THE MAXIMAL IDEAL SPACE OF A BANACH ALGEBRA OF MULTIPLIERS

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1. Introduction.

J. L. Taylor [12] has characterized the maximal ideal space of the convolution algebra $\mathcal{M}(G)$ of bounded regular Borel measures on a locally compact abelian topological group $G$ as the set of semi-characters on a compact abelian topological semigroup called the Taylor structure semigroup of $\mathcal{M}(G)$. A different construction of the Taylor structure semigroup has later been given by Ramirez [7] and Rennison [8], who exploit the natural $C^*$-algebra structure in the dual of $\mathcal{M}(G)$ regarded as the bidual of $C_0(G)$, the $C^*$-algebra of continuous functions on $G$ vanishing at infinity. They note that the strongly closed span $Q$ of the set of multiplicative linear functionals on $\mathcal{M}(G)$ is a sub-$C^*$-algebra of $\mathcal{M}(G)'$, and indentify the Taylor structure semigroup of $\mathcal{M}(G)$ with the maximal ideal space of $Q$.

For a commutative semi-simple Banach algebra $A$, denote by $\hat{A}$ the set of the Gelfand transforms of the elements of $A$ and by $A^m$ the set of functions on the spectrum of $A$ that keep $\hat{A}$ invariant by pointwise multiplication. Each $f \in A^m$ determines a bounded linear operator on $A$ and the set $A^m$ of such operators, the multiplier algebra of $A$, is a Banach algebra under the uniform operator norm. If $A$ is $L^1(G)$, the convolution algebra of Haar integrable functions on $G$, $A^m$ may be identified with $\mathcal{M}(G)$. In this paper we generalize the structure theory of $\mathcal{M}(G)$ sketched above to the multiplier algebras of certain commutative semi-simple Banach algebras with the distinctive feature that the strongly closed span $P$ of the set of multiplicative linear functionals has the structure of a commutative Banach algebra, too. Following Birtel [4], we embed $A^m$ in $P'$. We then define an Arens quotient product in the dual of $A^m$ originating from the product in $P$ and show — under some natural additional assumptions — that the spectrum of $A^m$ spans a commutative subalgebra $Q$ of $(A^m)'$. The spectrum of $Q$ generalizes the Taylor structure semigroup $S$ of $\mathcal{M}(G)$, as is indicated in section 5. The main theorem is theorem 4.7, in which it is assumed that $P$ is a $C^*$-algebra with

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identity, though most of the auxiliary results are proved under weaker hypotheses. Applied to the case of the group algebra $L^1(G)$, theorem 4.7 shows that the natural embedding of the dual group $\Gamma$ of $G$ into the spectrum of $M(G)$ may be interpreted as the dual mapping from $\Gamma$ into the set of semicharacters on $S$ of a continuous homomorphism from $S$ onto the Bohr compactification of $G$.

**Convention.** All Banach algebras considered in this paper are complex. For any commutative Banach algebra $A$, $\mathcal{A}(A)$ denotes the spectrum of $A$, that is, the set of non-zero multiplicative linear functionals on $A$. If $D \subseteq A'$, $[D]$ denotes the subspace of $A'$ generated by $D$, and $[D]^\perp$ its norm closure.

2. Arens products and quotient products.

2.1. R. Arens [1], [2] has extended the product of an arbitrary Banach algebra $A$ to its bidual $A''$ by the following rule. If $m : A \times A \to A$ denotes the product in $A$, a jointly continuous bilinear map $m^* : A' \times A \to A'$ may be defined by setting $m^*(x', x)y = x'm(x, y)$ for $x' \in A'$, $x, y \in A$. Iterating this procedure one obtains

$$m^{**} : A'' \times A' \to A', \quad m^{**}(x'', x')y = x''m^*(x', y),$$

and finally

$$m^{***} : A'' \times A'' \to A'', \quad m^{***}(y'', x'')x' = y''m^{**}(x'', x').$$

For any Banach algebra product $m$ we denote by $m^t$ the product for which $m^t(x, y) = m(y, x)$. A is called Arens regular, if $m^{t***} = m^{***}$. When no confusion can arise, any Banach algebra product will be denoted in the usual way by juxtaposition.

**Theorem 2.1.** Let $A$ be a Banach algebra and $E$ a subspace of $A'$. Denote by $E^\circ$ the annihilator of $E$ in $A''$. Consider the following five statements:

1. $E^\circ$ is a right ideal of $A''$ in the $m^{***}$-product,
2. $m^{**}(A'' \times E) \subseteq \overline{E}$,
3. $m^*(E \times A) \subseteq \overline{E}$,
4. $m^{**}(E^\circ \times \overline{E}) = \{0\}$,
5. $E^\circ$ is a left ideal of $A''$ in the $m^{***}$-product.

We have the implications (1) $\iff$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5), and if $A$ is Arens regular, all statements are equivalent.

**Proof.** (1) implies (2): Suppose $m^{t**}(x'', x) \notin \overline{E}$ for $x \in E$, $x'' \in A''$. 
It is a consequence of the Hahn–Banach theorem that we can find $y'' \in E^\circ$ such that

$$m^{***}(y'', x'') x' = y'' m^{**}(x'', x') = 0.$$ 

Thus $E^\circ$ cannot be a right ideal for $m^{***}$.

(2) implies (1): If $x'' \in E^\circ$, $y'' \in A''$ and $x' \in E$, we have, assuming (2),

$$m^{***}(x'', y'') x' = x'' m^{**}(y'', x') = 0,$$

as $E^\circ = E^\circ$.

