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ON DOUBLE DUALIZATION MONADS

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If $\mathscr{V}$ is a symmetric monoidal closed category in the sense of [2, Section III], and D is any object in it, the functor

$$-\pitchfork D\colon\ \mathscr{V} \to \mathscr{V}^{\mathrm{opp}}$$

(where $\pitchfork$ denotes the inner hom-functor of $\mathscr{V}$) is a (strong) left adjoint for

$$-\pitchfork D\colon\ \mathscr{V}^{\mathrm{opp}} \to \mathscr{V}\,,$$

and so gives rise to a strong monad $(-\pitchfork D)\pitchfork D$ on $\mathscr{V}$. If $\boldsymbol{T}=T,\eta,\mu$ is any other strong monad on $\mathscr{V}$, then giving a $\boldsymbol{T}$-algebra structure on D is equivalent to giving a transformation of monads

$$\tau\colon\ \boldsymbol{T} \to (-\pitchfork D)\pitchfork D \tag{0.1}$$

(Theorem 3.2 below). So the monad $(-\pitchfork D)\pitchfork D$ plays a role analogous to that of the ring of endomorphisms, $\mathrm{End}(A)$, of an abelian group: giving a Λ-module structure ξ on A is the same as giving a ring-homomorphism

$$\tau\colon\ \Lambda \to \mathrm{End}(A)\,.$$

We shall exploit this fact to define when two structures (for two possibly different monads) on a single object commute. In particular, we call an algebra commutative if the structure of the algebra commutes with itself. In the last section we prove that all algebras for a monad are commutative if and only if the monad is commutative in the sense of [4].

The notation and setting in the present paper is like in [4] and [5] which in turn is almost like in [2]. It should be possible to read this paper on the basis of knowledge of the concepts from [2] alone, although we shall need a few Lemmas and Definitions from [4] and [5]. Since the inner hom-functor $\hom_{\mathscr{V}}(-,-)$ (here denoted by $\pitchfork$ between the arguments) in this paper appears "iterated", it is convenient to avoid brackets by the convention

$$X\pitchfork Y\pitchfork Z = (X\pitchfork Y)\pitchfork Z\,.$$

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I owe credit to F. W. Lawvere for many interesting discussions. In particular, he called my attention to the existence of a τ like (0.1) which of course is fundamental for the present point of view.

1. Description of the monads.

Let D be an object in $\mathscr{V}$, fixed in what follows. Then we have a contravariant functor $-\pitchfork D$ from $\mathscr{V}_0$ to itself. The category $\mathscr{V}$ is a $\mathscr{V}$-category, and we can thus form the dual $\mathscr{V}$-category $\mathscr{V}^*$ by the recipe in Section III.2 of [2]; and, according to Section III.6 of the same paper, $-\pitchfork D$ is the functor part of a $\mathscr{V}$-functor $R = R^D$,

$$R\colon\ \mathscr{V}^* \to \mathscr{V}\,. \tag{1.1}$$

We use the notation R^D_{BA} (from [2]) for the strength of the functor R,

$$R^D_{BA}\colon\ A \pitchfork B \to (B \pitchfork D) \pitchfork (A \pitchfork D)\,.$$

The dual $\mathscr{V}$-functor R^* (formed according to Proposition III.2.2, p. 514 of [2]),

$$R^*\colon\ \mathscr{V} = \mathscr{V}^{**} \to \mathscr{V}^*\,, \tag{1.2}$$

is left adjoint to R in the strong sense, meaning that there exist $\mathscr{V}$-natural isomorphisms

$$\varphi_{A,B}\colon\ AR^* \pitchfork_* B \to A \pitchfork BR\,, \tag{1.3}$$

($\pitchfork_*$ denoting the hom-functor for $\mathscr{V}^*$, and $\mathscr{V}$-naturality meaning that axiom VN of [2, p. 466], is satisfied). The construction of $\varphi_{A,B}$ is as the composite arrow in

$$\begin{array}{ccccc} AR^* \pitchfork_* B = B \pitchfork (A \pitchfork D) & \xrightarrow{p^{-1}_{BAD}} & (B \otimes A) \pitchfork D \\ & & \Big\downarrow c \pitchfork 1 \\ A \pitchfork BR = A \pitchfork (B \pitchfork D) & \xleftarrow{p_{ABD}} & (A \otimes B) \pitchfork D\,, \end{array} \tag{1.4}$$

where c is the symmetry of $\mathscr{V}$, and p is the fundamental isomorphism for monoidal closed categories [2, p. 475]. According to Theorem III.7.4, p. 543, of [2], c and p are $\mathscr{V}$-natural, and so the composite (1.4) is $\mathscr{V}$-natural, by Theorem I.10.2, p. 466, of [2].

For our purposes, it is more convenient to restate the adjointness in terms of the front- and end-adjunctions. If one applies the underlying-functor V to (1.3), one gets an adjointness between the functors.

$$\mathscr{V}_0 \underset{-\pitchfork D}{\overset{-\pitchfork D}{\rightleftarrows}} \mathscr{V}_0^{\text{opp}};$$

the front adjunction y_A is then the value of the set mapping $\varphi_{A,AR^*}V$ on the element

$$1_{AR^*} = 1_{A \pitchfork D} \in (AR^* \pitchfork_* AR^*)V .$$

From Proposition III.7.9, p. 548, of [2], one directly infers

PROPOSITION 1.1. *The transformation* y_A *is* $\mathscr{V}$*-natural in* A (*and also* $\mathscr{V}$*-natural in* D *in the extraordinary sense defined in* [2, III.5]).

