MATH. SCAND. 28 (1971), 129–138

SOME EXTREMAL PROBLEMS FOR FUNCTIONS UNIVALENT IN AN ANNULUS

RENATE MCLAUGHLIN

1. Introduction.

Many extremal problems for functions univalent in an annulus have been solved (see, for example, [1] through [9], [12], [13]), and most extremal functions obtained so far are simple: the omitted continuum is either a line segment or a cirular arc. P. L. Duren [1] obtained a more complicated extremal function when he considered the problem of maximizing the distortion at a fixed point in a certain class of univalent functions: the omitted continuum starts as a line segment and then sprouts a fork. In this paper, we shall encounter extremal functions whose omitted continua rarely are parts of well known curves.

Let R denote the annulus $\{z\colon 0<r_0<|z|<1\}$, and let F denote the class of functions analytic and univalent in R and satisfying the following conditions:

$$|f(z)| < 1 \text{ for } z\in R, \quad |f(z)| = 1 \text{ for } |z|=1\ , \tag{1}$$

$$f(z) \neq 0 \text{ for } z\in R\ , \tag{2}$$

$$f(1) = 1\ . \tag{3}$$

Using a variational method of P. L. Duren and M. Schiffer [2], we investigate the maxima of the quantities $\arg[f(z)/z]$ and $\arg f'(z)$, where z is a fixed point in R and where f ranges over the class F. The variational method yields a differential equation for the continuum inside the unit disk that is omitted by an extremal function. For both rotation problems, the omitted continuum is a curve that starts at the origin and spirals without branching. We use a special parametrization of the omitted curve (a method first employed by Schiffer [13]) to obtain a differential equation for the extremal function. However, some of the parameters occurring in the differential equation can only be determined implicitly.

Let F_0 denote the class of functions analytic and univalent in R and satisfying conditions (1) and (2). We use the variational method of Duren and Schiffer [2] to answer questions concerning the spherical derivative

Received July 11, 1969; in revised form October 29, 1969.

of functions in F_0, and we raise some questions concerning the curvature, convexity, and starlikeness of the images of a circle $|z|=r$, $r_0<r<1$, under functions in F_0.

Some results of this paper are contained in the author's dissertation, written at the University of Michigan under the direction of Professor Peter L. Duren.

2. The basic tool.

Duren and Schiffer [2] showed that if f belongs to F, then for all sufficiently small positive values of ϱ, $V_\varrho \circ f$ belongs to F, where

$$V_\varrho(w) = w\left[1+\frac{a\varrho^2(1-w)}{(w-w_0)(1-w_0)\,w_0}+\frac{\bar{a}\varrho^2(1-w)}{(1-\overline{w}_0 w)(1-\overline{w}_0)\overline{w}_0}\right] + O(\varrho^3);$$

and if f belongs to F_0, then for all sufficiently small positive values of ϱ, $V_\varrho^{(0)} \circ f$ belongs to F_0, where

$$V_\varrho^{(0)}(w) = w\left[1+\frac{a\varrho^2}{(w-w_0)w_0}-\frac{\bar{a}\varrho^2 w}{(1-\overline{w}_0 w)\overline{w}_0}\right] + O(\varrho^3)\,.$$

Here a is a complex number ($|a|<1$) depending on ϱ, and w_0, $0<|w_0|<1$, is a point in the continuum omitted by f.

Clearly, the classes F and F_0 are compact. Therefore extremal functions exist, and we can compare an extremal function g with its "neighbors" $V_\varrho \circ g$, respectively, $V_\varrho^{(0)} \circ g$. If this comparison yields an inequality of the type

$$\operatorname{Re}[a\varrho^2 s(w_0)+O(\varrho^3)] \leqq 0$$

for all sufficiently small values of ϱ (s is analytic at w_0), then Schiffer's lemma [13] implies that the continuum omitted by the extremal function g is an analytic curve satisfying the differential equation

$$w'(t)^2\, s\big(w(t)\big) > 0\,.$$

The study of extremal problems thus leads us to quadratic differentials. Concerning quadratic differentials, we use the terminology of [10].

