IDEALS IN A C*-ALGEBRA

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Introduction.

The aim of this work is to continue the investigations of non-closed ideals in C*-algebras begun in [6]. In particular we shall, for a C*-algebra A, study the minimal dense ideal $K_A$, introduced in [9], and the "inductive limit" topology of $K_A$ given in [10]. Specializing in section 3 to C*-algebras with continuous trace we explore the non-commutative Gelfand transformation $\hat{x}: K_A \to K(\hat{A})$ given by $\hat{x}(\pi) = \text{tr}_{\pi(x)}$.

In [9, Theorem 1.5] the first author incorrectly stated that if A has continuous trace, then $K_A$ consists of the elements $\pi \in A$ such that the dimension of $\pi(x)$ is finite and bounded for $\pi \in \hat{A}$, and $\pi(x) = 0$ for $\pi$ outside some compact set in $\hat{A}$. Since $K_A$ is minimal dense it is true that $K_A$ is contained in the latter set, but we show by a counterexample that the inclusion may be proper, using homogeneous algebras whose corresponding fibre bundles have sufficiently many twists. Theorem 1.5 of [9] is cited in [12], but fortunately only the valid half of the theorem is needed for the conclusions.

It is a pleasure to thank H. Rischel, who provided and explained to us the building blocks of the above mentioned example.

We use the standard notation and terminology from [4]. Throughout the paper A will denote a C*-algebra.

1. Order ideals.

In [6] E. G. Effros set up a bijective correspondence between closed order ideals of $A^+$ and closed left ideals of $A$. The extension of this correspondence to non-closed order ideals was considered in [9], and the distinction between invariant and strongly invariant order ideals was clarified in [12]. The following theorem gives the complete extension of the Effros correspondence.

Theorem 1.1. Let $J$ be an order ideal of $A^+$ and define
\[ I = I(J) = \{ x \in A \mid x^* x \in J \} . \]

The map \( J \to I(J) \) is a bijection between order ideals of \( A^+ \) and left ideals of \( A \) satisfying:

1) If \( x \in I \) and \( \{ u_n \} \) is a bounded sequence in \( A \) such that \( \lim u_n x = A \), then \( \lim u_n x \in I \).

2) For any finite set \( \{ x_n \} \subseteq I \) there exists \( x \in I \) such that \( \sum x_n^* x_n = x^* x \).

Furthermore:

- \( J \) is invariant \( \iff \) \( I(J) \) is a two-sided ideal.
- \( J \) is strongly invariant \( \iff \) \( I(J) \) is self-adjoint \( \iff \) \( I(J) \) is positively generated.

**Proof.** Let \( \{ u_n \} \) be a sequence in the unit ball of \( A \) and assume \( x \in I(J) \), \( \lim u_n x = y \in A \). Then

\[ y^* y = \lim x^* u_n^* u_n x \leq x^* x \in J , \]

hence \( y \in I(J) \), so \( I(J) \) satisfies 1). That \( I(J) \) satisfies 2) as well is immediate from the definition.

Suppose now that \( I \) is a left ideal of \( A \) satisfying 1) and 2), and define

\[ J = \{ x^* x \in A^+ \mid x \in I \} . \]

Then \( J \) is a cone in \( A^+ \) since \( I \) is a linear space and satisfies 2). If \( 0 \leq y \leq x^* x \in J \), we put \( u_n = y^t (n^{-1} + x^* x)^{-1} x^* \). We have \( \| u_n \| \leq 1 \), and since \( (n^{-1} + x^* x)^{-1} x^* x \) is an approximative unit for the hereditary (=order-related [9], =facial [2]) \( C^* \)-subalgebra generated by \( x^* x \), we have \( \lim u_n x = y^t \in A \), hence \( y^t \in I \) and \( y \in J \) by condition 1). This shows that \( J \) is an order ideal, hence the correspondence, which is clearly injective, is a surjection on left ideals satisfying 1) and 2).

That \( J \) is invariant iff \( I(J) \) is a two-sided ideal follows from the expression

\[ (xy)^* (xy) = y^* (x^* x) y . \]

The equivalence between \( J \) strongly invariant and \( I(J) \) self-adjoint is seen by comparing

\[ x \in I(J) \iff x^* x \in J , \]

\[ x^* \in I(J) \iff xx^* \in J . \]

Clearly \( I(J) \) is self-adjoint if it is positively generated. Conversely, if \( I(J) \) is self-adjoint and \( x = x^* \in I(J) \), we have
\[ x = x_+ - x_- \quad \text{with} \quad x_+^2 + x_-^2 = x^2 \in J, \]
hence \( x_+, x_- \in I(J) \) and \( I(J) \) is positively generated.