The end-adjunction ε for the adjointness is, by inspecting (1.4), seen also to be y_A, this time considered as a morphism in $\mathscr{V}_0^{\mathrm{opp}}$ from $A \pitchfork D \pitchfork D$ to A. The diagram stating that ε is $\mathscr{V}$-natural in A is then, by replacing $X \pitchfork_* Y$ by $Y \pitchfork X$, exactly the diagram stating that y_A is $\mathscr{V}$-natural in A. So ε_A, too, is $\mathscr{V}$-natural in A, by the proposition.

By Theorem I.10.7, p. 469, of [2], one gets a hypercategory $\mathscr{V}_*$ (=associative 2-dimensional category) by taking the objects to be $\mathscr{V}$-categories, morphisms (arrows) to be $\mathscr{V}$-functors, and hypermorphisms (2-cells) to be $\mathscr{V}$-natural transformations. Since y and ε are $\mathscr{V}$-natural, it follows that they make R^* and R adjoint arrows in $\mathscr{V}_*$. Adjoint arrows in any hypercategory give, by composition, rise to a monad in that hypercategory. So we have the following proposition (which of course is also easy to prove by direct methods, without appeal to hypercategories):

PROPOSITION 1.2. *The data*

$$\text{(i)}\quad \begin{cases} (- \pitchfork D) \pitchfork D:\ \mathscr{V}_0 \to \mathscr{V}_0 \\ A \pitchfork B \xrightarrow{R^D_{BA}} (B \pitchfork D) \pitchfork (A \pitchfork D) \xrightarrow{R^D_{A \pitchfork D, B \pitchfork D}} \\ \qquad\qquad\qquad\qquad [(A \pitchfork D) \pitchfork D] \pitchfork [(B \pitchfork D) \pitchfork D] \end{cases}$$

$$\text{(ii)}\qquad y_A:\ A \to (A \pitchfork D) \pitchfork D$$

$$\text{(iii)}\qquad y_{A \pitchfork D} \pitchfork 1:\ \big(((A \pitchfork D) \pitchfork D) \pitchfork D\big) \pitchfork D \to (A \pitchfork D) \pitchfork D$$

define a $\mathscr{V}$*-monad on* $\mathscr{V}$, *that is,* (i) *defines a* $\mathscr{V}$*-functor* $T: \mathscr{V} \to \mathscr{V}$, *and* (ii) *and* (iii) *define* $\mathscr{V}$*-natural transformations*

$$\eta:\ 1 \to T, \qquad \mu:\ T.T \to T , \tag{1.5}$$

so that the following (*usual*) *equations hold for any* $A \in |\mathscr{V}_0|$:

$$\eta_A T.\mu_A = 1_{AT} = \eta_{AT}.\mu_A , \tag{1.6}$$

$$\mu_A T.\mu_A = \mu_{AT}.\mu_A . \tag{1.7}$$

REMARK 1.3. Applying the right adjoint $- \pitchfork D$ to an object B in $\mathscr{V}_0^{\mathrm{opp}}$, and to the end-adjunction y_B for B, gives, by general theory [3], an algebra for the monad $- \pitchfork D \pitchfork D$.

2. The twisting.

It is convenient to "express" the symmetry $c\colon A \otimes B \to B \otimes A$ assumed for $\mathscr{V}$ in terms of $\pitchfork$ alone. For any objects X,Y,Z in $\mathscr{V}$, we have a composite isomorphism which we denote $TW_{X,Y,Z}$,

$$\begin{array}{ccc} X \pitchfork (Y \pitchfork Z) & \xrightarrow{TW_{X,Y,Z}} & Y \pitchfork (X \pitchfork Z) \\ \Big\downarrow {\scriptstyle p^{-1}_{XYZ}} & & \Big\uparrow {\scriptstyle p_{YXZ}} \\ (X \otimes Y) \pitchfork Z & \xrightarrow{c \pitchfork 1} & (Y \otimes X) \pitchfork Z \,. \end{array}$$

It is clearly "involutory": $TW_{X,Y,Z} . TW_{Y,X,Z} = 1$. Furthermore, using Theorem III.7.4, p. 543, in [2], we see that TW is $\mathscr{V}$-natural in each variable. So it is also natural in the ordinary sense in all three variables.

Applying the "underlying" functor V to $TW_{X,Y,Z}$ gives a set mapping

$$Tw_{X,Y,Z}\colon\ \mathscr{V}_0(X, Y \pitchfork Z) \to \mathscr{V}_0(Y, X \pitchfork Z)$$

(in the sequel subscripts will often be omitted) which again is natural in all three variables.

The definition of $y_A{}^D$ can be rephrased in terms of the Tw-operation. We have

$$Tw(A \pitchfork D \xrightarrow{1} A \pitchfork D) = A \xrightarrow{y_A{}^D} (A \pitchfork D) \pitchfork D \,.$$

Also, the transformation λ, which is fundamental to [5] and to this paper can be defined in terms of Tw. Let T,st be a $\mathscr{V}$-functor from $\mathscr{V}$ to itself (see [2, p. 444]). Then define for any pair A,B of objects of $\mathscr{V}$ a morphism

$$\lambda_{A,B}\colon\ (A \pitchfork B)T \to A \pitchfork BT$$

by

$$\lambda_{A,B} = Tw\big(A \xrightarrow{y_A{}^B} (A \pitchfork B) \pitchfork B \xrightarrow{\text{st}} (A \pitchfork B)T \pitchfork BT\big) \,. \tag{2.1}$$

This is just a restatement of the definition of λ from [5]. In particular, $\lambda_{A,B}$ is $\mathscr{V}$-natural in both variables.