(2) implies (3): If $x' \in E$ and $\hat{x} \in A''$ is the canonical image of $x \in A$, we have for $y \in A$

$$m^*(x', x) y = x' m(x, y) = x' m^t(y, x) = \hat{x} m^*(x', y) = m^{**}(\hat{x}, x') y.$$

Hence $m^*(x', x) = m^{**}(\hat{x}, x') \in E$.

(3) implies (4): If $x'' \in E^\circ$, $x' \in E^\circ$ and $x \in A$, we have by (3),

$$m^{**}(x'', x') x = x'' m^*(x', x) = 0,$$

as $E^\circ = E^\circ$.

(4) implies (5): Take $x'' \in A''$, $y'' \in E^\circ$ and $x' \in E$. If (4) holds,

$$m^{***}(x'', y'') x' = x'' m^{**}(y'', x') = x''(0) = 0.$$

Thus $m^{***}(x'', y'') \in E^\circ$. Finally, if $A$ is Arens regular, (5) implies (1), since a left ideal for $m^{***} = m^{***}$ is a right ideal for $m^{***}$.

2.2. We assume in this subsection that $A$ is a commutative Banach algebra and $E$ a subspace of $A'$ such that the condition (2) of theorem 2.1 holds, that is, $m^{**}(A'' \times E) \subset E$. Then $m^*(\overline{E} \times A) \subset E$ by theorem 2.1 and the continuity of $m^*$. Thus we may define $m^{**}: E' \times \overline{E} \to A'$ by setting

$$m^{**}(F, x') x = F m^*(x', x) = x'' m^*(x', x) = m^{**}(x'', x') x,$$

where $F \in E'$ and $x''$ is its extension to $A'$ such that $\|x''\| = \|F\|$. Then $m^{**}$ is jointly continuous and $m^{**}(E' \times \overline{E}) \subset E$, so that we may define

$$m^{***}: E' \times E' \to E' \quad \text{by} \quad m^{***}(F, G) x' = F m^{**}(G, x').$$

By theorem 2.1., the ideal $E^\circ$ is a closed two-sided ideal in $(A'', m^{***})$. Let $p: A'' \to A''/E^\circ$ be the natural homomorphism. It is a well-known consequence of the Hahn–Banach theorem that $\varphi: A''/E^\circ \to E'$ defined by $\varphi \circ p(x'') = x''|E$ is a linear onto isometry.

**Theorem 2.2.** If $A''$ is given the product $m^{***}$, then $\varphi$ transfers to $m^{***}$ the natural product in $A''/E^\circ$. 
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Proof. If \( x'', y'' \in A'' \) and \( F = x''|E, G = y''|E \), we have for \( x' \in E \),

\[
m***(\varphi \circ p(x''), \varphi \circ p(y''))x' = Fm**(G, x') = x''m**(y'', x') = m***(x'', y'')x',
\]
and the theorem follows.

We call \( m*** \) the Arens quotient product in \( E' \), and denote usually \( m***(F, G) = F_* G \).

2.3. If \( A \) is a commutative Banach algebra and \( D \subset \Delta(A) \),

\[
m**(x'', \sum_{i=1}^n \lambda_i \alpha_i) = \sum_{i=1}^n \lambda_i x''(\alpha_i \alpha_i)
\]
for \( \lambda_i \in \mathbb{C}, \alpha_i \in D \), as is easily verified. Denote \( E = [D]^- \). By continuity, \( m**(A'' \times E) \subset E \), and the Arens quotient product in \( E' \) is given by

\[
F_* G(\sum_{i=1}^n \lambda_i \alpha_i) = \sum_{i=1}^n \lambda_i F(\alpha_i) G(\alpha_i)
\]
(cf. [4, p. 816]).

Theorem 2.3. Let \( A \) be a commutative Banach algebra and \( E = [D]^- \),
where \( D \subset \Delta(A) \). If \( E \) is a commutative Banach algebra having \( D \) as a multiplicative subsemigroup, then \( \Delta(E) \cup \{0\} \) with the weak* topology is a compact abelian topological semigroup under the Arens quotient product. If \( E \) has an identity \( e \in D \), \( \Delta(E) \) is a compact subsemigroup of \( \Delta(E) \cup \{0\} \).

Proof. If \( F, G \in \Delta(E), \lambda_i, \mu_j \in \mathbb{C}, \alpha_i, \beta_j \in D \)
for \( i = 1, \ldots, n, j = 1, \ldots, p \), then we have

\[
F_* G(\sum_{i=1}^n \lambda_i \alpha_i)(\sum_{j=1}^p \mu_j \beta_j) = F_* G(\sum_{i=1}^n \sum_{j=1}^p \lambda_i \mu_j \alpha_i \beta_j) = \sum_{i=1}^n \sum_{j=1}^p \lambda_i \mu_j F(\alpha_i) G(\beta_j)
\]

By continuity, \( F_* G \) extends to a multiplicative linear functional on \( E \).
If \( E \) has an identity \( e \in D \), \( F_* G \) is non-zero, as \( F_* G(e) = F(e) G(e) = 1 \).
It is well known that \( \Delta(E) \cup \{0\} \) is compact in the weak* topology
[9, p. 113], and \( \Delta(E) \) is compact if \( E \) has an identity. Since the Arens quotient product of \( E' \) can be defined by pointwise multiplication of the restrictions to \( D \) of the elements of \( E' \) and the elements of \( D \) suffice to define the relative weak* topology on \( \Delta(E) \cup \{0\} \) [9, pp. 112–113] the multiplication in \( \Delta(E) \cup \{0\} \) is jointly continuous.