**Remark.** If \( A \) is imbedded in a von Neumann algebra \( B \), then condition 1) is equivalent with

\[ \text{if } x \in I, \ u \in B, \ ux \in A \quad \Rightarrow \quad ux \in I. \]

An order ideal \( J \) of \( A^+ \) has roots if \( x \in J \) implies \( x^\alpha \in J \) for \( \alpha = \frac{1}{2} \) and hence for all \( \alpha \in \mathbb{R} \). An invariant order ideal with roots is strongly invariant. Since the property of having roots is equivalently expressed by \( J = I(J)^+ \), we have the following

**Corollary 1.2.** An order ideal \( J \) is invariant and has roots iff \( I(J) = \text{lin} J \).

An ideal generated by an invariant order ideal with roots is called **algebraic.** To give an idea of what an algebraic ideal can be like, consider the \( C^* \)-algebra \( C_0(\mathbb{R}) \) and the ideal consisting of those elements \( x \) such that \( px \in C_0(\mathbb{R}) \) for any polynomial \( p \). As the next results show, the class of algebraic ideals has properties very similar to those of the class of closed ideals.

**Proposition 1.3.** The class of algebraic ideals is a distributive lattice under sum and intersection.

**Proof.** By [12, Theorem 1.6] the class of strongly invariant order ideals is a distributive lattice under sum and intersection. To show that the invariant order ideals with roots form a sublattice assume that \( J_1 \) and \( J_2 \) have roots. Clearly, then \( J_1 \cap J_2 \) has roots. If \( x = y_1 + y_2 \), \( y_i \in J_i \), define

\[ u_{in} = (n^{-1} + x^\dagger)^{-1} y_i^\dagger. \]

Then

\[ (n^{-1} + x^\dagger)^{-1} x (n^{-1} + x^\dagger)^{-1} = (u_{1n} y_1^\dagger)(u_{1n} y_1^\dagger)^* + (u_{2n} y_2^\dagger)(u_{2n} y_2^\dagger)^*. \]

We have \( \|u_{in}\| \leq 1 \), and since \( (n^{-1} + x^\dagger)^{-1} x^\dagger \) is an approximative unit for the hereditary \( C^* \)-subalgebra generated by \( x \) we have \( u_{in} y_i^\dagger \) convergent to \( z_i \in A \). By condition 1) in theorem 1.1, \( z_i \in I(J_i) \) hence

\[ x^\dagger = z_1 z_1^* + z_2 z_2^* \in J_1 + J_2 \]

and we have shown that \( J_1 + J_2 \) has roots. The proposition now follows from the expressions.
\[ \text{lin} J_1 + \text{lin} J_2 = \text{lin} (J_1 + J_2), \]
\[ I(J_1) \cap I(J_2) = I(J_1 \cap J_2). \]

**Proposition 1.4.** If \( I_1 \) and \( I_2 \) are algebraic ideals, then
\[ I_1 I_2 = I_1 \cap I_2. \]

**Proof.** Put \( J_i = I_i^+ \). Then, since \( J_i \) has roots,
\[ I_1 I_2 \subseteq I_1 \cap I_2 = \text{lin} (J_1 \cap J_2) \subseteq \text{lin} (J_1 J_2) \subseteq \text{lin} J_1 \text{lin} J_2 = I_1 I_2. \]

**Corollary 1.5.** If \( I \) is algebraic, then \( I = I^n \).

**Corollary 1.6.** If \( I_1 \) is an algebraic ideal of an algebraic ideal \( I_2 \) of \( A \), then \( I_1 \) is an algebraic ideal of \( A \).

**Proof.** The only non-trivial thing to check is that \( I_1 \) is an ideal of \( A \). This follows from corollary 1.5 (cf. the implication 1.5.8 \( \Rightarrow \) 1.8.5 in [4]).

2. On the minimal dense ideal.

Let \( K_A \) be the intersection of all dense, hereditary two-sided ideals of \( A \). Then \( K_A \) is a dense algebraic ideal of \( A \). (Cf. [9], [10], [11], [12].)