There is a description of the Tw-operation in terms of morphisms already present in $\mathscr{V}_0$. Let $t\colon X \to Y \pitchfork Z$ be a morphism. Then $Tw(t)$ is the composite morphism

$$(2.2)\quad Y \xrightarrow{u} X \pitchfork (Y \otimes X) \xrightarrow{1 \pitchfork c} X \pitchfork (X \otimes Y) \to$$
$$\xrightarrow{1 \pitchfork (t \otimes 1)} X \pitchfork ((Y \pitchfork Z) \otimes Y) \xrightarrow{1 \pitchfork \mathrm{ev}} X \pitchfork Z\,,$$

where u and ev are the front- and end-adjunctions for the adjointness of the type

$$-\otimes A \;\dashv\; A \pitchfork -$$

(called u and t, respectively, in [2, p. 477]).

3. Double dualization monads and algebras.

Let $\boldsymbol{T} = ((T,\mathrm{st}),\eta,\mu)$ be a $\mathscr{V}$-monad on the $\mathscr{V}$-category $\mathscr{V}$, that is, T,st is a $\mathscr{V}$-functor from $\mathscr{V}$ to itself (in particular, $\mathrm{st}_{A,B}\colon A \pitchfork B \to AT \pitchfork BT$), and η and μ are $\mathscr{V}$-natural transformations

$$1_{\mathscr{V}} \xrightarrow{\eta} T \xleftarrow{\mu} T.T$$

satisfying the (usual) equations (1.6) and (1.7). Recall [3] that a $\boldsymbol{T}$-algebra is a pair (X,ξ), where $X \in \mathscr{V}_0$ and $\xi\colon XT \to X$ satisfies the unit- and associative laws

$$(3.1)\qquad \eta_X.\xi = 1, \quad \xi T.\xi = \mu_X.\xi;$$

a $\boldsymbol{T}$-homomorphism $(X,\xi) \to (X',\xi')$ is a morphism $f\colon X \to X'$ so that $fT.\xi' = \xi.f$.

Let (D,δ) be a $\boldsymbol{T}$-algebra. Construct a transformation $\tau\colon T \to (-\pitchfork D) \pitchfork D$ by putting τ_A equal to the composite

$$(3.2)\qquad AT \xrightarrow{y_A{}^{DT}} ((A \pitchfork D) \pitchfork D)T \xrightarrow{\lambda_{A \pitchfork D, D}}$$
$$(A \pitchfork D) \pitchfork DT \xrightarrow{1 \pitchfork \delta} (A \pitchfork D) \pitchfork D\,,$$

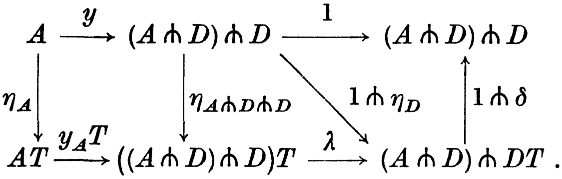
where λ is defined as in (2.1). Since y and λ are $\mathscr{V}$-natural in both variables, τ_A is $\mathscr{V}$-natural in A (and natural in the extraordinary sense with respect to (D,δ)).

Proposition 3.1. *The morphisms τ_A from* (3.2) *constitute a transformation of $\mathscr{V}$-monads.*

Proof. We have argued that τ_A is $\mathscr{V}$-natural in A. It remains to be proved that "τ commutes with the η's and μ's". To prove

$$\eta_A.\tau_A = y_A$$

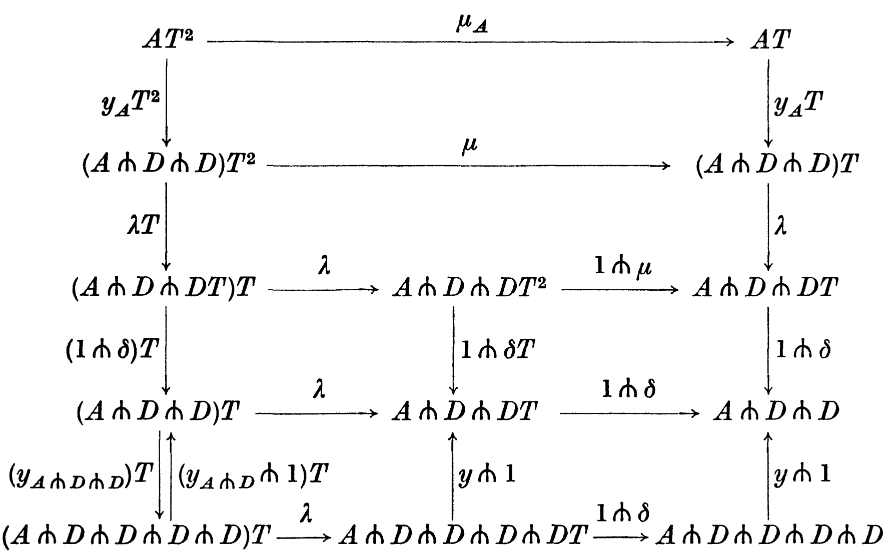
means proving commutativity of the outer diagram in