Example. The following example shows the role of the requirement
that the identity of \( E \) belong to \( D \). Take \( A = C^2 \) with componentwise
operations and norm \( \| (z_1, z_2) \| = \sup \{ |z_1|, |z_2| \} \). Then \( A' = \mathbb{C}^2 \) with the norm \( \| (z_1, z_2) \| = |z_1| + |z_2| \), and \( A' \) is generated by \( \Delta(A) = \{(1, 0), (0, 1)\} \). If \( A' \) is considered as a Banach algebra under componentwise operations, \( A' \) has the identity \((1, 1)\), but

\[
\Delta(A') = \{(1, 0), (0, 1)\} \subset A'' = A
\]

is not closed under (Arens) multiplication.

2.4. The next result is essentially indicated in [7, pp. 1395–1396], but we give it an integration free proof. A semicharacter on a topological semigroup is a non-zero continuous homomorphism into the multiplicative semigroup of complex numbers \( z \) with \( |z| \leq 1 \).

**Theorem 2.4.** Let \( A \) be a commutative semi-simple Banach algebra. Suppose that \( P = [\Delta(A)]^- \) is a commutative \( C^* \)-algebra having \( \Delta(A) \) as a multiplicative subsemigroup with identity. Then the semicharacters of \( \Delta(P) \) are precisely the Gelfand transforms of the elements of \( \Delta(A) \).

**Proof.** Since

\[
F_\* G(\sum_{i=1}^n \lambda_i x_i) = Fm**(G, \sum_{i=1}^n \lambda_i x_i) = F(\sum_{i=1}^n \lambda_i G(x_i)x_i)
\]

for \( F, G \in P', \lambda_i \in \mathbb{C} \) and \( x_i \in \Delta(A) \), it follows by continuity that the adjoint of the operator \( F \mapsto F_\* G \) maps \( P \) into itself. Hence the Arens quotient product in \( P' \) is separately \( \sigma(P', P) \)-continuous [11, p. 128]. As \( P \) is semi-simple, \( H = [\Delta(P)]^- \) is a separating subspace of \( P' \). Hence \( H \) is \( \sigma(P', P) \)-dense in \( P' \) [11, p. 125].

Let \( f: \Delta(P) \to \mathbb{C} \) be a semicharacter. If \( f \neq 0 \) is continuous, it is the Gelfand transform of some non-zero \( x' \in P \). It follows from theorem 2.3 that \( H \) is a subalgebra of \( P' \), and since \( f \) is multiplicative on \( \Delta(P) \), it is clear that the continuous linear functional defined by \( x' \) on \( P' \) is multiplicative on \( H \). Since the \( \sigma(P', P) \)-continuous functions \( F \mapsto F(x')G(x') \) and \( F \mapsto F_\* G(x') \) for fixed \( G \in H \) coincide on \( H \), we have \( F_\* G(x') = F(x')G(x') \) for all \( F \in P' \). Keeping next \( F \in P' \) fixed and letting \( G \) vary, we conclude that the functional defined by \( x' \) is multiplicative on the whole of \( P' \), and as \( A \) is homomorphically embedded in \( P' \), \( x' \in \Delta(A) \). As

\[
F_\* G(\alpha) = F(\alpha)G(\alpha) \quad \text{and} \quad |F(\alpha)| \leq \|\alpha\| \|F\| \leq 1
\]

for \( F, G \in \Delta(P), \alpha \in \Delta(A) \), it is, conversely, clear that the Gelfand transform of any \( \alpha \in \Delta(A) \) is a semicharacter of \( \Delta(P) \).
3. Preliminaries on multipliers.

3.1. In this section $A$ is a commutative semi-simple Banach algebra and $P=[\Delta(A)]^{-}$. We denote by $A^{m}$ the set of functions $f: \Delta(A) \to \mathbb{C}$ such that $f\mathcal{A} \in \mathcal{A}$ for all $x \in A$. Each $f \in A^{m}$ defines a unique bounded linear operator $T_{f}: A \to A$ such that $(T_{f}x) = f\mathcal{A}$. The set $A^{m} = \{T_{f} | f \in A^{m}\}$ with the operations induced by pointwise addition and multiplication in $A^{m}$ is a commutative semi-simple Banach algebra under the uniform operator norm. It is called the algebra of multipliers of $A$. For the basic theory of multiplier algebras we refer to [3]. Since the elements of $\Delta(A)$ are linearly independent, each $f \in A^{m}$ extends uniquely to a linear functional on $[\Delta(A)]$. If $A$ has a weak bounded approximate identity [4, p. 817] this extension is continuous and determines therefore a unique element $f$ of $P'$. Since $A$ is semi-simple, the mapping $x \mapsto \hat{x}$, where $\hat{x}$ denotes the continuous linear functional on $P$ defined by $x$, is injective. It is clearly norm decreasing. If it is a homeomorphism, $A$ is said to be topologically embeddable in $P'$.

**Theorem 3.1.** If $A$ has a weak bounded approximate identity and is topologically embeddable in $P'$, the mapping $T_{f} \to f$ is a topological isomorphism from $A^{m}$ into $P'$.

**Proof.** As noted in [4, p. 817], the mapping $T_{f} \to f$ is a continuous homomorphism, and obviously injective. Let $C > 0$ be such that $||\hat{x}|| > C ||x||$ for all $x \in A$. Then we have

$$||f|| \geq \sup_{x \in A, ||x|| \leq 1} ||f_{*}\hat{x}|| \geq \sup_{||x|| \leq 1} ||f_{*}\hat{x}|| \geq C \sup_{||x|| \leq 1} ||T_{f}x|| = C ||T_{f}||,$$

and the assertion follows.

