MATH. SCAND. 27 (1970), 113–131

ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF PARABOLIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

KJELL-OVE WIDMAN

1. Introduction.

The object of this paper is to investigate the behavior, as t tends to infinity, of weak solutions of parabolic equations with discontinuous coefficients,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_i}\left(a^{ij}(x,t)\frac{\partial u}{\partial x_j}\right) \tag{1.1}$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_i}\left(a^{ij}(x,t)\frac{\partial u}{\partial x_j}\right) + \frac{\partial}{\partial x_i}f^i + f\,. \tag{1.2}$$

It is shown that a solution of (1.1) in $\Omega\times(0,\infty)$ which satisfies $u(x,t)=0$ for $x\in\partial\Omega$ is bounded above by a function in t tending to zero like an exponential. Also, solutions of (1.2) which satisfy the same boundary conditions tend to zero, provided that f^i and f do. As a corollary we prove that if the a^{ij} and the boundary values of a solution u of (1.1) have limits as t tends to infinity, then u has a limit which is a weak solution of the limiting elliptic equation.

In the opposite direction it is shown that a non-negative solution of (1.1) is bounded below by an exponential in t, on subdomains whose distance to $\partial\Omega$ is positive. The proof of this is accomplished via some integral inequalities which should be of interest in themselves. In the case when the a^{ij} do not depend on t it is proved that, roughly speaking, a non-negative solution is bounded below in $\Omega\times(t_0,\infty)$, $t_0>0$, by a power of the corresponding solution of the heat equation. An additional assumption here is that the parabolicity of the equation is not too large. It is believed that this restriction is due only to the imperfections of the proof, but we have not been able to remedy these.

Theorems of the first type, i.e. upper bounds and convergence to the

Received September 13, 1969.

solution of the limiting equation, are of course well known in the case of non-divergence type equations with regular coefficients, see e.g. the book [6] by Friedman, Chap. 6, or [4] or [5]. In [2] Aronson indicated that the convergence to the steady state in our case follows from a theorem of his and Friedman's results. Our proof seems to be more direct, though.

Lower bounds with less explicit asymptotic behavior are known even for abstract equations in Banach space, see e.g. Agmon and Nirenberg [1]. Similar results for parabolic equations can be found in [6]. More precise estimates like the ones given in this paper seem not to have been known before for this class of equations.

2. Notations, definitions, and known properties.

Ω is to be a bounded domain with regular, say C^2, boundary $\partial\Omega$ in R^n, the points of which are denoted by $x=(x_1,\ldots,x_n)$. The product $\Omega\times(s,T)\subset R^{n+1}$ will be denoted by $\Omega^s{}_T$, with $\Omega_T=\Omega^0{}_T$, $0\leqq s<T\leqq\infty$. About the coefficients $a^{ij}(x,t)$ we assume that they are defined in Ω_∞, measurable, symmetric, that is, $a^{ij}=a^{ji}$, and that they satisfy the inequality

$$\lambda_1|\xi|^2 \leqq a^{ij}\xi_i\xi_j \leqq \lambda_2|\xi|^2, \quad \xi\in R^n,\ 0<\lambda_1\leqq\lambda_2<\infty . \tag{2.1}$$

We also put $\lambda=\lambda_1/\lambda_2$, and assume that $f^i\in L_2(\Omega_T)$, $f\in L_\infty(\Omega_T)$ for all T, $T<\infty$.

To define weak solutions we need some function spaces.

$\mathring{W}_2{}^1(\Omega)$ (and $W_2{}^1(\Omega)$) is the closure of $C_0{}^\infty(\Omega)$ (respectively $C^\infty(\Omega)$) in the norm

$$\|u\|^2{}_{W^1{}_2} = \int_\Omega u^2\,dx + \sum_{i=1}^n \int_\Omega u_i{}^2\,dx . \tag{2.2}$$