The left square commutes by naturality of η, the middle triangle by Lemma 1.6 in [5], and the right-hand triangle commutes by (3.1). To prove

$$\mu_A \cdot \tau_A = (\tau_A)T \cdot \tau_{A \pitchfork D \pitchfork D} \cdot (y_{A \pitchfork D} \pitchfork 1)$$

means proving commutativity of the outer diagram in



The top square commutes by naturality of μ; then comes a pentagon, and it commutes by Lemma 1.6 in [5]. The two next squares are commutative by naturality of λ and (3.1), respectively. The bottom right square obviously commutes; the bottom left square (reading upwards) again commutes by naturality of λ. Finally

$$(y_{A \pitchfork D \pitchfork D})T \cdot (y_{A \pitchfork D} \pitchfork 1)T = 1$$

comes from one of the standard equations between front- and end-adjunctions

$$y_{(A \pitchfork D)R} \cdot (\varepsilon_A \pitchfork_D)R^* = 1$$

for the fundamental adjointness (1.3). This proves Proposition 3.1.

We can perform a converse construction. If

$$\tau\colon\ T \to (-\pitchfork D)\pitchfork D \tag{3.3}$$

is a monad transformation, D can be endowed with a structure δ. This follows from the obvious observation that D carries a canonical algebra structure Δ for the $(-\pitchfork D)\pitchfork D$ monad. For, let I be the unit object which is part of the data for a closed category. Then $D \cong I \pitchfork D$, and so

$$y_I \pitchfork 1\colon\ I \pitchfork D \pitchfork D \pitchfork D \to I \pitchfork D$$

is an algebra structure on $I \pitchfork D$. Transporting it to D by means of $I \pitchfork D \cong D$ defines the structure Δ on D. An easy (and well-known) argument gives that if $\tau\colon\ T \to S$ is a monad transformation and X,ξ an S-algebra, then

$$XT \xrightarrow{\ \tau_X\ } XS \xrightarrow{\ \xi\ } X$$

makes X into a T-algebra. So in particular, with the τ of (3.3),

$$DT \xrightarrow{\ \tau_D\ } (D \pitchfork D)\pitchfork D \xrightarrow{\ \Delta\ } D \tag{3.4}$$

makes D into a T-algebra.

THEOREM 3.2. *There is a one-to-one correspondence between* $\boldsymbol{T}$*-algebra structures* δ *on* D *and* $\mathscr{V}$*-monad transformations from* $\boldsymbol{T}$ *to the double dualization monad for* D.

The theorem will follow from the above observations together with

PROPOSITION 3.3. *The passage from* τ *to* δ *and conversely describes a 1–1 correspondence between the set of maps* $\delta\colon\ DT \to D$ *and the set of* $\mathscr{V}$*-natural transformations* $T \to (-\pitchfork D)\pitchfork D$.

PROOF. Let us start with a map $\delta\colon\ DT \to D$. Then τ_A is given by (3.2). The clockwise composite in the following diagram is then the map $DT \to D$ constructed out of τ:

$$\begin{array}{ccccccc}
DT & \xrightarrow{\ yT\ } & (D\pitchfork D\pitchfork D)T & \xrightarrow{\ \lambda\ } & D\pitchfork D\pitchfork DT & \xrightarrow{\ 1\pitchfork\delta\ } & D\pitchfork D\pitchfork D \\
{\scriptstyle i_DT}\downarrow & & \cong\downarrow{\scriptstyle (i_D\pitchfork 1\pitchfork 1)T} & & \cong\downarrow & & \cong\downarrow \\
(I\pitchfork D)T & \xrightarrow{\ yT\ } & (I\pitchfork D\pitchfork D\pitchfork D)T & \xrightarrow{\ \lambda\ } & I\pitchfork D\pitchfork D\pitchfork DT & \xrightarrow{\ 1\pitchfork\delta\ } & I\pitchfork D\pitchfork D\pitchfork D \\
{\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle (y\pitchfork 1)T} & & \downarrow{\scriptstyle y\pitchfork 1} & & \downarrow{\scriptstyle y_I\pitchfork 1} \\
(I\pitchfork D)T & = & (I\pitchfork D)T & \xrightarrow{\ \lambda\ } & I\pitchfork DT & \xrightarrow{\ 1\pitchfork\delta\ } & I\pitchfork D \\
 & & \cong\downarrow{\scriptstyle (i_D{}^{-1})T} & & \cong\downarrow{\scriptstyle i_{DT}^{-1}} & & \cong\downarrow{\scriptstyle i_D{}^{-1}} \\
 & & DT & = & DT & \xrightarrow{\ \delta\ } & D\,.
\end{array}$$

In the first row the first diagram commutes by naturality of y, the second by naturality of λ, and the third one for obvious reasons. In the next row, the commutativity of the "triangle" is one of the adjunction equations for the fundamental adjointness; the first square commutes by naturality of λ, and the second one for obvious reasons.

Finally, the bottom "triangle" commutes by Lemma 1.7 in [5], and the bottom square by naturality of i. The counterclockwise composite in the diagram is δ. Conversely, if a $\mathscr{V}$-functor transformation $\tau\colon T \to (- \pitchfork D) \pitchfork D$ is given, we have to show that we get τ back again when applying the two processes. It can be done directly, but is quite elaborate. Instead, we shall derive from (the Eilenberg–Kelly version of) Yoneda's lemma, that the process leading from τ (assumed to be $\mathscr{V}$-natural) to δ is in fact one-to-one, onto; our process leading from δ to τ is then the inverse, since we already have seen that it is a one-sided inverse.