3.2. A commutative Banach algebra $A$ is called regular, if for any closed set $F \subset \Delta(A)$ and $\alpha \notin F$ there is $x \in A$ such that $\hat{x} | F \equiv 0$ and $x(\alpha) = 1$. Let $j_{A}(\infty)$ denote the set of elements of $A$ with Gelfand transforms of compact support. A function $f: \Delta(A) \to \mathbb{C}$ belongs locally to $\hat{A}$ at $\alpha \in \Delta(A)$, if there is a neighborhood $U$ of $\alpha$ and $x \in A$ such that $f | U = \hat{x} | U$. Augmented with theorem 3.1. above, the theorem in [4, p. 819], may be rephrased as follows:

**Theorem 3.2.** Suppose that $A$ is regular, topologically embeddable in $P'$, and has an approximate identity $\{u_{\alpha}\} \subset j_{A}(\infty)$. Then $f: \Delta(A) \to \mathbb{C}$ belongs to $A^{m}$ if and only if it can be extended to a continuous linear functional $f$ on $P$ and belongs locally to $A$ at each point of $\Delta(A)$. The correspondence $T_{f} \to f$ is a topological isomorphism from $A^{m}$ into $P'$.
4. The spectrum of the multiplier algebra.

4.1. Our main concern here is roughly to define a semigroup structure in \( \Lambda(\mathcal{A}^m) \) if one is given in \( \Lambda(\mathcal{A}) \). In view of theorem 3.2 it is reasonable to have first a closer look at the algebra of functions belonging locally to \( \hat{\mathcal{A}} \). — In this section we assume throughout that

a) \( \mathcal{A} \) is a commutative semi-simple Banach algebra, and

b) \( P = [\Lambda(\mathcal{A})]^* \) is a commutative Banach algebra under a product \( m \) such that

c) \( m(\Lambda(\mathcal{A}) \times \Lambda(\mathcal{A})) \subseteq \Lambda(\mathcal{A}) \).

For a function \( f: \Lambda(\mathcal{A}) \to \mathbb{C} \) and \( \alpha, \beta \in \Lambda(\mathcal{A}) \) we define \( f^\alpha(\beta) = f(\alpha \beta) \). A subset \( \mathcal{F} \) of \( \mathbb{C}^{\Lambda(\mathcal{A})} \) is translation invariant, if \( f^\alpha \in \mathcal{F} \) for all \( f \in \mathcal{F} \), \( \alpha \in \Lambda(\mathcal{A}) \).

**Theorem 4.1.** The mapping \( \xi \mapsto m(\alpha, \xi) \) on \( P \) is weak*-continuous for all \( \alpha \in \Lambda(\mathcal{A}) \) if and only if \( \hat{\mathcal{A}} \) is translation invariant.

**Proof.** When \( \mathcal{A} \) is regarded (algebraically) as a subspace of \( P' \), the adjoint \( m^*(\cdot, \alpha) \) of the norm continuous linear operator \( \xi \mapsto m(\alpha, \xi) \) maps \( \mathcal{A} \) into \( P' \). Since \( m^*(\hat{x}, \alpha) \beta = \hat{\xi}^\alpha(\beta) \) for all \( \beta \in \Lambda(\mathcal{A}) \), it follows therefore by linearity and continuity that \( m^*(\cdot, \alpha) \) maps \( \mathcal{A} \) into \( \mathcal{A} \) if and only if \( \hat{\xi}^\alpha \in \hat{\mathcal{A}} \) for all \( x \in \mathcal{A} \). But the condition \( m^*(\mathcal{A}, \alpha) \subseteq \mathcal{A} \) is equivalent to the \( \sigma(P, \mathcal{A}) \)-continuity of the operator \( \xi \mapsto m(\alpha, \xi) \) [11, p. 128], and the theorem is proved.

We assume henceforth that

d) \( \hat{\mathcal{A}} \) is translation invariant.

**Notation.** We denote by \( \mathcal{B} \) the subspace of \( P' \) consisting of those functionals whose restrictions to \( \Lambda(\mathcal{A}) \) belong locally to \( \hat{\mathcal{A}} \) at each point of \( \Lambda(\mathcal{A}) \). Let \( \mathcal{B}_0 \) denote the set of restrictions \( \xi' | \Lambda(\mathcal{A}) \), where \( \xi' \in \mathcal{B} \).

**Theorem 4.2.** \( \mathcal{B}_0 \) is translation invariant.

**Proof.** Let \( f \in \mathcal{B}_0 \) and \( \alpha \in \Lambda(\mathcal{A}) \). Since the multiplication in \( P \) is norm continuous, it is clear that \( f^\alpha \) is the restriction to \( \Lambda(\mathcal{A}) \) of a continuous linear form on \( P \). We show that \( f^\alpha \) belongs locally to \( \hat{\mathcal{A}} \) at \( \beta \in \Lambda(\mathcal{A}) \). There is a neighborhood \( U \) of \( \alpha \beta \) and \( x \in \mathcal{A} \) such that \( f|U = \hat{x}|U \). As the multiplication in \( \Lambda(\mathcal{A}) \) is separately continuous for the Gelfand topology by theorem 4.1, there is a neighborhood \( V \) of \( \beta \) such that \( \alpha V \subseteq U \). Then we have for any \( \gamma \in V \),

\[
f^\alpha(\gamma) = f(\alpha \gamma) = \hat{x}(\alpha \gamma) = \hat{x}^\alpha(\gamma),
\]

and the conclusion follows by the translation invariance of \( \hat{\mathcal{A}} \).
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Theorem 4.3. If $P$ is Arens regular, so that $(P'', m^{**})$ is commutative, the annihilator $B^\circ$ of $B$ is a closed ideal of $P''$ and $B' = B' = P''|B^\circ$ is a commutative Banach algebra under the Arens quotient product $m^{**}$ which, moreover, is separately $\sigma(B', B)$-continuous.