3. A rotation problem.

Let z_0 be a fixed point in the annulus R, and choose the branch of the logarithm for which $\operatorname{Im}\log 1 = 0$. Then the problem

(4) $$\max_{f\in F} \arg[f(z_0)/z_0] = \max_{f\in F} \operatorname{Im}\log[f(z_0)/z_0]$$

is meaningful. Let g be an extremal function for this problem. Then

$$\operatorname{Im} \log[g(z_0)/z_0] \geqq \operatorname{Im} \log[V_\varrho(g(z_0))/z_0] ,$$

which implies that

$$\operatorname{Re}\left\{a\varrho^2\left[\frac{i(1-\bar{c})}{(1-w_0\bar{c})(1-w_0)w_0}-\frac{i(1-c)}{(c-w_0)(1-w_0)w_0}\right] + O(\varrho^3)\right\} \leqq 0 ,$$

where $c=g(z_0)$. Hence the continuum omitted by an extremal function for problem (4) satisfies the differential equation $w'(t)^2 s(w(t))>0$, where

$$(5) \qquad s(w) = i\,\frac{2c-1-|c|^2+w(2\bar{c}-1-|c|^2)}{w(1-w)(c-w)(1-\bar{c}w)} .$$

Note that the solution curves for this differential equation are symmetric with respect to the unit circle.

In the open unit disk, $s(w)$ has no zeros and only simple poles at $w=0$ and $w=c$. Therefore one trajectory terminates at $w=0$, one trajectory terminates at $w=c$, and these are the only points in the unit disk where a trajectory can terminate. In order to find the limiting tangential direction of the trajectory terminating at the origin, we set $w=\omega^2$. The differential equation $w'(t)^2 s(w(t))>0$ becomes

$$\left(\frac{d\omega}{dt}\right)^2 i\,\frac{2c-1-|c|^2+w(2\bar{c}-1-|c|^2)}{(1-w)(c-w)(1-\bar{c}w)} > 0;$$

hence

$$\omega'(t)^2 i(2c-1-|c|^2)c^{-1} > 0 \quad \text{for } \omega=0 .$$

It follows that the limiting tangential direction at the origin in the w-plane is that of the number

$$i[c(1+|c|^2)-2|c|^2] .$$

In particular, if c is positive, a trajectory leaves the origin in the direction of the positive imaginary axis. Similarly, we find that the limiting tangential direction of the trajectory terminating at $w=c$ is that of the number $-ic$.

A look at the direction field of the differential equation $w'(t)^2 s(w(t))>0$ reveals that the continuum omitted by an extremal function for problem (4) is an arc (its length is determined by the modulus of R) that starts at the origin and spirals outward without branching.

Let Γ denote the continuum omitted by the extremal function g. Using a technique first employed by Schiffer [13, pp. 444–446], we shall obtain a differential equation for g from the differential equation for Γ.

Note first that according to the symmetry principle, every function $f \in F$, defined for $r_0 < |z| \leqq 1$, can be extended to a function (call it f also) analytic and univalent in $r_0 < |z| < 1/r_0$.

It is easy to prove that $w(t) = g(r_0 e^{it})$ is a parametrization of the curve Γ and that $w'(t) = i r_0 e^{it} g'(r_0 e^{it})$. With this parametrization and the abbreviation $z = r_0 e^{it}$, the differential equation for Γ becomes

$$-z^2 g'(z)^2 s(g(z)) \geqq 0 . \tag{6}$$

The left-hand side of (6) is a function of z, defined for $r_0 \leqq |z| \leqq 1/r_0$; we abbreviate it by $H(z)$. Thus

$$H(z) = -z^2 g'(z)^2 s(g(z))$$

is meromorphic and satisfies the conditions

$$H(1/\bar{z}) = \overline{H(z)}$$

and

$$H(z) \geqq 0 \quad \text{for } |z| = r_0 \text{ and } |z| = 1/r_0 .$$

Moreover, $H(z)$ has simple poles at $z = 1$, $z = z_0$, and $z = 1/\bar{z}_0$. At the point on $|z| = r_0$ corresponding to the origin in the w-plane, $H(z)$ has a removable singularity, since $g(z)$ and $g'(z)$ both vanish.

