr compactification of a topological group \((G, \mathcal{T})\) as the completion of \(G\) with respect to a group topology \(\mathcal{T}_B\) (the Bohr topology) which is coarser than \(\mathcal{T}\). The purpose of this note is to prove that the general description of \(\mathcal{T}_B\) can be simplified in amenable groups, i.e. groups admitting an invariant mean on the space of bounded left uniformly continuous functions. The result can be read as follows: \(W\) is a \(\mathcal{T}_B\)-neighbourhood of \(e\) if and only if there is a symmetric relatively dense \(\mathcal{T}\)-neighbourhood \(V\) of \(e\) with \(V^* \subseteq W\). Though stated in another way, this has earlier been proved by E. Følner for abelian groups ([2, Theorem 1] and [3]), and his ideas are used extensively.

Section 1 contains the needed results concerning the Bohr compactification; with a slight modification of proof we get a simpler characterization of \(\mathcal{T}_B\) than in [1]. The connections between the upper and the lower mean values, invariant means and distinguished subsets of a group have been studied by E. Følner and in the abelian case by P. Tomter. Section 2 is devoted to this. In Section 3 the characterization of the Bohr topology in amenable topological groups is given.

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1. The Bohr compactification.

In [1] the Bohr compactification \(\hat{G}\) of a topological group \((G, \mathcal{T})\) is obtained as the Hausdorff completion of \(G\) with respect to the finest uniform structure \(\mathcal{U}\) on \(G\) satisfying

\begin{align}
\tag{1.1} & \mathcal{U} \text{ is totally bounded}, \\
\tag{1.2} & \mathcal{U} \text{ is compatible with the group structure, i.e. the group operations are uniformly continuous}, \\
\tag{1.3} & \mathcal{U} \text{ defines a topology on } G \text{ coarser than } \mathcal{T}.
\end{align}

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Proposition 1 of [1] tells us that a uniform structure satisfying (1.2) is completely determined by the associated group topology on \( G \), and therefore it suffices to study the finest group topology on \( G \) satisfying the analogous of (1.1), (1.2), and (1.3).

Recall that a subset \( A \) of \( G \) is called left (right) relatively dense if there is a finite set \( \{a_1, \ldots, a_n\} \) in \( G \) such that \( G = \bigcup_{i=1}^{n} a_i A \) (\( G = \bigcup_{i=1}^{n} A a_i \)). If \( A \) is both left and right relatively dense, \( A \) is called relatively dense. The right uniform structure of a topological group is totally bounded if and only if the left uniform structure is, and this is the case if and only if each neighbourhood of \( e \) is relatively dense. It is well known that in this case the left and right uniform structures coincide. A proof of this fact is not so easily traced in the literature, so we include one for completeness.

**Lemma 1.** If \((G, \mathcal{T})\) is a totally bounded topological group, \( \mathcal{U}_1 (\mathcal{U}_r) \) the left (right) uniform structure, then \( \mathcal{U}_1 = \mathcal{U}_r \), and the group operations are uniformly continuous.

**Proof.** It is an easily established fact that the group operations are uniformly continuous if and only if \( \mathcal{U}_1 = \mathcal{U}_r \), and this is the case if and only if \( G \) admits a fundamental system of neighbourhoods of \( e \) whose members \( V \) are all *invariant* in the sense that \( xVx^{-1} = V \) for every \( x \) in \( G \).

Let \( U \) be an arbitrary neighbourhood of \( e \). Choose a symmetric neighbourhood \( V \) of \( e \) such that \( V^3 \subset U \). Now \( G = \bigcup_{i=1}^{n} a_i V \) for some \( a_1, \ldots, a_n \in G \). Let

\[ V_1 = \bigcap_{i=1}^{n} a_i V a_i^{-1} \quad \text{and} \quad W = \bigcup_{x \in G} x^{-1} V_1 x. \]

Then \( W \) is an invariant neighbourhood of \( e \). If \( y \in V_1 \) and \( x \) is arbitrary, we have \( x \in a_i V \) for some \( i \). Now

\[ x^{-1} y x \in (a_i V)^{-1} (a_i V a_i^{-1}) a_i V = V^3 \subset U, \]

so \( W \subset U \), and the lemma is proved.