Proof. By theorem 4.2, $m^*(B, \alpha) \subset B$ for each $\alpha \in \Delta(A)$, and by linearity and continuity, $m^*(B \times P) \subset \bar{B}$. It follows from theorem 2.1 that $B^\circ$ is an ideal and the Arens quotient product, obviously commutative, is defined. Since

$$(F \star G) \xi' = Fm^{**}(G, \xi') \quad \text{for } F, G \in \bar{B}', \xi' \in \bar{B},$$

$$m^{**}(G, \xi') = m^{**}(\xi'', \xi') \in \bar{B}$$

(theorem 2.1), where $G = \xi''|P$, the adjoint of the operator $F \mapsto F \star G$ maps $\bar{B}$ into itself, whence the separate $\sigma(B', B)$-continuity of $m^{**}$ [11, p. 128].

4.2. As in theorem 4.3, we generally identify $B'$ and $\bar{B}'$ as Banach spaces (note, however, that the weak* topologies on $B'$ and $\bar{B}'$ are distinct unless $B$ is closed). As $\bar{B}$ is a commutative Banach algebra in the Arens quotient product of $P'$, $\Delta(\bar{B})$ is then identified with the set of continuous multiplicative linear functionals on $B$.

Theorem 4.4. If $P$ is Arens regular, then $\Delta(\bar{B}) \cup \{0\}$ is a multiplicative subsemigroup of $B'$. If the function $e(\alpha) \equiv 1$ belongs to $B_0$, then $\Delta(\bar{B})$ is a subsemigroup of $\Delta(\bar{B}) \cup \{0\}$.

Proof. Let $F, G \in \Delta(\bar{B}) \cup \{0\}$. We show that $m^{**}(F, G) = F \star G$ is multiplicative on $B$. For $f, g \in B$ and $\alpha, \beta \in \Delta(A)$ we have

$$m^*(fg, \alpha)\beta = fgm(\alpha\beta) = fm(\alpha\beta)gm(\alpha\beta) = m^*(f, \alpha)\beta m^*(g, \alpha)\beta.$$ 

Thus

$$m^*(fg, \alpha) = m^*(f, \alpha)m^*(g, \alpha),$$

where $m^*(f, \alpha)$ and $m^*(g, \alpha)$ belong to $B$ by theorem 4.2. Therefore,

$$m^{**}(G, fg) \alpha = Gm^*(fg, \alpha) = Gm^*(f, \alpha)Gm^*(g, \alpha)$$

$$= m^{**}(G, f) \alpha m^{**}(G, g) \alpha$$

for all $\alpha \in \Delta(A)$, so that

$$m^{**}(G, fg) = m^{**}(G, f) m^{**}(G, g),$$

where $m^{**}(G, f)$ and $m^{**}(G, g)$ belong to $\bar{B}$ by theorems 4.2 and 2.1.
Hence we have
\[ F_\ast G(fg) = Fm^{**}(G,fg) = Fm^{**}(G,f)Fm^{**}(G,g) = F_\ast G(f)F_\ast G(g). \]

Finally, if \( F \) and \( G \) are non-zero and \( e \in B_0 \), where \( e(\alpha) = 1 \), then
\[ m^{**}(G,e)\alpha = Gm^*(e,\alpha) = G(e) = 1 \]
for all \( \alpha \in \Lambda(A) \). It follows that
\[ F_\ast G(e) = Fm^{**}(G,e) = F(e) = 1. \]

4.3. In important applications, for example when \( A \) is the group algebra of a locally compact abelian group, \( P \) has the structure of a \( C^* \)-algebra. The next result is concerned with this situation.

**Theorem 4.5.** If, in addition to the assumptions a)—d), an involution \( \xi \mapsto \xi^* \) is defined on \( P \) such that \( P \) becomes a \( C^* \)-algebra, then \( B' \) is a commutative \( C^* \)-algebra with identity \( u \). If \( \xi' \in \overline{B} \), then \( \xi^\prime \in \overline{B} \) where \( \xi'(\xi) = \overline{\xi'(\xi^*)} \), and the involution \( F \mapsto F^* \) in \( B' = \overline{B} \) is given by
\[ F^*(\xi') = \overline{F(\xi')} \]
The involution in \( B' \) is \( \sigma(B',\overline{B}) \)-continuous. The subspace \( Q = [\Lambda(\overline{B})] \) is a sub-\( C^* \)-algebra of \( B' \). If \( P \) has an identity, its canonical image in \( B' \) is \( u \) and \( u \in Q \).