It is convenient to look at $H(z)$ in the u-plane, where $u = \log z$. To this end, we set

$$G(u) = H(e^u) .$$

The principal branch of the logarithm maps the annulus $r_0 \leqq |z| \leqq 1/r_0$ onto the rectangle

$$\log r_0 \leqq \operatorname{Re} u \leqq -\log r_0, \quad 0 \leqq \operatorname{Im} u < 2\pi .$$

Clearly, we can extend $G(u)$ to the entire strip $\log r_0 \leqq \operatorname{Re} u \leqq -\log r_0$ by the rule $G(u) = G(u + 2\pi i)$. Note that

$$G(-\bar{u}) = H(e^{-\bar{u}}) = \overline{H(e^u)} = \overline{G(u)} .$$

Since $G(u)$ takes only nonnegative values on the lines $\operatorname{Re} u = \log r_0$ and $\operatorname{Re} u = -\log r_0$, we may apply the symmetry principle repeatedly and extend the domain of G to the entire u-plane.

The function G is now meromorphic and doubly periodic in the u-plane with periods $2\omega_1 = 2\pi i$ and $2\omega_2 = -2 \log r_0$. In each period parallelogram, G has three simple poles. Computations show that the residue of G at $u = 0$ is $2ig'(1)$, the residue of G at $u = \log z_0$ is $-i z_0 b c^{-1}$, and the residue of G at $u = -\log \bar{z}_0$ is $-i \bar{z}_0 \bar{b} \bar{c}^{-1}$, where $b = g'(z_0)$. But an elliptic function whose periods and principal parts are known has a representation in terms of the Weierstrass ζ-function and its derivatives [11, p. 182, Theorem 5.13]. We find that

$$G(u) = K + 2ig'(1)\zeta(u) - iz_0bc^{-1}\zeta(u - \log z_0) - i\bar{z}_0\bar{b}\bar{c}^{-1}\zeta(u + \log\bar{z}_0) ,$$

where K is a constant and $\zeta(u)$ is the Weierstrass ζ-function constructed from the periods $2m\pi i - 2n \log r_0$. We now have the following differential equation for g:

$$(7) \qquad g'(z)^2 \frac{2c - 1 - |c|^2 + g(z)[2\bar{c} - 1 - |c|^2]}{g(z)[1 - g(z)][c - g(z)][1 - \bar{c}g(z)]}$$
$$= iz^{-2}[K + 2ig'(1)\zeta(\log z) - iz_0bc^{-1}\zeta(\log[z/z_0]) - i\bar{z}_0\bar{b}\bar{c}^{-1}\zeta(\log z\bar{z}_0)] .$$

The differential equation (7) contains the parameters c, z_0bc^{-1}, $g'(1)$, and K. It is easy to show that K is real and that $\mathrm{Re}(z_0bc^{-1}) = g'(1)$. Sufficiently many relations involving the remaining parameters are readily available, but an explicit determination does not seem possible.

4. Another rotation problem.

We now turn to finding the maximum of the quantity

$$\arg f'(z_0) = \mathrm{Im} \log f'(z_0) ,$$

for a fixed z_0 in R. Because of the well known identity

$$\lim_{z\to z_1} \arg[f(z) - f(z_1)] - \lim_{z\to z_1} \arg[z - z_1] = \arg f'(z_1) ,$$

the argument of $f'(z_1)$ is the difference of the arguments of the tangent vectors to a curve in the z-plane and its image in the w-plane. Since each $f \in F$ maps the circle $|z| = 1$ onto the circle $|w| = 1$ such that $f(1) = 1$, we have that $\arg f'(1) \equiv 0 \pmod{2\pi}$ for each $f \in F$. (Implicit here is an extension of f, by reflection, to $r_0 < |z| < 1/r_0$.) Thus we can choose the branch of the logarithm for which $\mathrm{Im} \log f'(1) = 0$, and the problem

$$(8) \qquad \max_{f\in F} \mathrm{Im} \log f'(z_0)$$

is meaningful.

Let g be an extremal function for problem (8). Then

$$\begin{aligned}\mathrm{Im} \log g'(z_0) &\geqq \mathrm{Im} \log (V_\varrho \circ g)'(z_0) \\ &= \mathrm{Im} \log V_\varrho'(g(z_0)) + \mathrm{Im} \log g'(z_0) .\end{aligned}$$

Setting $g(z_0) = c$, we obtain the inequality

$$\operatorname{Re}[a\varrho^2 s(w_0)+O(\varrho^3)] \leqq 0 ,$$

where

$$s(w) = i\frac{1}{w(1-w)(c-w)^2(1-\bar{c}w)^2}P(w)$$

and

$$P(w) = a_0+a_1w+\bar{a}_1w^2+\bar{a}_0w^3$$

with $a_0=2c(c-|c|^2)$, $a_1=1+4|c|^2+|c|^4-4c-2c|c|^2$. Hence the continuum Γ omitted by the extremal function g satisfies the differential equation $w'(t)^2 s(w(t))>0$. Note that the solution curves of this differential equation are symmetric with respect to the unit circle.