The problem of finding the finest uniform structure on \( G \) satisfying (1.1), (1.2), and (1.3) is therefore reduced to find the finest group topology on \( G \) coarser than the original one such that each neighbourhood is relatively dense. Following the proof of [1, Theorem 1] we now conclude:

**Theorem 1.** For a topological group \((G, \mathcal{T})\) let \( \mathcal{T}_B \) be the finest group topology on \( G \) coarser than \( \mathcal{T} \) defining a totally bounded uniform struc-
ture. This uniform structure (denoted $\mathcal{U}_B$) is the finest satisfying (1.1), (1.2), and (1.3), and the neighbourhood system of $e$ associated with $\mathcal{T}_B$ consists of those subsets $V$ of $G$ which admit a sequence $\{V_n\}$ of sets such that:

(1.4) $V_1^2 \subset V$ and $V_{n+1}^2 \subset V_n$ for $n = 1, 2, \ldots$.

(1.5) Every $V_n$ is a symmetric and relatively dense $\mathcal{T}$-neighbourhood of $e$.

The topology $\mathcal{T}_B$ is called the Bohr topology on $G$, and from [1] it is known that the $\mathcal{U}_B$-uniformly continuous functions are exactly the almost periodic functions on $G$.

2. Invariant means and related subsets of the group.

On $\text{BR}(G)$ (= the set of bounded real valued functions on $G$) we define the right upper mean value $\bar{M}$ by

$$\bar{M}(f) = \inf \{\sup_{x \in G} \sum_i \alpha_i f(xa_i) : a_i \in G, \alpha_i > 0, \sum_i \alpha_i = 1\}.$$ 

The right lower mean value $\underline{M}$ is defined by $\underline{M}(f) = -\bar{M}(-f)$.

**Lemma 2.** The right upper mean value $\bar{M}$ has the following properties:

(2.1) $\inf_{x \in G} f(x) \leq \bar{M}(f) \leq \sup_{x \in G} f(x)$.

(2.2) $\bar{M}(\lambda f) = \lambda \bar{M}(f)$ for $\lambda \geq 0$.

(2.3) $\bar{M}(f_a) = \bar{M}(f)$ for $a \in G$; $f_a$ is the function $f_a(x) = f(xa)$.

(2.4) $\bar{M}(f-f_a) \leq 0$.

(2.5) $\bar{M}(f+g) \leq \bar{M}(f) + \bar{M}(g)$ if $G$ is abelian.

**Proof.** Only part (2.4) needs a proof. Take $a_1 = a$, $a_{k+1} = a_k a$. Then

$$\bar{M}(f-f_a) \leq \sup_{x \in G} n^{-1} \sum_{i=1}^{n} (f-f_a)(xa_i)$$

$$= \sup_{x \in G} n^{-1} (f(xa_1) - f(xa_{n+1})) \leq 2n^{-1} \|f\|_{\infty}$$

This holds for any $n$, and (2.4) follows.

If $A$ is a subset of $G$ and $\chi_A$ is its characteristic function, it is easy to see that $A$ is right relatively dense if and only if $\bar{M}(\chi_A) > 0$. (This observation is due to Følner.) Sets with positive upper mean value have been studied by P. Tomter [6] in the abelian case, and we will transfer his ideas to arbitrary groups.
DEFINITION. A subset $A$ of $G$ is called left (right) relatively accumulating if there is a positive integer $n_0$ such that for any non-negative integer $m$, at least $m + 1$ of any $mn_0 + 1$ left (right) translates of $A$ have a common, non-empty intersection. If $A$ is left and right relatively accumulating, $A$ is called relatively accumulating.