**Proposition 2.1.** If \( B \) and \( C \) are \( C^* \)-subalgebras of \( A \) contained in \( K_A \), then the hereditary \( C^* \)-subalgebra generated by \( B \) and \( C \) is also contained in \( K_A \).

**Proof.** It suffices to prove that the closed order ideal \( J \) generated by \( B^+ \) and \( C^+ \) is contained in \( K_A^+ \). Since the order ideal generated by \( B^+ \) and \( C^+ \) is dense in \( J \) by [6, Theorem 2.5], there exists for any \( x \in J \) a sequence \( \{x_n\} \subset A \) converging to \( x \) such that
\[ x_n \leq \alpha_n y_n + \beta_n z_n, \]
\[ \alpha_n, \beta_n \in \mathbb{R}^+, \quad y_n \in B^+, \quad z_n \in C^+, \quad ||y_n|| = ||z_n|| = 1. \]

Define \( y = \sum 2^{-n} y_n, \ z = \sum 2^{-n} z_n \). Then \( y \in B^+, \ z \in C^+ \), hence \( y + z \in K_A^+ \) and \( x \) is contained in the closed order ideal generated by \( y + z \). By [11, Proposition 4] this order ideal is contained in \( K_A^+ \).

**Proposition 2.2.** Let \( A \) and \( B \) be \( C^* \)-algebras and let \( \Phi: K_A \rightarrow B \) be a morphism. Then \( \Phi \) extends canonically to a morphism \( \tilde{\Phi}: A \rightarrow B \), and if \( C = \tilde{\Phi}(A) \), then \( \Phi(K_A) = K_C \).
PROOF. Since $\Phi$ is norm continuous on every $C^*$-subalgebra of $K_A$ by [4, Proposition 1.3.7], we have $\|\Phi\| \leq 1$ on $K_A$. Hence $\Phi$ extends to a morphism $\bar{\Phi}$ of $A$. The last statement follows from [11, Corollary 6].

**Corollary 2.3.** If $K_A$ and $K_B$ are isomorphic as involutive algebras, then $A$ and $B$ are isomorphic $C^*$-algebras.

In [10, Theorem 2.1] a vector-space topology $\tau$ was defined on $K_A$, such that in the commutative case $\tau$ was the usual inductive limit topology on functions with compact supports. It was proved in [12, Theorem 2.4] that $\tau$ is the weakest locally convex topology on $K_A$ for which all invariant convex functionals on $K_A^+$ are continuous.

**Lemma 2.4.** Let $\Phi : K_A \to K_B$ be a linear positive and surjective map such that

1) $\forall x \in K_A \exists y \in K_B$:
   $\Phi(x^*x) = y^*y, \quad \Phi(xx^*) = yy^*$,

2) $\forall a \in K_A^+ \forall y \in K_B \exists x \in K_A$:
   $y^*y \leq \Phi(a) \Rightarrow x^*x \leq a, \quad \Phi(xx^*) = yy^*$.

Then $\Phi$ is open and continuous in the respective $\tau$-topologies.

**Proof.** For any invariant convex functional $\varrho$ on $K_B^+$ the composite map $\varrho \circ \Phi$ is an invariant convex functional on $K_A^+$ by 1), hence $\Phi$ is continuous. If $\varrho$ is any invariant convex functional on $K_A^+$, define

$$\sigma(b) = \inf \{ \varrho(a) \mid \Phi(a) = b \}.$$  

Then clearly $\sigma$ is a convex functional on $K_B^+$. If $y, z \in K_B$ and $y^*y \leq z^*z$, then for any $\varepsilon > 0$ there exists $a \in K_A^+$ such that

$$\sigma(z^*z) + \varepsilon > \varrho(a), \quad \Phi(a) = z^*z.$$  

There exists $x \in K_A$ satisfying 2), hence

$$\sigma(z^*z) + \varepsilon > \varrho(x^*x) = \varrho(xx^*) \geq \sigma(yy^*).$$

Since $\varepsilon$ is arbitrary this proves that $\sigma$ is invariant.

Finally we have by definition of $\sigma$ that

$$\Phi \{ a \in K_A^+ \mid \varrho(a) < 1 \} = \{ b \in K_B^+ \mid \sigma(b) < 1 \}.$$  

This proves that $\Phi$ is open.
Theorem 2.5. Let $\Phi: A \to B$ be a surjective morphism. Then the restriction of $\Phi$ to $K_A$ is an open and continuous map onto $K_B$ in the respective $\tau$-topologies.