In the open unit disk, $s(w)$ has a simple pole at $w=0$ and a pole of order 2 at $w=c$. The method used in Section 3 shows that the trajectory terminating at the origin has as limiting tangential direction that of the number $-i(1-c)$ (observe that $-i(1-c)$ always lies in the lower half-plane). It follows from [10, p. 32, Theorem 3.4] that at $w=c$, the trajectories of $w'(t)^2 s(w(t))>0$ behave locally like logarithmic spirals. Because the polynomial P satisfies the condition $\overline{P(1/\bar{w})}=w^{-3}P(w)$, it has either no root or exactly one root in $|w|<1$. Apparently both possibilities actually occur. We plotted the direction field of $w'(t)^2 s(w(t))>0$ for several choices of the parameter c. In each case, the omitted curve Γ bent from the origin toward $w=c$ in such a way that the argument of the tangent vectors changed monotonically.

As in Section 3, the parametrization $w(t)=g(r_0e^{it})$ yields a differential equation for g, namely

$$\begin{aligned} g'(z)^2 \frac{P(g(z))}{g(z)[1-g(z)][c-g(z)]^2[1-\bar{c}g(z)]^2} \\ = iz^{-2}[K+2ig'(1)\zeta(\log z)-i(1+z_0db^{-1})\zeta(\log[z/z_0])- \\ -\overline{i(1+z_0db^{-1})}\zeta(\log z\bar{z}_0)-i\wp(\log[z/z_0])+i\wp(\log z\bar{z}_0)] ,\end{aligned}$$

where $b=g'(z_0)$, $d=g''(z_0)$, and K is a constant. ζ and $\wp$ denote the Weierstrass ζ- and $\wp$-functions, constructed from the periods $2m\pi i+2n\log r_0$. It is easy to see that K is real and that $\operatorname{Re}(1+z_0db^{-1})=g'(1)$. Relations involving the remaining parameters can be obtained easily, but in this general setting, it does not seem possible to determine these parameters explicitly.

5. The spherical derivative.

In this section, we ask for the maximum and minimum of the spherical derivatives $|f'(z_0)|/(1+|f(z_0)|^2)$, $f\in F_0$, at a fixed point $z_0\in R$. Note that the spherical derivative of a function f is unchanged if we replace f by some rotation $e^{i\theta}f$. Thus we may assume, without loss of generality, that $z_0 = r > 0$ and that $f(r) > 0$.

Let g be an extremal function for the minimum problem, that is,

$$\frac{|g'(r)|}{1+|g(r)|^2} = \min_{f\in F_0} \frac{|f'(r)|}{1+|f(r)|^2}.$$

A comparison of g with the functions $V_\varrho^{(0)}\circ g$ leads to the following differential equation for the continuum Γ omitted by g:

$$w'(t)^2 \frac{1+aw(t)+w(t)^2}{w(t)\,[c-w(t)]^2[1-cw(t)]^2} > 0\,, \tag{9}$$

where $c = g(r) > 0$ and $a = (1+c^2)(1-6c^2+c^4)/4c^3$. (When Duren [1] asked for the maximal and minimal distortion of functions in F_0, he obtained a differential equation similar to (9). In his case, the constant a has the value $(1-4c^2-c^4)/2c^3$.)

It is easy to check that $w(t) = ct$, $0 < t < 1$, is a trajectory of (9). Hence the spherical derivative attains its minimal value at a fixed point for a radial slit mapping.

If we reverse the inequality sign in (9), we obtain the differential equation for the continuum Γ omitted by an extremal function g for the maximum problem.