$\mathring{W}_2{}^{1,1}(\Omega_T)$ (and $W_2{}^{1,1}(\Omega_T)$) is the closure of the span of $C_0{}^\infty(\Omega)\times C^\infty(\Omega_T)$ (respectively $C^\infty(\Omega_T)$) in the corresponding norm, that is, (2.2) with integration with respect to t added. Further, $\mathring{V}_2{}^{1,0}(\Omega_T)=\mathring{V}_2{}^1(\Omega_T)$ (and $V_2{}^1(\Omega_T)$) will be the closure of $\mathring{W}_2{}^{1,1}(\Omega_T)$ (respectively $W_2{}^{1,1}(\Omega_T)$) in the norm

$$\|u\|^2{}_{V_2{}^1} = \max_{0\leq t\leq T}\int_\Omega u^2\,dx + \sum_{i=1}^n \iint_{\Omega_T} u_i{}^2\,dx dt .$$

We observe that $\int_\Omega u^2 dx$ is a continuous function of t for $u\in V_2{}^1(\Omega_T)$.

We are now able to make the following definition:

u is a solution of (1.2) *satisfying the boundary conditions* $u(x,t)=0$, $x \in \partial\Omega$, $u(x,0)=u_0(x) \in L_2(\Omega)$ *if* $u \in \mathring{V}_2^1(\Omega_T)$ *for all* $T>0$ *and*

$$\iint_{\Omega_T} \{-u\varphi_t + a^{ij}u_i\varphi_j + f^i\varphi_i + f\varphi\}\,dx\,dt + \\ + \int_\Omega u(x,T)\varphi(x,T)\,dx - \int_\Omega u_0(x)\varphi(x,0)\,dx = 0 \tag{2.3}$$

for all $\varphi \in \mathring{W}_2^{1,1}(\Omega_T)$.

If u belongs to $\mathring{W}_2^{1,1}(\Omega_T)$ we can use a more convenient definition, namely

$$\iint_{\Omega^{t_1}{}_{t_2}} \{u_t\varphi + a^{ij}u_i\varphi_j + f^i\varphi_i + f\varphi\}\,dx\,dt = 0 \tag{2.4}$$

for any non-negative t_1 and t_2, and for any function φ in $\mathring{V}_2^1$. One gets (2.4) from (2.3) by taking $T=t_1$ and $T=t_2$ in (2.3), subtracting, and integrating by parts.

Solutions of (1.1) of course satisfy (2.3), or (2.4), with $f^i \equiv f \equiv 0$.

By a solution of (1.1) satisfying arbitrary boundary conditions we mean a function u, continuous in the closure of $\Omega^{t_0}{}_T$, $t_0>0$, belonging to $V_2^1(\Omega_T')$ and satisfying (2.3) (with Ω_T replaced by Ω'_T) for all $T>0$ and all subdomains $\Omega' \Subset \Omega$.

It is known (see Ladyženskaya et al [7, p. 181]) that for any initial values $u_0 \in L_2(\Omega)$ there is a unique $u \in \mathring{V}_2^1$ satisfying (2.3). This solution does have a very weak derivative in the t-direction ([7, p. 189]) but we shall not need it. If the a^{ij}, u_0, f^i, and f are regular enough, then the solution belongs to $\mathring{W}_2^{1,1}(\Omega_T)$ ([7, p. 209]).

The following approximation theorem is very useful.

If the sequence of coefficients $a^{(m)ij}$ are uniformly bounded, satisfy (2.1), *and converge to a^{ij} almost everywhere, and $u_0^{(m)} \to u_0$ in $L_2(\Omega)$, $f^{(m)i} \to f^i$ in $L_2(\Omega_T)$, and $f^{(m)} \to f$ in $L_p(\Omega_T)$, p large enough, for all $T>0$, then for the solution $u^{(m)}$ of*

$$\begin{aligned} u_t^{(m)} &= (a^{(m)ij}u_i^{(m)})_j + (f^{(m)i})_i + f^{(m)}, \\ u^{(m)}(x,0) &= u_0^{(m)}(x), \\ u^{(m)}(x,t) &= 0, \quad x \in \partial\Omega, \end{aligned}$$

we have that $u^{(m)} \to u$ in $\mathring{V}_2^1(\Omega_T)$ for all T.