Proof. That $\Phi$ satisfies 1) is obvious, and 2) follows from [2, Lemme 4.1].

3. For $C^*$-algebras with continuous trace.

Throughout this section we shall assume that $A$ is a $C^*$-algebra with continuous trace. In contrast to $CCR$-algebras and algebras of type I, it need not be true that $C^*$-subalgebras of $A$ have continuous trace. However the following result holds.

Proposition 3.1. If $B$ is a hereditary $C^*$-subalgebra of $A$, then $B$ has continuous trace.

Proof. By [10, Theorem 1.6] any irreducible representation of $B$ is the restriction of some irreducible representation $(\pi,H)$ of $A$ to the subspace $\pi(B)H$, and this restriction map induces a homeomorphism between $\hat{B}$ and $\hat{A} \setminus \text{hull}B$. It follows that an element $x \in B$ has bounded and continuous trace on $\hat{B}$ iff $x$ as an element of $A$ has bounded and continuous trace on $\hat{A}$.

Proposition 3.2. If $I$ is a closed two-sided ideal of $A$, then $I$ and $A/I$ have continuous trace.

Proof. The first statement is a corollary of proposition 3.1. To prove the second let $\Phi: A \to A/I$ be the natural morphism. By [4, Proposition 3.2.1] any irreducible representation of $A/I$ arises from an irreducible representation $(\pi,H)$ of $A$ for which $\pi(I)=0$, and since this induces a homeomorphism between $(A/I)^\wedge$ and hull $I$, we see that if $x \in A$ has bounded and continuous trace on $\hat{A}$, then $\Phi(x)$ has bounded and continuous trace on $(A/I)^\wedge$.

We define the map $^\wedge: K_A \to K(\hat{A})$ by

$$^\wedge(\pi) = \text{tr}\pi(x), \quad x \in K_A, \pi \in \hat{A}.$$  

Since $K_A$ is minimal dense, this is well defined and by [3, Lemme 23] it is a positive, linear and surjective map. When $A$ is commutative, $^\wedge$ is the Gelfand transformation, and even though $^\wedge$ is not multiplicative in the non-commutative case, it seems to be the best substitute we can
get. By the Dauns–Hofmann theorem [5, p. 379] there exists for any \( x \in A \) and \( f \in C^0(\hat{A}) \) an element in \( A \) denoted \( f \cdot x \) such that \( f(\pi)x - f \cdot x \in \ker \pi \) for all \( \pi \in \hat{A} \). It is immediate that we have

\[
(f \cdot x)^\wedge = f \hat{x}, \quad f \in C^0(\hat{A}), \quad x \in K_A.
\]

With the correspondence in mind between invariant \( C^* \)-integrals on \( A \) and Radon measures on \( \hat{A} \), proved in [3, Théorème 1], it is only natural that we have

**Theorem 3.3.** The map \( ^\wedge : K_A \to K(\hat{A}) \) is open and continuous in the respective \( \tau \)-topologies.

**Proof.** That \( ^\wedge \) satisfies condition 1) of lemma 2.4 is trivial. To prove that \( ^\wedge \) satisfies condition 2) as well, assume \( a \in K_A^+, \ f \in K(\hat{A})^+, \ f \leq \hat{a} \). Define \( x_n = (n^{-1} + \hat{a})^{-1} f \cdot a \). Then \( \{x_n\} \) converges to an element \( x \leq a \) such that \( \hat{x} = f \). Hence lemma 2.4 applies and the theorem follows.

Clearly \( ^\wedge \) is not injective although \( x \geq 0 \) and \( \hat{x} = 0 \) imply \( x = 0 \). Thus \( ^\wedge \) divides \( K_A^+ \) in equivalence classes, where \( x, y \in K_A^+ \) are equivalent when \( \hat{x} = \hat{y} \). By the Riesz decomposition for \( C^* \)-algebras [12, Proposition 1.1] we may define another equivalence relation on \( A^+ \) by putting \( x \sim y \) if there exists a finite set \( \{z_n\} \subset A \) such that

\[
x \in \Sigma z_n^* z_n, \quad y = \Sigma z_n z_n^*.
\]

In terms of these definitions the following theorem can now be stated:

**Theorem 3.4.** For \( x, y \in K_A^+ \) the following conditions are equivalent:

1. \( \hat{x} = \hat{y} \),
2. \( x \sim y \).