**Proof.** It is shown by Civin and Yood [5, p. 869] that \( P''' \) is Arens regular and is in fact isometrically isomorphic to the von Neumann algebra \( \mathcal{A} \) enveloping \( P \), i.e. the von Neumann algebra generated by the image of \( P \) in the universal representation (cf. [6, p. 236]). In particular, \( P''' \) is a commutative \( C^* \)-algebra with identity having \( B^\circ \) as a closed ideal (theorem 4.3). Hence \( B' \) is also self-adjoint, and \( P''/B^0 = B' \) is a commutative \( C^* \)-algebra with identity [6, proposition 1.8.2., p. 17]. It is shown by Ramirez [7, p. 1392] that the involution \( \xi'' \mapsto (\xi'')^* \) induced by \( \mathcal{A} \) on \( P''' \) can be defined by
\[ (\xi'')(\xi') = \overline{\xi''(\xi')}, \quad \text{where} \quad \xi'(\xi) = \overline{\xi'(\xi^*)}. \]
Another way of seeing this is to note that the above involution is weak* continuous and coincides with the original involution of \( P \) on the weak* dense subspace \( P \) of \( P''' \), whereas the weak* topology of \( P''' \) corresponds to the weak operator topology of \( \mathcal{A} \) [6, p. 237], for which the involution in \( \mathcal{A} \) is continuous. To the quotient algebra the involution is transferred as follows: If \( F \in B' \), \( F^* = (\xi'')^*|B \), where \( \xi'' \in P''' \) is any exten-
sion of $F$. If we now had $\xi' \notin \bar{B}$ for some $\xi' \notin \bar{B}$, we could find $\xi'' \in B^0$ such that $\xi''(\bar{\xi'}) \neq 0$. Then $\xi''|\bar{B} = 0$, and $\xi''(\bar{\xi'}) = (\xi''|\bar{B})*\xi' = 0$, which is a contradiction. It follows that the involution can be defined as stated in the theorem. It is then immediate that the involution is $\sigma(B', \bar{B})$-continuous. It follows from theorem 4.4 that $Q$ is a subalgebra of $B'$. Now let $F \in \Lambda(\bar{B})$ and $\xi', \eta' \in \bar{B}$. Then

$$\bar{\xi'} \bar{\eta'}(\alpha) = \bar{\xi'}(\bar{\eta'}(\alpha)) = \bar{\xi'}(\bar{\alpha}) \bar{\eta'}(\bar{\alpha}) = \bar{\xi'}(\alpha) \bar{\eta'}(\alpha)$$

for $\alpha \in \Lambda(A)$, so that $\bar{\xi'} \bar{\eta'} = \bar{\xi'} \bar{\eta'}$. Since

$$F^*(\bar{\xi'} \bar{\eta'}) = \bar{F}(\bar{\xi'} \bar{\eta'}) = \bar{F}(\bar{\xi'}) \bar{F}(\bar{\eta'}) = F^*(\bar{\xi'}) F^*(\bar{\eta'}) ,$$

$F^* \in \Lambda(\bar{B})$, and by the continuity of the involution $Q$ is a self-adjoint subalgebra of $B'$. Finally, if $P$ has an identity $e$, a straightforward verification shows that its canonical image $\dot{e}$ in $P''$ is an identity for $P''$. Clearly, the natural homomorphism maps $\dot{e}$ onto the uniquely determined identity $u$ of $P''/B^0 = B'$, and since each $\alpha \in \Lambda(A)$ may be regarded as an element of $\Lambda(\bar{B})$, $u \in Q$.

4.4. For $\xi \in P = [\Lambda(A)]^-$ and $f \in B$ we define

$$\Phi(\xi)f = f(\xi) .$$

Then $\Phi$ is a norm decreasing injection from $P$ into $Q = [\Lambda(\bar{B})]^-$, since $\Phi_\alpha \in \Lambda(\bar{B})$ for all $\alpha \in \Lambda(A)$. The range of $\Phi$ is a subalgebra of $B'$ and $\Phi$ is a homomorphism onto its range, as is readily seen by the definition of the Arens quotient product, noting that the Arens product in $P''$ extends the product of $P$.

**Theorem 4.6.** The adjoint $\Phi^*: Q' \to P'$ of $\Phi$ is a homomorphism for the Arens quotient products of $Q'$ and $P'$.

**Proof.** Let $F, G \in Q'$ and $\alpha \in \Lambda(A)$. Then $\Phi_\alpha \in \Lambda(\bar{B})$ and

$$\Phi^*(F_*G)\alpha = (F_*G)\Phi_\alpha = F(\Phi_\alpha)G(\Phi_\alpha)$$

$$= \Phi^*(F)\alpha \Phi^*(G)\alpha = (\Phi^*F_* \Phi^*G)\alpha .$$

Thus $\Phi^*(F_*G) = \Phi^*F_* \Phi^*G$.

4.5. If the assumptions of theorem 3.2 are fulfilled, then $B_0 = A^m$ and $B = \bar{B}$, which simplifies some notations. We conclude this section with a theorem in which we collect the most important results of the above theory in a specialized situation.
Theorem 4.7. Let $A$ be a commutative, regular, semi-simple Banach algebra with an approximate identity $\{u_\alpha\} \subset j_A(\infty)$. Suppose that $P = [\Delta(A)]^*$ is a commutative $C^*$-algebra having $\Delta(A)$ as a multiplicative subsemigroup with identity, such that $\hat{A}$ is translation invariant. Suppose, furthermore, that $A$ is topologically embeddable in $P$. Then $Q = [\Delta(A^m)]$ is a commutative $C^*$-algebra with identity having $\Delta(A^m)$ as a multiplicative subsemigroup. With their respective Gelfand topologies and Arens quotient products, $\Delta(P)$ and $\Delta(Q)$ are compact abelian topological semigroups, and $\Delta(A)$ (resp. $\Delta(A^m)$) may be identified, via the Gelfand transform, with the set of the semicharacters on $\Delta(P)$ (resp. on $\Delta(Q)$). There is a continuous homomorphism $\Psi$ from $\Delta(Q)$ onto $\Delta(P)$ such that the mapping $\Psi' : \Delta(A) \rightarrow \Delta(A^m)$ defined by
\[
(\Psi'(\alpha))^\wedge = \hat{\alpha} \circ \Psi
\]
is a topological isomorphism onto an open subsemigroup of $\Delta(A^m)$.