This theorem allows us to use definition (2.4) by first approximating the equation, and the solution, carrying out the necessary partial integrations with respect to t, and then letting $m \to \infty$. Since the results will not contain any t-derivative of u, the procedure is perfectly legitimate. We shall indicate at what stages in the proofs we let the approximation parameter tend to infinity.

Other important properties of solutions of (1.1) are that they are locally bounded, Hölder continuous, and satisfy the maximum principle

$$u(x,T) \leqq \max_{\partial\Omega_T \cap \{t<T\}} u(x,t)$$

(see [7, p. 220]). From [7, p. 239] it also follows that a function $u \in \mathring{V}_2{}^1$ satisfying (2.3) is actually continuous in the closure of $\Omega^{t_0}{}_T$, $t_0 > 0$, so our definitions above are not contradictory. Finally we have the Harnack inequality for non-negative solutions of (1.1):

$$\max_{\Omega' \times (\tau,\tau+\tau_1)} u \leqq \gamma \min_{\Omega' \times (\tau+2\tau_1,\tau+3\tau_1)} u, \quad \tau \geqq \tau_0 > 0, \Omega' \Subset \Omega ,$$

where γ depends on n, λ_1, λ_2, τ_0, τ_1, Ω, and Ω'. This inequality was proved by Moser [8] in the case $u \in W_2{}^{1,1}$, but since γ does not depend on u_t the result holds in the general case, as was shown via the approximation argument by Aronson [3].

3. The upper bound and convergence to the steady state.

To be able to formulate our theorem we note that

$$\mu_0 = \inf \left(\int_\Omega u_x{}^2 dx \Big/ \int_\Omega u^2 dx \right), \tag{3.1}$$

where u ranges over $\mathring{W}_2{}^1$ and μ_0 is the smallest non-zero eigenvalue of $\Delta u + \mu u = 0$.

THEOREM 1. *Let u be a solution of* (1.2), $u(x,t) = 0$ *for* $x \in \partial\Omega$, *and assume that*

$$\iint_{\Omega^\tau{}_{\tau+1}} (f^i)^2 \, dx dt \to 0, \quad \max_{\Omega^\tau{}_\infty} |f| \to 0 \quad \text{as } \tau \to \infty .$$

Then $u(x,t) \to 0$ uniformly for $x \in \Omega$ as $t \to \infty$. If $f^i \equiv f \equiv 0$ we have more precise information:

$$\int_\Omega u^2(x,t) \, dx \leqq e^{-2\lambda_1 \mu_0 t} \int_\Omega u_0{}^2(x) \, dx$$

and

$$\max_{x \in \Omega} |u(x,t)| \leqq K e^{-\lambda_1 \mu_0 t}, \quad t \geqq t_0 > 0 .$$

Here K depends on λ, n, t_0, Ω, and $\int_\Omega u_0{}^2 dx$.

PROOF. We first establish two formulas. One gets them by assuming u to belong to $\mathring{W}_2^{1,1}(\Omega_T)$, and putting $\varphi = u$ and $\varphi = h^{-1}(t-\tau)u$ respectively in (2.4), with $h, \tau > 0$. Integration by parts with respect to t between τ and $\tau + h$ then gives us

$$\tag{3.2} \tfrac{1}{2}\int_\Omega u^2(x,\tau+h)\,dx - \tfrac{1}{2}\int_\Omega u^2(x,\tau)\,dx = -\iint_{\Omega^\tau{}_{\tau+h}} a^{ij}u_iu_j\,dx\,dt + \iint_{\Omega^\tau{}_{\tau+h}} f^i u_i\,dx\,dt + \iint_{\Omega^\tau{}_{\tau+h}} fu\,dx$$

and

$$\tag{3.3} \tfrac{1}{2}\int_\Omega u^2(x,\tau+h)\,dx = \frac{1}{2h}\iint_{\Omega^\tau{}_{\tau+h}} u^2\,dx\,dt - \frac{1}{h}\iint_{\Omega^\tau{}_{\tau+h}} a^{ij}u_iu_j(t-\tau)\,dx\,dt + \frac{1}{h}\iint_{\Omega^\tau{}_{\tau+h}} f^iu_i(t-\tau)\,dx\,dt + \frac{1}{h}\iint_{\Omega^\tau{}_{\tau+h}} fu(t-\tau)\,dx\,dt\,.$$

To establish (3.2) and (3.3) in the general case we use the approximation argument mentioned in section 2.