**Proof.** (2) \( \Rightarrow \) (1) is obvious. Since \( x + y \in K_A^+ \) there exist by definition two finite sets \( \{a_m\} \) and \( \{b_m\} \) in \( A^+ \) such that

\[
x + y \leq \Sigma a_m, \quad [a_m] \leq b_m.
\]

Put \( b = \Sigma b_m \) and let \( I \) be the closed two-sided ideal of \( A \) generated by \( b \). Then \( I \) has continuous trace by proposition 3.2, and \( x, y \in K_I^+ \). Since each set

\[
\mathcal{T}_n = \{ \pi \in \hat{I} \mid \|\pi(b)\| \geq n^{-1} \}
\]

is compact by [4, Proposition 3.3.7] and since \( \bigcup \mathcal{T}_n = \hat{I} \), the proof of (1) \( \Rightarrow \) (2) is reduced to the case where \( \hat{A} \) is \( \sigma \)-compact.
For any $\pi_0 \in \hat{\mathbb{A}}$ there exists by [4, Proposition 4.5.3] an element $e \in A^+$ such that $\pi(e)$ is a one-dimensional projection in a neighbourhood $\mathcal{O}$ of $\pi_0$. From the way $e$ is constructed in [4, Lemme 4.4.2] we see that one may assume $e \in K_{A^+}$. Let $J$ denote the strongly invariant order ideal in $A^+$ generated by $e$ and let $I$ denote the closed two-sided ideal of $A$ generated by $e$. Then $J$ consists of the elements $a \in A^+$ such that there exist $b \in A^+$ and $\alpha \geq 0$ with

$$ a \sim b, \quad b \leq \alpha e , $$

while $I$ can be described as the elements $a \in A$ such that for any $\pi \in \hat{\mathbb{A}}$

$$ \pi(e) = 0 \Rightarrow \pi(a) = 0 . $$

We claim that for any $a \in K_{A^+}$ and any positive function $f \in C^b(\hat{\mathbb{A}})$ with support in $\mathcal{O}$ we have

$$(*) \quad f \cdot a \in J .$$

Since the elements in $A^+$ satisfying $(*)$ constitute an order ideal it is enough to prove the formula for $a \in K_{\alpha^+}$, hence we may assume $[a] \leq b$ for some $b \in A^+$. Since $\hat{\mathcal{O}} = 1$ on the support of $f$, this gives

$$ [f \cdot a] \leq \hat{\mathcal{O}} \cdot b . $$

Now $\hat{\mathcal{O}} \cdot b \in I$ hence $f \cdot a \in K_{I^+}$ by definition, and since $J$ is dense in $I^+$, we have $K_{I^+} \subseteq J$ and the claim is established.

Since $\hat{\mathbb{A}}$ is locally compact and $\sigma$-compact, it is paracompact and normal, hence the covering by sets of the form $\mathcal{O}$ has a locally finite refinement $\{ \mathcal{U}_i \}$ and there exists a partition of unity $\{ f_i \}$ subordinate to the covering $\{ \mathcal{U}_i \}$ (see [8]). For each $i$ we select $e_i \in K_{A^+}$ such that $\pi(e_i)$ is a one-dimensional projection for $\pi \in \mathcal{U}_i$.

Now consider the pair $x, y \in K_{A^+}$. By $(*)$ there exist for each $i$ elements $a_i, b_i \in A^+$ and constants $\alpha_i, \beta_i$ such that

$$ f_i \cdot x \sim a_i, \quad f_i \cdot y \sim b_i , $$

$$ a_i \leq \alpha_i e_i, \quad b_i \leq \beta_i e_i . $$

Since $\hat{\mathcal{O}} = \mathcal{O}$, we have $\hat{\mathcal{O}}_i = \mathcal{O}_i$, but as $\dim \pi(e_i) = 1$ for any $\pi \in \mathcal{U}_i$, this implies $a_i = b_i$, hence $f_i \cdot x \sim f_i \cdot y$. Since the covering is locally finite and $\hat{\mathcal{O}}$ has compact support, only finitely many elements $f_i \cdot x$ and $f_i \cdot y$ are non-zero, hence

$$ x = \sum f_i \cdot x \sim \sum f_i \cdot y = y . $$

It is interesting to compare the above theorem with the result of Dixmier which one can read out of [3, Lemme 21], namely that if $x, y \in A^+$
have bounded and continuous trace on $\hat{A}$, then $x = y$ iff there exists a sequence $\{z_t\} \subset A$ such that $x = \sum z_t^* z_t$, $y = \sum z_t z_t^*$. Clearly, it is the minimality condition on $K_A$ which allows us to pass from a convergent sequence to a finite number of elements when $x, y \in K_A^+$.