The coefficient a in the numerator of (9) is a monotone decreasing function of c, and a can assume values in the interval $(-2, \infty)$. Let c_0 denote the value of c for which $a = 2$. (The number c_0 is the smallest positive root of the equation

$$c^4 - 2c^3 - 2c^2 - 2c + 1 = 0\,,$$

and one can show that $1/3 < c_0 < 7/20$.) We distinguish three cases.

i) $c = c_0$. The entire negative real axis and the entire unit circle, except for the point $w = -1$, are trajectories.

ii) $c_0 < c < 1$. The entire negative real axis and a portion of the unit circle, symmetric with respect to the real axis and containing $w = 1$, are trajectories.

iii) $0 < c < c_0$. The entire unit circle is a trajectory, and the segment $(w_1, 0)$ is a trajectory, where $w_1 = \frac{1}{2}(-a + (a^2-4)^{\frac{1}{2}})$ (recall that $a > 2$ in this case). At w_1, three trajectories terminate, and their limiting tangential directions are 120° apart. The solution curve to the differential equation is a forked curve, similar to the one obtained by Duren [1].

As in the case of the maximal distortion at a fixed point, the spherical derivative at a fixed point r attains its maximum for a function g that maps the annulus R onto the unit disk slit radially from 0 to $-g(r)$, provided r is sufficiently large (say, $r \geqq r_1$). For $r_0 < r < r_1$, the omitted line segment sprouts a fork.

Extremal functions maximizing the distortion at $z=r$ and those maximizing the spherical derivative at $z=r$ differ in the sense that a radial slit mapping yields a maximum for the spherical derivative "more often": if a radial slit map maximizes the distortion at r for $r_2 \leqq r < 1$, then a radial slit map maximizes the spherical derivative at r for $r_1 \leqq r < 1$, where $r_1 < r_2$.

6. Other extremal problems.

Each $f \in F_0$ maps the circle $|z|=1$ onto the circle $|w|=1$ whose interior is a convex domain. One can ask whether there exists a number r_1, $r_0 \leqq r_1 < 1$, such that each $f \in F_0$ maps each circle $|z|=r$, $r_1 < r < 1$, onto a curve whose interior is convex. Similarly, one can ask whether there exists a number r_2, $r_0 \leqq r_2 < 1$, such that each $f \in F_0$ maps each circle $|z|=r$, $r_2 < r < 1$, onto a curve whose interior is starlike with respect to the origin.

The questions in the preceding paragraph lead to the consideration of the maximum and minimum of the quantities

$$\text{(10)} \qquad \operatorname{Re}(zf'(z)/f(z)),$$

$$\text{(11)} \qquad \operatorname{Re}(1+zf''(z)/f'(z)),$$

where z is a fixed point in R. Expressions (10) and (11) remain unchanged if we replace f by some rotation $e^{i\theta}f$; hence we may assume that $z=r>0$ and $f(r)>0$.

Suppose the function g, $g \in F_0$, maximizes (10). Set $c=g(r)$ and $b=g'(r)$. The method of Section 2 shows that the continuum omitted by g satisfies the differential equation $w'(t)^2 s(w(t)) < 0$, where

$$\text{(12)} \qquad s(w) = \frac{a_0 - 2a_1 w + \bar{a}_0 w^2}{w(c-w)^2(1-cw)^2}$$

with $a_0 = b + \bar{b}c^2$ and $a_1 = bc + \bar{b}c$.

Since the roots of the polynomial $a_0 - 2a_1 w + \bar{a}_0 w^2$ lie on the unit circle $|w|=1$, a trajectory in the open unit disk can terminate only at $w=0$ or at $w=c$. The limiting tangential direction of the trajectory terminating at the origin is that of the number $-\bar{a}_0$. In particular, if b is positive, the omitted continuum is a segment $[-a,0]$.

If g minimizes (10), we only have to reverse the inequality sign in the differential equation. In particular, if b is positive, the omitted continuum is a segment $[0,a]$, $0<a<c$.

Suppose now that g maximizes (11), with $z=r$. Again we set $c=g(r)$ and $b=g'(r)$. The method of Section 2 shows that the omitted continuum satisfies the differential equation $w'(t)^2 s(w(t))>0$, where

$$s(w) = \frac{P(w)}{w(c-w)^3(1-cw)^3}, \tag{13}$$

and

$$P(w) = a_0+a_1w+a_2w^2+\bar{a}_1w^3+\bar{a}_0w^4 \tag{14}$$

with $a_0=-\bar{b}c^3$, $a_1=b+3\bar{b}c^2$, and $a_2=-3c(b+\bar{b})$. The limiting tangential direction of the trajectory terminating at the origin is that of the number $-b$. In particular, if b is positive, the omitted continuum is a segment $[-a,0]$.