If $f^i \equiv f \equiv 0$ we leave (3.2) as it is; if not take $h = 1$, use the parabolicity (2.1) and Young's inequality to see that the right hand side of (3.2) is less than or equal to

$$-\lambda_1 \iint_{\Omega^\tau{}_{\tau+1}} u_x{}^2\,dx\,dt + \tfrac{1}{4}\lambda_1 \iint_{\Omega^\tau{}_{\tau+1}} u_x{}^2\,dx\,dt + \lambda_1^{-1}\iint_{\Omega^\tau{}_{\tau+1}} (f^i)^2\,dx\,dt + \tfrac{1}{4}\mu_0\lambda_1 \iint_{\Omega^\tau{}_{\tau+1}} u^2\,dx\,dt + (\mu_0\lambda_1)^{-1}\max_{\Omega^\tau{}_{\tau+1}}|f|^2\,\mathrm{mes}(\Omega)\,.$$

After using (3.1) and rearranging, (3.2) becomes

$$\tag{3.4} \iint_{\Omega^\tau{}_{\tau+1}} u_x{}^2\,dx\,dt \leqq K\int_\Omega u^2(x,\tau)\,dx - K\int_\Omega u^2(x,\tau+1)\,dx + C_\tau\,,$$

where we have put

$$C_\tau = K\iint_{\Omega^\tau{}_{\tau+1}} (f^i)^2\,dx\,dt + K\max_{\Omega^\tau{}_\infty}|f|^2$$

with $C_\tau \to 0$ as $\tau \to \infty$. Similar manipulations show that from (3.3) we get

$$\tag{3.5} \int_\Omega u^2(x,\tau+1)\,dx \leqq K\iint_{\Omega^\tau{}_{\tau+1}} u^2\,dx\,dt + K\iint_{\Omega^\tau{}_{\tau+1}} u_x{}^2\,dx\,dt + C_\tau \leqq K\iint_{\Omega^\tau{}_{\tau+1}} u_x{}^2\,dx\,dt + C_\tau\,,$$

where the last inequality follows after recourse to (3.1). Here K depends on λ_1, μ_0, and Ω only. Combining (3.4) and (3.5) we get

$$\int_\Omega u^2(x,\tau+1)\,dx \leqq \frac{K}{K+1}\int_\Omega u^2(x,\tau)\,dx + C_\tau\,. \tag{3.6}$$

This inequality implies, via an elementary argument, that $\int_\Omega u^2(x,\tau)\,dx \to 0$ as $\tau\to\infty$. To conclude that $u\to 0$ we utilize the following inequality

$$\begin{aligned} u^2(x,T) \leqq K_\tau \iint\limits_{\Omega^{T-\tau}{}_T} u^2\,dx\,dt + K_\tau \iint\limits_{\Omega^{T-\tau}{}_T} (f^i)^2\,dx\,dt\, + \\ + K_\tau \max\nolimits_{\Omega^{T-\tau}{}_T} |f|^2,\ T\geqq\tau,\ x\in\Omega\ , \end{aligned} \tag{3.7}$$

valid for non-negative solutions in $\mathring{V}_2{}^1$ of (1.2). The inequality (3.7) was essentially proved by Moser [8], in the case $f^i\equiv f\equiv 0$. The proof in the general case is only slightly more complicated, so we omit it.

The first part of the theorem evidently follows from (3.7) if $u\geqq 0$. If u is not non-negative we solve the boundary value problem with initial values $u_0{}^+=\sup(u_0,0)$ and $u_0{}^-=\inf(u_0,0)$ respectively, and $u^\pm(x,t)=0$, $x\in\partial\Omega$. By the maximum principle $u^-\leqq u\leqq u^+$, and since u^+ and u^- tend to zero, the same is true for u.