We shall finally give an example of a $C^*$-algebra with continuous trace and homogeneous of degree two. By [7, Theorem 3.2] such an algebra is isomorphic with the family of continuous cross-sections of a suitable fibre bundle which we shall first describe. Let $P^n$ denote the complex $n$-dimensional projective space, i.e. the set of one-dimensional subspaces $\pi \subset \mathbb{C}^{n+1}$. The total space of the bundle is the set

$$E = \{(a, b, c, d, \pi) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \times P^n \mid c \in \pi, d^* \in \pi\},$$

where $d^*$ means complex conjugation at each coordinate. The base space is $P^n$ and the projection $p : E \to P^n$ is projection on the last coordinate. Clearly, then for any $\pi \in P^n$ the fibre $p^{-1}(\pi)$ is homeomorphic with $C^4$, hence with $M_2$, and if $\gamma_j$ is the open subset of points of $P^n$ whose $j$th homogeneous coordinates are non-zero, then $p^{-1}(\gamma_j)$ is homeomorphic with $M_2 \times \gamma_j$. It follows that the system $\mathcal{B}_n = (E, P^n, M_2, p)$ is a fibre bundle, where the bundle group is the subgroup of automorphisms of $M_2$ induced by inner transformations by the unitaries of the form

$$u_\theta = \begin{pmatrix} \exp i\theta & 0 \\ 0 & \exp -i\theta \end{pmatrix}. $$

We shall write the elements of $E$ in the form

$$e = \begin{pmatrix} a & c \\ d & b \end{pmatrix}, \pi$$

and we can then restore the matrix operations in the fibre over each $\pi$ by the definitions

$$ee' = \begin{pmatrix} aa' + c \cdot d' & ac' + cb' \\ da' + bd' & d \cdot c' + bb' \end{pmatrix}, \pi; \quad e^* = \begin{pmatrix} a^* & d^* \\ c^* & b^* \end{pmatrix}, \pi.$$

If we choose a unit vector $v$ in $\pi$, then $e$ is represented by the matrix $e \in M_2$, where

$$e = \begin{pmatrix} a & c \\ d & b \end{pmatrix}; \quad e = \begin{pmatrix} a & cv \\ dv^* & b \end{pmatrix}, \pi.$$

Since any other representative of $e$ is of the form $u_\theta^* e u_\theta$, the norm of $e$ depends only on $e$ and is denoted $\|e\|$.

Now let $A_n$ be the set of continuous cross-sections of $\mathcal{B}_n$, that is, the
set of continuous maps $x: \mathbb{P}^n \to E$ such that $p(x(\pi)) = \pi$ for all $\pi \in \mathbb{P}^n$. With the definitions
\[ xy(\pi) = x(\pi)y(\pi), \quad x^*(\pi) = x(\pi)^*, \quad \|x\| = \sup \|x(\pi)\|, \]
$A_n$ becomes a $C^*$-algebra. By [7, Theorem 3.2] $A_n$ is homogeneous of degree two and $\hat{A}_n = \mathbb{P}^n$.

We define elements $x_n, y_n \in A_n^+$ by
\[ x_n(\pi) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad y_n(\pi) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \pi, \]
and claim the following

**Lemma 3.5.** If $z_i \in A_n$, $i = 1, 2, \ldots, k$, and $\sum z_i z_i^* = x_n$, $\sum z_i^* z_i = y_n$, then $k > n$.