Proof. By virtue of theorem 3.2, $A^m$ may be identified with $B$ (cf. subsection 4.1), which is closed, as $A^m$ is a Banach algebra. By theorem 4.5, $Q$ is a commutative $C^*$-algebra having $\Delta(A^m)$ as a multiplicative subsemigroup with identity by theorem 4.4 (the Arens regularity of $P$ was noted in the proof of theorem 4.5). Theorem 2.3 then shows that $\Delta(P)$ and $\Delta(Q)$ are compact abelian topological semigroups and by theorem 2.4, $(\Delta(A))^\wedge$ (resp. $(\Delta(A^m))^\wedge$) is precisely the set of semicharacters on $\Delta(P)$ (resp. on $\Delta(Q)$). Consider the mapping $\Phi$ introduced in section 4.4. As $\Phi$ is a homomorphism and maps the identity of $P$ onto that of $Q$, $F \circ \Phi$ is a non-zero multiplicative linear functional on $P$ for any $F \in \Delta(Q)$. Thus the adjoint $\Phi^*$ of $\Phi$ maps $\Delta(Q)$ into $\Delta(P)$. We define $\Psi = \Phi^*|\Delta(Q)$. By theorem 4.6 $\Psi$ is a homomorphism. It is continuous for the weak* topologies of $\Delta(Q)$ and $\Delta(P)$ since $\Phi^*$ is so [11, p. 128]. Since $\Phi$ is injective, $\Psi(\Delta(Q)) = \Delta(P)$, as may be seen by an argument used in [6, p. 17], in the proof of proposition 1.8.1. If $\alpha \in \Delta(A)$, then $\hat{\alpha} \circ \Psi$ is a continuous function on $\Delta(Q)$, and therefore the Gelfand transform of an element $\Psi'(\alpha)$ of $Q$. We show that $\Psi'|\Delta(A) = \Phi|\Delta(A)$, whence in particular $\Psi'(\Delta(A)) \subset \Delta(A^m)$ (this inclusion also follows from the fact that $\hat{\alpha} \circ \Phi$ is a semicharacter on $\Delta(Q)$). If $F \in \Delta(Q)$, then
\[
F(\Psi'(\alpha)) = (\hat{\alpha} \circ \Psi)F = \hat{\alpha}(\Psi F) = (\Psi F)\alpha = F(\Phi \alpha),
\]
and the semi-simplicity of $Q$ implies that $\Psi''$ coincides with $\Phi$ on $\Delta(A)$. But it follows from the proof of theorem 1 in [3, p. 204], that $\Phi|\Delta(A)$ is a homeomorphism onto an open subset of $\Delta(A^m)$. 
5. Application to harmonic analysis.

5.1. Let $\Gamma$ be a locally compact abelian topological group with a fixed Haar measure, denoted by $dx$, and dual group $\Gamma$. If $A = L^1(\Gamma)$, $\Lambda(\Gamma)$ may be identified with $\Gamma$ and $P = [\Lambda(\Gamma)]^{-1}$ with $AP$, the $C^*$-algebra of the almost periodic functions on $\Gamma$. The Gelfand transform on $\Gamma$ is then the Fourier transform:

$$f(\gamma) = \int_\Gamma f(x)(x,\gamma) \, dx \text{ for } f \in L^1(\Gamma), \gamma \in \Gamma.$$  

It is a consequence of Eberlein's theorem [10, p. 32] that $L^1(\Gamma)$ is topologically, even isometrically, embeddable in $AP'$. Also the other assumptions of theorem 4.7 are well-known basic results of harmonic analysis. Since the multiplier algebra of $L^1(\Gamma)$ is isometrically isomorphic to the convolution measure algebra $M(\Gamma)$ [10, p. 74], theorem 4.7 yields a connection between $\Gamma$ and $\Lambda(M\Gamma)$. In this case $\Lambda(P) = \Lambda(AP)$ is the Bohr compactification of $\Gamma$.

5.2. Ramírez [7] and Rennison [8] have studied $\Lambda(M\Gamma)$ by considering the Arens product in $M\Gamma' = C_0(\Gamma)''$, which makes $M\Gamma'$ isomorphic to the enveloping von Neumann algebra of $C_0(\Gamma)$. In the next theorem we show that the $C^*$-algebra structure thereby introduced in $M\Gamma'$ is the same as that which is obtained by interpreting $M\Gamma$ as the multiplier algebra of $L^1(\Gamma)$ and using the techniques of section 4. For $\mu \in M\Gamma$, denote by $\hat{\mu}$ its Fourier–Stieltjes transform, i.e.

$$\hat{\mu}(\gamma) = \int_\Gamma (x,\gamma) \, d\mu(x) \text{ for } \gamma \in \Gamma,$$

and by $\hat{\mu}$ the continuous linear functional defined by $\hat{\mu}$ on $AP$. Then the mapping $\mu \mapsto \hat{\mu}$ is a linear isometry into $AP'$ [10, p. 32]. We denote by $B$ the image of $M\Gamma$ in $AP'$.

**Theorem 5.1.** When $M\Gamma' = C_0(\Gamma)''$ is regarded as the enveloping von Neumann algebra of $C_0(\Gamma)$ and $B'$ is given the $C^*$-algebra structure introduced in theorem 4.5, then the adjoint of the operator $\mu \mapsto \hat{\mu}$ is a $C^*$-algebra isomorphism from $B'$ onto $M\Gamma'$.