First notice that the Tw-operation can be used to establish a bijection (also called Tw)

$$\mathscr{V}\text{-}Nat(G, (-)F \pitchfork D) \cong \mathscr{V}\text{-}Nat(F, (-)G \pitchfork D)\,,$$

where $\mathscr{A}$ is a $\mathscr{V}$-category and

$$F\colon\ \mathscr{A} \to \mathscr{V} \quad \text{and} \quad G\colon\ \mathscr{A}^* \to \mathscr{V}$$

are $\mathscr{V}$-functors. For, let a τ be given on the left; construct $\hat{\tau}$ on the right by putting

$$\hat{\tau}_X = (\tau_X)Tw_{XG,XF,D}, \quad X \in |\mathscr{A}|\,.$$

Then $\hat{\tau}_X$ is $\mathscr{V}$-natural if and only if

$$(3.5) \qquad (\hat{\tau}_X)\iota\colon\ I \to XF \pitchfork (XG \pitchfork D)$$

is $\mathscr{V}$-natural in X, by Lemma III.7.8, p. 547, of [2]. But (3.5) can be described as the composite

$$I \xrightarrow{(\tau_X)\iota} XG \pitchfork (XF \pitchfork D) \xrightarrow{TW_{XG,XF,D}} XF \pitchfork (XG \pitchfork D)\,,$$

which as a composite of $\mathscr{V}$-natural transformations is $\mathscr{V}$-natural. From the involutory property of Tw it follows that the established correspondence is one-to-one, onto.

In particular, we have an isomorphism

$$(3.6) \qquad \mathscr{V}\text{-}Nat(T, (- \pitchfork D) \pitchfork D) \xrightarrow[\cong]{Tw} \mathscr{V}\text{-}Nat(- \pitchfork D, (-)T \pitchfork D)\,,$$

but for the right-hand side here we can use Eilenberg–Kelly's Yoneda lemma (Theorem I.8.6, p. 457, in [2]) to get a bijection Γ, displayed as the first arrow in

$$\text{(3.7)}\qquad \begin{array}{ccccc} \mathscr{V}\text{-}Nat(-\pitchfork D,(-)T\pitchfork D) & \underset{\cong}{\xrightarrow{\ \Gamma\ }} & \mathscr{V}_0(I,DT\pitchfork D) & \underset{\cong}{\xrightarrow{\ Tw\ }} & \mathscr{V}_0(DT,I\pitchfork D) \\ & & & & \Big\downarrow {\scriptstyle \mathscr{V}_0(1,i_D{}^{-1})\ \cong} \\ & & & & \mathscr{V}_0(DT,D)\,. \end{array}$$

Proposition 3.3 then follows from

LEMMA 3.4. *The bijection* (3.6) *followed by the bijection* (3.7) *sends* τ *to the structure* $\tau_D.\Delta$ *displayed in* (3.4).

PROOF. By (2.2), $\hat{\tau}_D$ is the composite from $D\pitchfork D$ to the lower right-hand corner in the diagram (3.8), p. 160.

The whole clockwise composite is $(\hat{\tau})\Gamma$. The whole counter-clockwise composite is, again by (2.2),

$$(\tau_D.j_D\pitchfork 1)Tw = (\tau_D.\Delta.i_D)Tw\,.$$

But (3.8) on p. 160 commutes by naturality of u and c and (extraordinary) naturality of ev. So

$$(\tau)Tw\Gamma = (\hat{\tau})\Gamma = (\tau_D.\Delta.i_D)Tw\,.$$

Applying Tw to this equation and multiplying on the right by $i_D{}^{-1}$ gives the equality claimed in the lemma.

REMARK 3.5. Since $(B\pitchfork D, y_B\pitchfork 1)$ is an algebra for the monad $-\pitchfork D\pitchfork D$ (Remark 1.3), Proposition 3.1 implies that there exists a transformation of $\mathscr{V}$-monads

$$\zeta\colon\ -\pitchfork D\pitchfork D \to -\pitchfork(B\pitchfork D)\pitchfork(B\pitchfork D)\,.$$

We can give a simple direct description of ζ. Consider

$$\text{(3.9)}\qquad X\pitchfork D\pitchfork D \xrightarrow{\ L^B\ } B\pitchfork(X\pitchfork D)\pitchfork(B\pitchfork D) \xrightarrow{\ TW\pitchfork 1\ } X\pitchfork(B\pitchfork D)\pitchfork(B\pitchfork D)\,.$$

It is clearly $\mathscr{V}$-natural in X. So to see that (3.9) actually is ζ_X, it suffices, by Proposition 3.3, to see that it gives rise to the structure $y_B\pitchfork 1$ on $B\pitchfork D$, (since $y_B\pitchfork 1$ was used to define ζ). To see this is fairly easy, using II.3.20, p. 480, and axioms CC 1 and CC 2 of [2].