Following [6, pp. 26–27] it is easily seen that
(a) if $A$ is left relatively dense, then $A$ is right relatively accumulating,
(b) if $A$ is right relatively accumulating, then $A^{-1}A$ is right relatively dense,
(c) $A$ is right relatively accumulating if and only if $\overline{M(\chi_A)} > 0$.
(b) was first proved by Følner.

REMARK. In connection with (a), note that a left relatively dense subset is not necessarily left relatively accumulating. An example of von Neumann can be used, take $G$ to be the free group of two generators $a$ and $b$, and let $A$ be the set of elements beginning with $a$ or $a^{-1}$ when written as reduced words. $G = A \cup aA$, so $A$ is left relatively dense. But $A$ is not left relatively accumulating, for instance any two distinct members of the collection $\{A, bA, \ldots, b^nA\}$ have empty intersection.

Now let $E$ be some linear space of complex valued functions on $G$ which contains the constants and is closed under complex conjugation and right translations (that is, $f \in A$, $a \in G \Rightarrow f_a \in E$). A linear functional $m$ on $E$ is called a right invariant mean (RIM) if

\begin{align}
(2.6) \quad m(\bar{f}) &= m(f), \\
(2.7) \quad \inf_{x \in G} f(x) &\leq m(f) \leq \sup_{x \in G} f(x) \quad \text{for any real valued } f \in E, \\
(2.8) \quad m(f_a) &= m(f).
\end{align}

Left invariant means are defined analogously, and if $m$ is both left and right invariant, it is called an invariant mean. If $m$ is a RIM, and if $f \in E$ is real valued, we have

\begin{align}
m(f) &= m(\sum \alpha_i f_a) \leq \sup_{x \in G} \sum \alpha_i f(xa_i)
\end{align}

for any convex combination $\sum \alpha_i f_a$ of translates of $f$. Thus $m(f) \leq \overline{M}(f)$, and we can conclude that

\begin{align}
\overline{M}(f) \leq m(f) \leq \overline{M}(f).
\end{align}

If $\overline{M}$ is subadditive on $E'$ (= the real functions in $E$), the Hahn–Banach theorem implies the existence of a linear functional $m$ satisfying $m(f) \leq$
\( \bar{M}(f) \) for \( f \in E' \). Applying (2.4) we find that \( m \) is a RIM on \( E' \), and \( m \) can uniquely be extended to a RIM on \( E \). In particular the space of all bounded complex valued functions on an abelian group will admit an invariant mean.

**Definition.** A topological group \( G \) is called *amenable* if there is a RIM on \( \text{UCB}_1(G) \) (= the set of left uniformly continuous bounded complex valued functions on \( G \)). (A complex valued function \( f \) is left uniformly continuous if for each \( \varepsilon > 0 \) there is a neighbourhood \( U \) of \( e \) such that \( |f(x) - f(y)| < \varepsilon \) if \( x^{-1}y \in U \).)

A locally compact group is usually called amenable if there is a RIM on \( L^\infty(G) \), but it is known that in this case the two definitions coincide. The results in Section 3 are valid not only for locally compact groups, and our choice of definition of amenability is motivated only by what is needed there.

A RIM is usually not strictly positive on positive, non-zero continuous functions. This is the case if and only if \( G \) is totally bounded.

Take \( P(G) \) to be the space of linear combinations of continuous positive definite functions. Over \( P(G) \) and over the space of almost periodic functions a RIM \( m \) coincides of course with the unique invariant mean defined on these spaces. If \( \varphi \) is positive definite, it is proved in [5, p. 59] that

\[
(2.10) \quad m(\varphi) = \inf \{ \sum_{i,j=1}^n a_i a_j \varphi(s_i^{-1}s_j) : s_i \in G, \ a_i > 0, \ \sum_i a_i = 1 \}.
\]

3. The Bohr topology in amenable topological groups.

We are now going to show that the characterization of the Bohr neighbourhoods given in Theorem 1 can be improved in amenable groups, in fact we shall prove that it suffices to have a finite chain of subsets of the sort described. As before, \( \mathcal{T}_B \) denotes the Bohr topology.