To prove the second part of the theorem we define $g(\tau)=\int_\Omega u^2(x,\tau)\,dx$ and note that (3.2) and (3.3) imply

$$g(\tau+h) - g(\tau) \leqq -2\lambda_1 \iint\limits_{\Omega^\tau{}_{\tau+h}} u_x{}^2\,dx\,dt$$

and

$$g(\tau+h) \leqq h^{-1} \iint\limits_{\Omega^\tau{}_{\tau+h}} u^2\,dx\,dt\,.$$

By (3.1),

$$g(\tau+h) \leqq h^{-1}\mu_0{}^{-1}\iint u_x{}^2\,dx\,dt \leqq \tfrac{1}{2}\,h^{-1}\mu_0{}^{-1}\lambda_1{}^{-1}[g(\tau)-g(\tau+h)]\,. \tag{3.8}$$

From this we conclude that $g'(\tau)\leqq -2\mu_0\lambda_1 g(\tau)$ from which follows

$$g(t) \leqq g(0)e^{-2\lambda_1\mu_0 t} \tag{3.9}$$

provided that g is absolutely continuous, which is easily seen to be the case. (Observe that it follows from (3.2) that g is monotonic.) However, one can dispense with this by noting that from (3.8)

$$g(\tau+h) \leqq (1+2h\mu_0\lambda_1)^{-1}g(\tau)$$

which implies, with $h = t/\nu$, $\tau = 0,1,\ldots,\nu-1$,

$$g(t) \leqq (1+2\mu_0\lambda_1\nu^{-1})^{-\nu}g(0),$$

and this gives (3.9) in the limit as $\nu \to \infty$. Finally the last part of the theorem follows from (3.7) as above.

THEOREM 2. *Let u be a solution of* (1.1) *in Ω_∞, such that u is continuous in the closure of $\Omega^{t_0}{}_T$, $t_0 > 0$, for all $T < \infty$, and such that the boundary values $u|_{\partial\Omega}$ tend to g as $t \to \infty$. If $a^{ij}(x,t) \to \bar{a}^{ij}(x)$ uniformly, that is,*

$$\sup_\Omega |a^{ij}(x,t) - \bar{a}^{ij}(x)| \to 0 \quad \text{as } t \to \infty,$$

then $u(x,t) \to w(x)$, where $w(x)$ is the weak solution in Ω of $(\bar{a}^{ij}w_i)_j = 0$, $x \in \Omega$, $w = g$ on $\partial\Omega$.

PROOF. It is sufficient to assume that $u|_{\partial\Omega}$ is the trace of a function in $W_2{}^1(\Omega)$ for each t, and that

$$\max_{\partial\Omega} \left|\partial u/\partial t|_{\partial\Omega}\right| \to 0 \quad \text{as } t \to \infty,$$

since we can approximate $u|_{\partial\Omega}$ with two functions h_1, h_2 satisfying these requirements and with the properties that $h_1 \leqq u|_{\partial\Omega} \leqq h_2$ and $h, h_2 \to g$. Then we solve the boundary value problem with h_1 and h_2, and with the theorem being true for these two solutions, it is also true for u by the maximum principle.

By a similar argument, using the maximum principle for elliptic equations, we can assume that g is the trace of a $W_2{}^1(\Omega)$-function and hence that $u|_{\partial\Omega}$ belongs to a bounded subset of $W_2{}^1(\Omega)$ for $0 < t_0 \leqq t < \infty$.

Now let $\bar{w}(x,t)$ be the solution of $(\bar{a}^{ij}\bar{w}_i)_j = 0$, $\bar{w} = u|_{\partial\Omega}$ on $\partial\Omega$, for each t. By the maximum principle for elliptic equations we have that

$$\max_\Omega |\bar{w}_t| \leqq \max_{\partial\Omega} |\partial u/\partial t| \to 0 \quad \text{and} \quad \max_\Omega |\bar{w} - w| \to 0$$

as $t \to \infty$. Hence it is sufficient to show that $u - \bar{w} \to 0$.