**Proof.** Let $\varphi$ be the adjoint of the inverse isometry $\hat{\mu} \mapsto \mu$, and denote $E = \varphi(C_0)$, where $C_0 = C_0(\Gamma)$ is canonically regarded as a subalgebra of $M\Gamma'$, whose $C^*$-algebra structure is defined analogously with that of $P''$ in the proof of theorem 4.5. Clearly, $\varphi$ is an isometric vector
space isomorphism onto $B'$. For $f \in C_0$ denote $\hat{f} = \varphi(f)$. We first show that $\varphi|C_0$ is an algebra isomorphism onto $E$. Denote the pointwise product in $AP$ by $m$. Then we have for $\alpha, \beta \in \Gamma$ and $\mu \in M = M(G)$,

$$m^* (\mu, \alpha) \beta = \mu(\alpha \beta) = \int \limits_{\hat{G}} (x, \alpha) (x, \beta) \, d\mu(x).$$

By the uniqueness of the Fourier-Stieltjes transform [10, p. 29],

$$m^* (\mu, \alpha) = \hat{\alpha} \mu,$$

where $\hat{\alpha} \mu$ is the measure $f \mapsto \mu(\hat{\alpha}f)$. Therefore, if $g \in C_0$,

$$m^{**}(\hat{g}, \hat{\mu}) \alpha = \hat{g} m^* (\hat{\mu}, \alpha) = \check{\alpha} \mu(g) = \int \limits_{\hat{G}} (x, \alpha) g(x) \, d\mu(x),$$

and by the same uniqueness theorem $m^{**}(\hat{g}, \hat{\mu}) = g \hat{\mu}$, where $g \mu$ is the measure $f \mapsto \mu(gf)$. It follows that

$$(\hat{f} \ast \hat{g}) \hat{\mu} = \hat{f} m^{**}(\hat{g}, \hat{\mu}) = \hat{f} (\hat{g} \mu) = (g \mu)f = \mu(fg),$$

i.e.

$$\varphi(fg) = \varphi f \ast \varphi g.$$

As $B$ is closed, the involution in $B'$ is weak* continuous (theorem 4.5), and multiplication separately weak* continuous (theorem 4.3). The corresponding statements are valid for $M'$ (see the proof of theorem 4.5). Since $\varphi$ is continuous with respect to the weak* topologies of $M'$ and $B'$ [11, p. 128], the mappings

$$S \mapsto \varphi(fS) \quad \text{and} \quad S \mapsto \varphi(f) \ast \varphi(S)$$

on $M'$ for fixed $f \in C_0$ are so, too. As they coincide by (1) on the weak* dense subspace $C_0$ of $M'$,

$$\varphi(fS) = \varphi f \ast \varphi S \quad \text{for all} \quad S \in M(G)'.$$

Fixing $S$ and replacing $f$ by a variable $T \in M'$, it is seen by the same argument that $\varphi$ is an algebra homomorphism. To show that $\varphi$ transfers the involution $S \mapsto S^*$ in $M(G)'$ to the involution $F \mapsto F^*$ defined in theorem 4.5, we first note that $C = C(G)$, the Banach space of bounded continuous functions on $G$ may be regarded as a subspace of the dual of $M(G)$, and since $C_0$ is $\sigma(C, M)$-dense in $C$ [11, p. 125], there is, in particular, for each $\gamma \in \Gamma$ a net $\{f_\gamma\}$ of elements of $C_0$ converging to $\gamma$ with respect to $\sigma(C, M)$. Using the Jordan decomposition of a measure into a linear combination of positive measures and the decomposition of the integrand into its real and imaginary parts for each of the positive
measures separately, we infer that the net \( \{f_\delta\} \) converges to \( \tilde{\gamma} \) for \( \sigma(C, M) \). Therefore,

\[
\hat{\mu}(\gamma) = \int_G (x, \gamma) \, d\hat{\mu}(x) = \lim_{\delta} \int_G f_\delta(x) \, d\hat{\mu}(x) = \lim_{\delta} \int_G f_\delta(x) \, d\mu(x) = \int_G (x, \gamma) \, d\mu(x) = \hat{\mu}(\tilde{\gamma}),
\]

and we have for \( \sum \lambda_i \gamma_i \in AP, i = 1, 2, \ldots, n \),

\[
\hat{\mu}(\sum \lambda_i \gamma_i) = \hat{\mu}(\sum \lambda_i \tilde{\gamma}_i) = \sum \lambda_i \hat{\mu}(\tilde{\gamma}_i) = \sum \lambda_i \hat{\mu}(\gamma_i) = \hat{\mu}(\sum \lambda_i \gamma_i).
\]

By continuity, \( \hat{\mu} = \hat{\mu} \), so that

\[
(\varphi S^*)\hat{\mu} = S^* \mu = \overline{S(\mu)} = \varphi S(\hat{\mu}) = \varphi S(\hat{\mu}) = \varphi S^* \hat{\mu},
\]

that is

\[
\varphi S^* = (\varphi S^*).
\]

We have proved that \( \varphi \) is a \( C^* \)-algebra isomorphism, and so is its inverse as stated in the theorem.

**Corollary.** If \( A = L^1(G) \) in theorem 4.7, then \( \Lambda(Q) \) is (topologically isomorphic to) the Taylor structure semigroup of \( M(G) \).

**Proof.** The corollary follows from the above theorem and theorem 6.5 in [8].

**References**


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