**Theorem 2 A.** Let \((G, \mathcal{T})\) be a topological group satisfying the following condition:

(A) The right upper mean value \( \bar{M} \) is subadditive over the space of real valued functions in \( \text{UCB}_1(G) \).

Then a subset \( W \) of \( G \) is a \( \mathcal{T}_B \)-neighbourhood of \( e \) if and only if there is a right relatively accumulating subset \( E \) of \( G \) and a \( \mathcal{T} \)-neighbourhood \( V \) of \( e \) such that \( (V^{-1}E^{-1}EV)^2 \subset W \).
Theorem 2 B. Suppose \((G, \mathcal{F})\) is a topological group satisfying:

(B) There is a right invariant mean on UCB\(_t\)(G), that is, \(G\) is amenable.

Then a subset \(W\) of \(G\) is a \(\mathcal{F}_B\)-neighbourhood of \(e\) if and only if there is a right relatively dense subset \(E\) of \(G\) and a \(\mathcal{F}\)-neighbourhood \(V\) of \(e\) such that \((V^{-1}E^{-1}EV)^2 \subset W\).

Proofs. From Theorem 1 it is easy to see that in both cases the condition is necessary. Conversely, suppose \(E\) and \(V\) have the stated properties. Following Fölner [2] we shall construct a non-zero almost periodic function vanishing outside \((V^{-1}E^{-1}EV)^2\), and this will give the conclusion.

There is a left uniformly continuous function \(h: G \to [0, 1]\) with \(h(e) = 1\) and \(h(x) = 0\) for \(x \notin V\).

Define \(j(x) = \sup_{y \in E} h(y^{-1}x)\). Then \(j \in \text{UCB}_t(G)\), \(j(x) = 1\) for \(x \in E\) and \(j(x) = 0\) for \(x \notin EV\).

If (A) is satisfied, the subadditivity of \(\overline{M}\) implies (via the Hahn–Banach theorem) that there is a right invariant mean \(m\) on \(\text{UCB}_t(G)\), and \(m\) can be chosen such that \(m(j) = \overline{M}(j)\), cf. [3]. Further

\[ m(j) = \overline{M}(j) \geq \overline{M}(\chi_E) > 0, \]

since \(E\) is right relatively accumulating in this case.

If (B) is satisfied, we have

\[ m(j) \geq \overline{M}(j) \geq \overline{M}(\chi_E) > 0, \]

since \(E\) is right relatively dense in case (B).

Hence in both cases we have a right invariant mean \(m\) on \(\text{UCB}_t(G)\) with \(m(j) > 0\).

The left uniform continuity of \(j\) implies that the function \(\varphi\) defined by

\[ \varphi(x) = m(j_x j) = m_t[j(t x) j(t)] \]

is continuous, and straightforward calculations show that \(\varphi\) is positive definite. Further, \(\varphi(x) \geq 0\) for any \(x\), and \(\varphi(x) = 0\) for \(x \notin V^{-1}E^{-1}EV\).

We want to show that \(m(\varphi) > 0\), and use the expression (2.10). If \(\{a_i\}_1^n\) are positive numbers with \(\sum a_i = 1\) and \(\{s_i\}_1^n\) are elements from \(G\), then by the right invariance of \(m\) we find that

\[ \sum_i a_i a_j \varphi(s_i^{-1}s_j) = m_t[(\sum_i a_i j(t s_i))^2] \geq (m_t[\sum_i a_i j(t s_i)])^2 = m(j)^2. \]

Thus \(m(\varphi) \geq m(j)^2 > 0\).