Since $\bar{u} = u - \bar{w} \in \mathring{V}_2{}^1$ we can put $\varphi = \bar{u}$ in (2.4), and then after partial integration with respect to t

$$\begin{aligned} \lambda_1 \iint_{\Omega^\nu{}_{\nu+1}} u_x{}^2\,dx\,dt &\leqq \iint_{\Omega^\nu{}_{\nu+1}} a^{ij}u_iu_j\,dx\,dt \\ &\leqq \tfrac{1}{2}\iint_{\Omega^\nu{}_{\nu+1}} a^{ij}u_iu_j\,dx\,dt + \qquad (3.10) \\ + \tfrac{1}{2}\iint_{\Omega^\nu{}_{\nu+1}} a^{ij}\bar{w}_i\bar{w}_j\,dx\,dt &+ \tfrac{1}{2}\int_\Omega u^2\,dx\Big|_\nu^{\nu+1} - \int_\Omega u\bar{w}\,dx\Big|_\nu^{\nu+1} - \iint_{\Omega^\nu{}_{\nu+1}} u\bar{w}_t\,dx\,dt. \end{aligned}$$

This inequality being true for regular a^{ij} it follows for general coefficients by the approximation argument referred to above. Since u and $\overline{w}_t$ are bounded, and $\overline{w} \in W_2^1(\Omega)$ uniformly in t, it follows from (3.10) that

$$\iint_{\Omega^\nu_{\nu+1}} u_x^2 \, dx dt \quad \text{is bounded as } t \to \infty \, .$$

Now we note that $\overline{u}$ satisfies the equation

$$\overline{u}_t = (\overline{a}^{ij}\overline{u}_i)_j + [(a^{ij}-\overline{a}^{ij})u_i]_j - \overline{w}_t \, ,$$

that is, (1.2) with $f^j = (a^{ij}-\overline{a}^{ij})u_i$ and $f = -\overline{w}_t$. By the assumptions about a^{ij}, and what was said above, f^i and f satisfy the requirements of Theorem 1, and hence $\overline{u} \to 0$, which proves the theorem.

4. The lower bound I.

THEOREM 3. *Let u be a non-negative solution of* (1.1) *with initial values $u_0 \in L_2$. Then for every compact subset Ω' of Ω,*

$$u(x,t) \geqq Ke^{-2\alpha\mu_0 t}, \qquad \alpha < \lambda_2^{-1}, \ x \in \Omega', \ t \geqq t_0 > 0 \, ,$$

where K depends on u_0, Ω, Ω', α, λ_1, λ_2, t_0, and n.

The proof of Theorem 3 uses the a priori inequalities in

THEOREM 4. *Let u be a non-negative solution of* (1.1) *with initial values $u_0 \in L_2$. Let v be the solution of the heat equation $v_t = \lambda_1 \Delta v$ with lateral boundary values zero and initial values u_0. Then*

(i) $$\iint_{\Omega_\infty} v_x^2 u^{-\alpha} \, dx\, dt \leqq K \int_\Omega u_0^{2-\alpha} \, dx \, ,$$

(ii) $$\iint_{\Omega_\infty} a^{ij} u_i u_j \, v^2 u^{-2-\alpha} \, dx dt \leqq K \int_\Omega u_0^{2-\alpha} \, dx \, ,$$

(iii) $$\int_\Omega v^2 u^{-\alpha} \, dx \leqq K \int_\Omega u_0^{2-\alpha} \, dx \, ,$$

(iv) $$\iint_{\Omega_\infty} v^2 u^{-\alpha} \, dx dt < \infty \, ,$$

for all $\alpha < \lambda$.

We shall first see how Theorem 3 follows from Theorem 4 (iii).

PROOF OF THEOREM 3. From (iii) it follows that for a fixed t there is at least one point in Ω' such that $v^2 u^{-\alpha} \leqq K$ for K large enough, K independent of t. By the Harnack-Moser inequality we have

$$\begin{aligned}\min_{\Omega'} u(x,t+\tau_1) &\geqq \gamma^{-1} \max_{\Omega'} u(x,t)\\ &\geqq \gamma^{-1} \min_{\Omega'} v^{2\alpha^{-1}}(x,t) \geqq K e^{-2\lambda_1\mu_0\alpha^{-1}t}, \qquad \alpha<\lambda ,\end{aligned}$$

which is equivalent to the statement of the theorem. Here we have used the estimate $v(t) \geqq K e^{-\lambda_1\mu_0 t}$, $x \in \Omega'$, which easily follows from the Fourier representation of v.