$$
\begin{array}{ccccccc}
I & \xrightarrow{j_D} & D \pitchfork D & \xrightarrow{u} & DT \pitchfork ((D \pitchfork D) \otimes DT) & \xrightarrow{1 \pitchfork c} & DT \pitchfork (DT \otimes (D \pitchfork D)) \\
\Big\downarrow{\scriptstyle u} & & & & & & \Big\downarrow{\scriptstyle 1 \pitchfork (\tau_D \otimes 1)} \\
DT \pitchfork (I \otimes DT) \xrightarrow{1 \pitchfork c} DT \pitchfork (DT \otimes I) & \xrightarrow{1 \pitchfork (\tau_D \otimes 1)} & & & DT \pitchfork ((D \pitchfork D \pitchfork D) \otimes I) & \xrightarrow{1 \pitchfork (1 \otimes j_D)} & DT \pitchfork ((D \pitchfork D \pitchfork D) \otimes (D \pitchfork D)) \\
 & & & & \Big\downarrow{\scriptstyle 1 \pitchfork ((j_D \pitchfork 1) \otimes 1)} & & \Big\downarrow{\scriptstyle 1 \pitchfork \mathrm{ev}} \\
 & & & & DT \pitchfork ((I \pitchfork D) \otimes I) & \xrightarrow{1 \pitchfork \mathrm{ev}} & DT \pitchfork D\,.
\end{array}
\tag{3.8}
$$

4. Commuting structures.

In this section, we define and study the notion of commutation of structures on an object (with respect to two, possibly different, monads). There are two approaches to this (they can be shown to be equivalent). The one to be used here, defines the notion in terms of $\otimes$, and gives the desired results in a fairly straightforward way. The other describes commutativity in terms of $\pitchfork$; this is the approach of a preliminary draft [6] of the paper, and is much more complicated. Of course, one might define the notion of *symmetric closed category* without mentioning $\otimes$, in which case the $\pitchfork$-method could be applied, the $\otimes$-method not.

Recall [4] that the functor part T a $\mathscr{V}$-monad $\boldsymbol{T}$ on $\mathscr{V}$ has two canonical closed (or monoidal) structures

$$\psi^T, \tilde{\psi}^T\colon \ AT\otimes BT \to (A\otimes B)T\,.$$

Definition 4.1. Let $\boldsymbol{T}_0$, $\boldsymbol{T}_1$, and $\boldsymbol{S}$ be $\mathscr{V}$-monads on $\mathscr{V}$, and let $\tau_i\colon \boldsymbol{T}_i \Rightarrow \boldsymbol{S}$, $i=0,1$, be $\mathscr{V}$-monad transformations. We say that τ_0 *commutes with* τ_1 if the following diagram commutes for all $A,B \in |\mathscr{V}_0|$:

$$AT_0\otimes BT_1 \xrightarrow{(\tau_0)_A\otimes(\tau_1)_B} AS\otimes BS \underset{\tilde{\psi}^S}{\overset{\psi^S}{\rightrightarrows}} (A\otimes B)S\,, \tag{4.1}$$

T_i being the functor part of $\boldsymbol{T}_i$, S the functor part of $\boldsymbol{S}$.

Proposition 4.2. *The notion of structures commuting is symmetric.*

Proof. This follows since $c.\psi^S_{B,A} = \tilde{\psi}^S_{A,B}.(c)S$ by definition of $\tilde{\psi}$ in terms of ψ, where again c denotes the symmetry.

Proposition 4.3. *The notion of commutation is stable under left and right composition, that is, if*

$$\boldsymbol{T}_i' \overset{\varrho_i}{\Rightarrow} \boldsymbol{T}_i \overset{\tau_i}{\Rightarrow} \boldsymbol{S} \overset{\sigma}{\Rightarrow} \boldsymbol{S}'$$

are transformations of monads, $i=0,1$, and τ_0 commutes with τ_1, then $\varrho_0.\tau_0.\sigma$ commutes with $\varrho_1.\tau_1.\sigma$.

Proof. It is obvious for composing on the left. For composition by σ on the right, the result follows if we know that the diagram below (and the similar diagram with $\tilde{\psi}$ instead of ψ) commutes:

$$\begin{array}{ccc} AS\otimes BS & \xrightarrow{\psi^S} & (A\otimes B)S \\ {\scriptstyle \sigma_A\otimes\sigma_B}\downarrow & & \downarrow{\scriptstyle \sigma_{A\otimes B}} \\ AS'\otimes BS' & \xrightarrow{\psi^{S'}} & (A\otimes B)S'\,. \end{array} \tag{4.2}$$

But these commutativities are immediate, using Lemma 1.1 in [4] together with the definition of ψ and $\tilde{\psi}$.

One might thus define the notion of Freyd tensor product of $\mathscr{V}$-monads on $\mathscr{V}$. Call $\bar{\tau}_i$: $\boldsymbol{T}_i \Rightarrow \boldsymbol{T}_0 \otimes \boldsymbol{T}_1$, $i=0,1$, a tensor product of monads if

1) $\bar{\tau}_0$ and $\bar{\tau}_1$ are commuting $\mathscr{V}$-monad transformations, and

2) to any other pair τ_i: $\boldsymbol{T}_i \Rightarrow \boldsymbol{S}$ of commuting $\mathscr{V}$-monad transformations, there exists a unique $\mathscr{V}$-monad transformation σ: $\boldsymbol{T}_0 \otimes \boldsymbol{T}_1 \to \boldsymbol{S}$ with $\bar{\tau}_i.\sigma = \tau_i$, $i=0,1$. (If such a $\boldsymbol{T}_0 \otimes \boldsymbol{T}_1$ exists, it is essentially unique.)