**Proof.** Clearly we have
\[ z_i(\pi) = \begin{pmatrix} 0 & c_i(\pi) \\ 0 & 0 \end{pmatrix}, \quad \pi, \]
where each $c_i$ is continuous and $\sum |c_i|^2 = 1$. If $S^{2n+1}$ is identified with the set
\[ \{a = (a_0, a_1, \ldots, a_n) \in \mathbb{C}^{n+1} \mid \sum |a_j|^2 = 1\} \]
and $\Psi: S^{2n+1} \to \mathbb{P}^n$ is given by $\Psi(a) = Ca$, then each composite map $c_i \circ \Psi: S^{2n+1} \to \mathbb{C}^{n+1}$ satisfies $c_i(\Psi(a)) \in \Psi(a)$. It follows that there exist continuous functions $\Phi_i: S^{2n+1} \to \mathbb{C}$ such that $c_i(\Psi(a)) = \Phi_i(a)$. Since $\sum |\Phi_i(a)|^2 = 1$, this defines a map $\Phi: S^{2n+1} \to S^{2k-1}$. However $\Psi(a) = \Psi(-a)$ and hence $\Phi(a) = -\Phi(-a)$. By the Borsuk–Ulam theorem [13, Corollary 5.8.8.] this implies $2k - 1 \geq 2n + 1$.

Now define $A_0 = \bigoplus_0 A_n$ (see [4, 1.9.14]). Then $A_0$ is homogeneous of degree two and $\hat{A}_0 = \bigcup \mathbb{P}^n$. Since $A_0$ is an ideal in the full direct sum of the $A_n$, and since $q = \sum x_n$ is a projection in this sum, we infer from [4, Corollaire 1.8.4] that the set $A = A_0 + Cq$ is a $C^*$-algebra. It is immediately verified that $\hat{A} = \bigcup \mathbb{P}^n \cup \{\pi_\infty\}$, where $\pi_\infty(A) = A/A_0 = \mathbb{C}$ and $\hat{A}$ is a one-point compactification of $A_0$. Clearly $A$ has continuous trace.

**Proposition 3.6.** There exists a $C^*$-algebra $A$ with continuous trace for which $K_A$ is properly contained in the set of elements $x \in A$ such that
\[ \sup \dim \pi(x) < \infty \quad \text{and} \quad \hat{x} \in K(\hat{A}). \]
PROOF. Take $A$ as above and consider the set $J$ consisting of elements $y \in A^+$ such that there exist a finite set $\{z_i\} \subseteq A$ and $\alpha \in \mathbb{R}^+$ satisfying

$$y = \sum z_i^* z_i, \quad \sum z_i z_i^* \leq \alpha q.$$ 

By definition $J$ is the strongly invariant order ideal generated by $q$. Since $q$ as a projection belongs to $K_A$, we have $J \subseteq K_A^+$. However, $q$ is not contained in any closed two-sided ideal of $A$, since $\pi(q) \neq 0$ for each $\pi \in \hat{A}$. Hence $J$ is dense in $A^+$ and so $J = K_A^+$.

Now consider $y = \sum n^{-1} y_n$. Then $\dim \pi(y) \leq 1$ for any $\pi \in \hat{A}$, and $\hat{g}$ is clearly continuous, hence $\hat{g} \in K(\hat{A})$ since $\hat{A}$ is compact. However, by lemma 3.5. there cannot exist any finite set $\{z_i\} \subseteq A$ such that $y = \sum z_i^* z_i, \quad \sum z_i z_i^* \leq \alpha q$. Hence $y \notin K_A$ and the proposition follows.

Apart from disproving [9, Theorem 1.5] (cf. the introduction) the above proposition also provides a negative answer to the problem raised in [4, 4.7.24].

If $\tilde{A}_0$ denotes the $C^*$-algebra obtained by adjoining an identity to $A$, then trivially $K_{\tilde{A}_0} = \tilde{A}_0$. This gives an example of a $C^*$-algebra which is CCR and has a Hausdorff spectrum, but for which theorem 3.4 is false.

Furthermore if $A''$ denotes the enveloping von Neumann algebra of $A$ and $I$ is the smallest ideal of $A''$ containing $K_A$, introduced in [1, Section 3], our example answers to the negative the question whether one always has $I \cap A = K_A$. To see this we observe that the projections $x_n, y_n \in A_n$ are equivalent in $A_n''$ since they are abelian and have central support 1. Hence there exists $v_n \in A_n''$ such that $x_n = v_n v_n^*$, $y_n = v_n^* v_n$. Define $v = \sum n^{-1} v_n \in A''$. Then $v^* v = y$ and $vv^* \leq q$, hence $y \in I \cap A \setminus K_A$.

REFERENCES


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