PROOF OF THEOREM 4. We first note that (iv) follows from (iii) and Theorem 1. In fact, for $\alpha<\lambda$ we have by (iii),

$$\int_\Omega v^2 u^{-\alpha-\varepsilon}\, dx \leqq K \int_\Omega {u_0}^{2-\alpha-\varepsilon}\, dx$$

for $\alpha+\varepsilon<\lambda$. But by Theorem 1

$$\int_\Omega v^2 u^{-\alpha}\, dx = \int_\Omega v^2 u^{-\alpha-\varepsilon} u^\varepsilon\, dx \leqq K e^{-\lambda_1\mu_0\varepsilon t} \int_\Omega {u_0}^{2-\alpha-\varepsilon}\, dx$$

for $t \geqq t_0 > 0$. Integration over t gives the result.

The next step will be to prove that (ii) and (iii) follow from (i). To that effect put $q=(u+\varepsilon)^{-1}$, $\varphi=v^2q^{1+\alpha}$ in (2.4):

$$\iint_{\Omega_T} (u_t v^2 q^{1+\alpha} - (1+\alpha) a^{ij} u_i u_j v^2 q^{2+\alpha} + 2a^{ij} u_i v_j v q^{1+\alpha})\, dx\,dt = 0$$

Integration by parts with respect to t and Cauchy's inequality give, with $q_0=(u_0+\varepsilon)^{-1}$,

$$\begin{aligned}&\alpha^{-1}\int_T v^2 q^\alpha\, dx + (1+\alpha)\iint_{\Omega_T} a^{ij} u_i u_j v^2 q^{2+\alpha}\, dx\,dt\\ &\qquad \leqq \alpha^{-1}\int_T {u_0}^2 {q_0}^\alpha\, dx + \iint_{\Omega_T} a^{ij} u_i u_j v^2 q^{2+\alpha}\, dx\,dt + \iint_{\Omega_T} a^{ij} v_i v_j q^\alpha\, dx\,dt\end{aligned}$$

or

$$\alpha^{-1}\int_T v^2 q^\alpha dx + \alpha \iint_{\Omega_T} a^{ij} u_i u_j v^2 q^{2+\alpha}\, dx\,dt \leqq \alpha^{-1}\int_\Omega {u_0}^{2-\alpha} dx + \lambda_2 \iint_{\Omega_T} {v_x}^2 q^\alpha\, dx\,dt .$$

Here the T subscript at the x-integrals means that the integration takes place over Ω with $t=T$.

This is the time to let the approximation parameter tend to infinity, after which we let $\varepsilon \to 0$, only to find that we have proved (ii) and (iii).

The corner stone, and the difficult part of this section, is the proof of (i). We first note that it is no restriction to assume that u has lateral boundary values zero, i.e. that $u \in \mathring{V}_2^1(\Omega_T)$. For technical reasons we shall need that $u \leqq 1$ in the proof. To circumvent this we first prove that (i) holds with $M^{-1}\bar{v}$ and $M^{-1}\bar{u}$, where these functions are determined as the solutions of $v_t = \lambda_1 \Delta v$ and (1.1) respectively, with initial values $M^{-1}\bar{u}_0 = M^{-1}\min(u_0, M)$, M constant > 0. By the homogeneity we then get

$$\iint_{\Omega_\infty} \bar{v}_x^2 \bar{u}^{-\alpha}\, dx\, dt \leqq K \int_\Omega \bar{u}_0^{2-\alpha}\, dx ,$$

from which by the maximum principle

$$\iint_D \bar{v}_x^2 u^{-\alpha}\, dx\, dt \leqq K \int_\Omega \bar{u}_0^{2-\alpha}\, dx \leqq K \int_\Omega u_0^{2-\alpha}\, dx ,$$

for every compact subset D of Ω_∞. But $\bar{v}_x^2 \to v_x^2$ as $M \to \infty$, uniformly on every D, and (i) follows. Hence we shall henceforth assume that $0 \leqq u \leqq 1$.