Recall [4] that a monad $\boldsymbol{T}$ was termed commutative if $\psi^T = \tilde{\psi}^T$. In the terminology of Definition 4.1 this then just says that the identity transformation on $\boldsymbol{T}$ commutes with itself. So from Proposition 4.3, one derives

PROPOSITION 4.4. *If* $\boldsymbol{T}$ *is a commutative* $\mathscr{V}$*-monad, then any* $\mathscr{V}$*-monad transformation*

$$\tau: \quad \boldsymbol{T} \Rightarrow \boldsymbol{S}$$

commutes with itself, $\boldsymbol{S}$ *being an arbitrary* $\mathscr{V}$*-monad.*

DEFINITION 4.5. Let $\boldsymbol{T}_0, \boldsymbol{T}_1$ be $\mathscr{V}$-monads on $\mathscr{V}$, and let an object D have $\boldsymbol{T}_i$-structure δ_i: $DT_i \to D$, $i=0,1$. Let τ_i: $\boldsymbol{T}_i \Rightarrow - \pitchfork D \pitchfork D$ be the corresponding $\mathscr{V}$-monad transformations. Then δ_0 is said to commute with δ_1 if τ_0 and τ_1 commute, and D, δ_0 is called a commutative $\boldsymbol{T}_0$-algebra if τ_0 commutes with itself.

By Propositions 4.2 and 4.3 we get

PROPOSITION 4.6. *The notion of commuting structures* δ_0, δ_1 *on an object* D *is symmetric. Further, it is stable under left composition, that is, if* ϱ_i: $\boldsymbol{T}_i' \Rightarrow \boldsymbol{T}_i$, $i=0,1$, *are* $\mathscr{V}$*-monad transformations, and* δ_0, δ_1 *commute (as above), then the* $\boldsymbol{T}_0'$ *(resp.* $\boldsymbol{T}_1'$*) structures on* D

$$(\varrho_i)_D.\delta_i: \quad DT_i' \to D$$

commute.

Let δ_0, δ_1 and τ_0, τ_1 be as in Definition 4.5, and let B be an arbitrary object. By Remark 3.5 we have a $\mathscr{V}$-monad transformation

$$\zeta: \quad - \pitchfork D \pitchfork D \to - \pitchfork (B \pitchfork D) \pitchfork (B \pitchfork D) ,$$

and therefore, by composition, $\mathscr{V}$-monad transformations

$$\tau_i.\zeta: \quad \boldsymbol{T}_i \to - \pitchfork (B \pitchfork D) \pitchfork (B \pitchfork D), \quad i=0,1 , \tag{4.3}$$

which by Theorem 3.2 means that we have a $\boldsymbol{T}_i$-structure on $B \pitchfork D$ "induced by δ_i", $i=0,1$. If δ_0 commutes with δ_1, that is, τ_0 commutes with τ_1, then by Proposition 4.3, the two transformations in (4.3) also commute, so that we have

PROPOSITION 4.7. *If the $\boldsymbol{T}_0$-structure δ_0 on D commutes with the $\boldsymbol{T}_1$-structure δ_1 on D, then also the induced structures on $B \pitchfork D$ commute.*

One may describe the induced structure on $B \pitchfork D$ directly as the composite

$$(B \pitchfork D)T \xrightarrow{\ \lambda\ } B \pitchfork DT \xrightarrow{\ 1 \pitchfork \delta\ } B \pitchfork D\,.$$

The proof of this fact involves a medium sized diagram. We omit it.

The complete link between the notions of commuting structures, commuting $\mathscr{V}$-monad transformations, and commutative $\mathscr{V}$-monads is given in

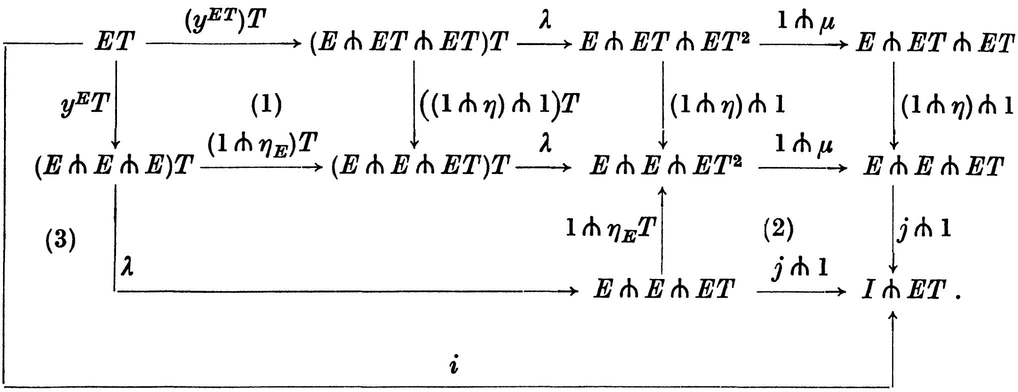
THEOREM 4.8. *Let $\boldsymbol{T}$ be a $\mathscr{V}$-monad on $\mathscr{V}$. Then the following three statements are equivalent:*

(i) *every algebra (D,δ) for $\boldsymbol{T}$ is commutative;*
(ii) *every $\mathscr{V}$-monad transformation with domain $\boldsymbol{T}$ commutes with itself;*
(iii) *$\boldsymbol{T}$ is commutative in the sense of* [4].