If g minimizes (11), we reverse the inequality sign in the differential equation. If b is positive, the omitted continuum either is a segment $[0,a]$ (c must be large with respect to the modulus of R), or it is a segment $[0,a]$ with a fork at a.

Unfortunately, in none of the four cases above is it clear that an extremal function exists with the property that $b=g'(r)>0$. For nonreal values of b, the solution curves of the differential equations are no longer easy to identify.

The curvature of the image of a circle $|z|=r$ under the function f at $f(z_0)$, $|z_0|=r$, is given by the expression

$$\frac{1}{|z_0f'(z_0)|}\operatorname{Re}\left(1+z_0\frac{f''(z_0)}{f'(z_0)}\right). \tag{15}$$

It is reasonable to ask for functions in F_0 that maximize or minimize (15). Again we may assume that $z_0=r>0$ and $f(r)>0$.

If g is such an extremal function, the continuum omitted by g satisfies the differential equation $w'(t)^2s(w(t))>0$ (if g maximizes (15)) or $w'(t)^2s(w(t))<0$ (if g minimizes (15)). Here $s(w)$ is determined by (13) and (14) with

$$\begin{aligned} a_0 &= 2c^3(dc|b|-r\bar{b}), \\ a_1 &= dc|b|(1-6c^2-3c^4)+2rb+6r\bar{b}c^2, \\ a_2 &= d|b|(-1+3c^2+9c^4+c^6)-6rc(b+\bar{b}), \end{aligned}$$

and

$$c = g(r), \quad b = g'(r), \quad d = |b|^{-1}\,\mathrm{Re}\big(1+rg''(r)/g'(r)\big)\,.$$

As before, we find that the solution curve containing the origin is easy to identify, if b is positive. But we do not know whether such an extremal function exists.

REFERENCES

1. P. L. Duren, *Distortion in certain conformal mappings of an annulus*, Michigan Math. J. 10 (1963), 431–441.
2. P. L. Duren and M. Schiffer, *A variational method for functions schlicht in an annulus*, Arch. Rational Mech. Anal. 9 (1962), 260–272.
3. D. Gaier, *Untersuchungen zur Durchführung der konformen Abbildung mehrfach zusammenhängender Gebiete*, Arch. Rational Mech. Anal. 3 (1959), 149–178.
4. D. Gaier and F. Huckemann, *Extremal problems for functions schlicht in an annulus*, Arch. Rational Mech. Anal. 9 (1962), 415–421.
5. F. W. Gehring and G. af Hällström, *A distortion theorem for functions univalent in an annulus*, Ann. Acad. Sci. Fenn. Ser. A. I. (Math.), no. 325 (1963), 15 pp.
6. H. Grötzsch, *Über einige Extremalprobleme der konformen Abbildung* I, S.-B. Sächs. Akad. Wiss. Leipzig Math.-Phys. Kl. 80 (1928), 367–376.
7. H. Grötzsch, *Über einige Extremalprobleme der konformen Abbildung* II, S.-B. Sächs. Akad. Wiss., Leipzig, Math.-Phys. Kl. 80 (1928), 497–502.
8. F. Huckemann, *Über einige Extremalprobleme bei konformer Abbildung eines Kreisringes*, Math. Z. 80 (1962), 200–208.
9. F. Huckemann, *Extremal elements in certain classes of conformal mappings of an annulus*, Acta Math. 118 (1967), 193–221.
10. J. A. Jenkins, *Univalent functions and conformal mapping* (Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F. 18), Second Printing, Springer-Verlag, New York, 1965.
11. A. I. Markushevich, *Theory of functions of a complex variable* III (translated by Richard A. Silverman), Prentice–Hall, Inc., Englewood Cliffs, New Jersey, 1967.
12. Z. Nehari, *Conformal mapping*, McGraw–Hill, New York, 1952.
13. M. Schiffer, *A method of variation within the family of simple functions*, Proc. London Math. Soc. (2) 44 (1938), 432–449.

THE UNIVERSITY OF MICHIGAN, FLINT COLLEGE,
FLINT, MICHIGAN 48503, U.S.A.