Over \(P(G)\) the functional \(m\) can be used to define a convolution

\[ f * g(x) = m_t[f(t) g(t^{-1} x)]. \]
R. Godement has proved that $f * g$ will be almost periodic [5, p. 63]. Thus the function $\psi = \varphi * \varphi$ will be positive definite and almost periodic. Further $\psi(x) \geq 0$ for all $x$, $\psi(x) = 0$ for $x \in (V^{-1}E^{-1}EV)^2$ and
\[
\psi(e) = m(|\varphi|^2) \geq |m(\varphi)|^2 > 0.
\]
The set
\[
W_0 = \{x \in G : |\psi(x) - \psi(e)| < \psi(e)\}
\]
is a $\mathcal{T}_B$-neighbourhood of $e$, since $\psi$ is almost periodic. Further
\[
W_0 \subset (V^{-1}E^{-1}EV)^2 \subset W,
\]
so $W$ is a $\mathcal{T}_B$-neighbourhood of $e$.

Theorem 2 B can be given in a weaker form which makes clear the connection with Theorem 1.

**Corollary 1.** If $(G, \mathcal{T})$ is an amenable topological group, then a subset $W$ is a $\mathcal{T}_B$-neighbourhood of $e$ if and only if there is a symmetric, relatively dense $\mathcal{F}$-neighbourhood $V$ of $e$ with $V^7 \subset W$.

**Proof.** If $V^7 \subset W$, take $E = V$ and let $U$ be a $\mathcal{F}$-neighbourhood of $e$ satisfying $UU^{-1} \subset V$. Apply Theorem 2 B with $E$ and $U$, and the conclusion follows from
\[
(U^{-1}E^{-1}EU)^2 \subset U^{-1}E^{-1}EV E^{-1}EU \subset V^7 \subset W.
\]

In abelian groups we can simplify even more, and since condition (A) always holds in this case, we have:

**Corollary 2.** If $(G, \mathcal{T})$ is an abelian topological group, a subset $W$ is a $\mathcal{T}_B$-neighbourhood of $0$ if and only if there is a symmetric, relatively accumulating $\mathcal{F}$-neighbourhood $U$ of $0$ such that $U^5 \subset W$.

**Proof.** Let $V$ be a symmetric neighbourhood of $0$ with $V^4 \subset U$, and take $E = U$ in Theorem 2 A.

If $G$ is a discrete group, we may take $V = \{e\}$ in Theorem 2 A. In this case the conditions (A) and (B) are equivalent [4, Theorem 1], and we have

**Corollary 3.** If $G$ is an amenable discrete group, then a subset $W$ is a $\mathcal{T}_B$-neighbourhood if and only if there is a right relatively accumulating subset $E$ of $G$ with $E^{-1}EE^{-1}E \subset W$. 
For an amenable topological group let $n$ be the minimal number such that $V^n$ is a Bohr neighbourhood whenever $V$ is a symmetric, relatively dense neighbourhood of $e$. We have seen that in general $n \leq 7$, $n \leq 5$ for abelian groups and $n \leq 4$ for discrete groups. A natural question is whether this number can be reduced for some special groups. The following example pointed out to us by J. F. Aarnes, shows that in general we have $n > 1$.

Take the discrete group of integers $\mathbb{Z}$, and let $V = \{0, \pm 1, \pm 3, \pm 5, \ldots \}$. This set is symmetric and relatively dense. Since the characters on a group are almost periodic, the subset

$$U = \{n \in \mathbb{Z} : |e^{n\pi i} - 1| < 1\} = \{0, \pm 2, \pm 4, \pm 6, \ldots \}$$

is a Bohr neighbourhood of 0. Since $U \cap V = \{0\}$, $V$ is not a Bohr neighbourhood. Hence for $\mathbb{Z}$ we have $2 \leq n \leq 4$. For the real numbers with the usual topology a similar argument shows that $2 \leq n \leq 5$.

Another question naturally arises, if $G$ is not amenable: will then such an $n$ exist, or perhaps the finite chain characterization of the Bohr neighbourhoods (at least for locally compact groups) is equivalent to amenability? The answers to these questions are not known to the author.

REFERENCES


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