PROOF. Proposition 4.4 establishes (iii) $\Rightarrow$ (ii), and (ii) $\Rightarrow$ (i) is trivial in view of Definition 4.5. Finally assume (i) for the algebra $(D,\delta) = ((A \otimes B)T, \mu_{A \otimes B})$, and let τ be the corresponding $\mathscr{V}$-monad transformation $\boldsymbol{T} \Rightarrow (- \pitchfork D) \pitchfork D$. Denote the two closed structures on $(- \pitchfork D) \pitchfork D$ by ψ^{DD}, $\tilde{\psi}^{DD}$, respectively. Then by assumption, the clockwise composite in the following diagram commutes:

$$\begin{array}{ccc}
AT \otimes BT & \xrightarrow{\ \tau_A \otimes \tau_B\ } & (A \pitchfork (A \otimes B)T \pitchfork (A \otimes B)T) \otimes ((B \pitchfork (A \otimes B)T \pitchfork (A \otimes B)T) \\
\psi^T \Big\downarrow\Big\downarrow \tilde{\psi}^T & & \psi^{DD} \Big\downarrow\Big\downarrow \tilde{\psi}^{DD} \\
(A \otimes B)T & \xrightarrow{\ \tau_{A \otimes B}\ } & (A \otimes B) \pitchfork (A \otimes B)T \pitchfork (A \otimes B)T \\
 & & \Big\downarrow 1 \pitchfork \eta \pitchfork 1 \\
i \Big| \cong & * & (A \otimes B) \pitchfork (A \otimes B) \pitchfork (A \otimes B)T \\
 & & \Big\downarrow j \pitchfork 1 \\
\Big\lfloor & \longrightarrow & I \pitchfork (A \otimes B)T\,.
\end{array}$$

Hence, by (4.2), the left hand composite commutes, provided * commutes. But using the definition of τ (and writing E for $A \otimes B$), * is the outer diagram of



Diagrams with no number commute by naturality. The diagram (1) commutes by extraordinary naturality of y, and (2) commutes by a monad law. Finally, (3) commutes using Lemma 1.7 in [5], naturality of λ, and the equation

(4.4) $$y.j \pitchfork 1 = i\,.$$

To prove (4.4), apply Tw to it and use naturality of Tw with respect to j. Since $Tw(y) = 1_{D \pitchfork D}$, the left hand side gives just j. To see that

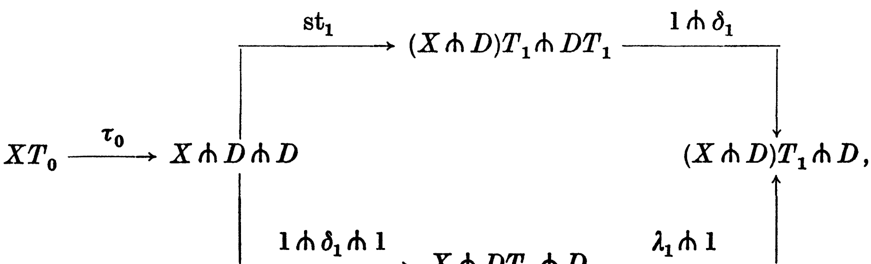
$$Tw(i) = j\,,$$

apply (II.3.15), (II.3.17), and Proposition III.1.1 of [2].

Let us finally, without proof, state how the notion of commutation of structures can be defined in terms of the $\pitchfork$-structure. Let

$$\delta_0\colon\ DT_0 \to D, \quad \delta_1\colon\ DT_1 \to D$$

be $\boldsymbol{T}_0$ and $\boldsymbol{T}_1$-structures on D. Then δ_0 and δ_1 commute in the sense of Definition 4.5 if and only if the following diagram commutes for all X:



where τ_0 is the monad transformation associated to δ_0, and st_1 is the strength of T_1. Note that the equalizer of the square (if it exists) is the "subobject of homomorphisms from $X \pitchfork D$ to D" (with respect to the T_1-structure δ_1 on D and the structure induced by δ_1 on $X \pitchfork D$). The "subobject of homomorphisms" was used by Bunge in [1] and Linton in [8] to define the $\mathscr{V}$-category of algebras for a $\mathscr{V}$-monad, and by the author in [5] to define the closed category of monads for a commutative $\mathscr{V}$-monad on $\mathscr{V}$. For the special case of the equalizer for the square diagram above, one may even prove that it defines a submonad of $- \pitchfork D \pitchfork D$, "the dual of T_1 with respect to D"; compare also [7].

REFERENCES

1. M. Bunge, *Relative functor categories and categories of algebras*, J. Algebra 11 (1969), 64–101.
2. S. Eilenberg and G. M. Kelly, *Closed categories*, Proc. Conf. Categorical Algebra, LaJolla 1965, Springer-Verlag, Berlin · Heidelberg · New York, 1966, 421–562.
3. S. Eilenberg and J. C. Moore, *Adjoint functors and triples*, Illinois J. Math. 9 (1965), 381–398.
4. A. Kock, *Monads on symmetric monoidal closed categories*, Arch. Math. (Basel) 21 (1970), 1–10.
5. A. Kock, *Closed categories generated by commutative monads*, J. Austr. Math. Soc. (to appear).
6. A. Kock, *On double dualization monads*, Aarhus Universitet, Preprint Series 1968/69, No. 38. (Mimeographed.)
7. F. W. Lawvere, *Ordinal sums and equational doctrines*, Seminar on Triples and Categorical Homology Theory, ETH 1966/67, edited by B. Eckmann (Lecture Notes in Mathematics 80), Springer Verlag, Berlin · Heidelberg · New York, 1969, 141–155.
8. F. E. J. Linton, *Relative functorial semantics*, Proc. Conf. Batelle Mem. Inst. June 24–July 19, 1968, Volume 3, *Category Theory, Homology Theory and their Applications* (Lecture Notes in Mathematics 99), Springer-Verlag, Berlin · Heidelberg · New York, 1969, 384–418.

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