We introduce a positive ε in the integral and use Taylor's formula, again putting $q = (u+\varepsilon)^{-1}$

$$\iint_{\Omega_T} v_x^2 q^\alpha\, dx\, dt = \sum_{\nu=0}^{\infty} \frac{\alpha^\nu}{\nu!} \iint_{\Omega_T} v_x^2 (\log q)^\nu\, dx\, dt = \sum_{\nu=0}^{\infty} \frac{\alpha^\nu}{\nu!} I_\nu .$$

The trick is to estimate the growth of the integrals I_ν by deriving an iterative inequality for them. Thus integration by parts and Young's inequality give (we suppress summation over i while the integration always is over Ω_T)

$$\begin{aligned}
I_{\nu+1} &= \iint v_x^2 (\log q)^{\nu+1}\, dx\, dt \\
&= (\nu+1) \iint v_i u_i v q (\log q)^\nu\, dx\, dt - \iint v \Delta v (\log q)^{\nu+1}\, dx\, dt \\
&\leqq \tfrac{1}{2}(\nu+1)\nu^{-1} \iint v_x^2 (\log q)^{\nu+1}\, dx\, dt + \\
&\quad + \tfrac{1}{2}(\nu+1)\nu \iint u_x^2 v^2 q^2 (\log q)^{\nu-1}\, dx\, dt - \lambda_1^{-1} \iint v v_t (\log q)^{\nu+1}\, dx\, dt .
\end{aligned}$$

To estimate the second term in the last membrum we put

$$\varphi = v^2 q (\log q)^\nu$$

in (2.4). Then

$$
\begin{aligned}
(4.1)\qquad & \nu \iint u_x^2 v^2 q^2 (\log q)^{\nu-1}\, dx\, dt \\
& \leqq \nu \lambda_1^{-1} \iint a^{ij} u_i u_j v^2 q^2 (\log q)^{\nu-1}\, dx\, dt \\
& = \lambda_1^{-1} \iint u_t v^2 q (\log q)^{\nu}\, dx\, dt + 2\lambda_1^{-1} \iint a^{ij} u_i v_j v q (\log q)^{\nu}\, dx\, dt - \\
& \quad - \lambda_1^{-1} \iint a_{ij} u_i u_j v^2 q^2 (\log q)^{\nu}\, dx\, dt \\
& \leqq \lambda_1^{-1} \iint u_t v^2 q (\log q)^{\nu}\, dx\, dt + \lambda_1^{-1} \iint a^{ij} u_i u_j v^2 q^2 (\log q)^{\nu}\, dx\, dt + \\
& \quad + \lambda_1^{-1} \iint a^{ij} v_i v_j (\log q)^{\nu}\, dx\, dt - \lambda_1^{-1} \iint a^{ij} u_i u_j v^2 q^2 (\log q)^{\nu}\, dx\, dt .
\end{aligned}
$$

The second and the fourth term in the last membrum cancel, and the third term is $\leqq \lambda^{-1} I_\nu$. In the term containing v_t above we integrate by parts with respect to t:

$$
\begin{aligned}
& -\lambda_1^{-1} \iint v v_t (\log q)^{\nu+1}\, dx\, dt \\
& \qquad \leqq \tfrac{1}{2}\lambda_1^{-1} \int_\Omega u_0^2 (\log q_0)^{\nu+1} - \tfrac{1}{2}(\nu+1)\lambda_1^{-1} \iint v^2 u_t q (\log q)^{\nu}\, dx\, dt .
\end{aligned}
$$

Collecting terms we get

$$
\begin{aligned}
I_{\nu+1} \leqq\ & \tfrac{1}{2}(\nu+1)\nu^{-1} I_{\nu+1} + \tfrac{1}{2}(\nu+1)\lambda_1^{-1} \iint u_t v^2 q (\log q)^{\nu}\, dx\, dt + \tfrac{1}{2}(\nu+1)\lambda^{-1} I_\nu + \\
& + \tfrac{1}{2}\lambda_1^{-1} \int_\Omega u_0^2 (\log q_0)^{\nu+1}\, dx - \tfrac{1}{2}(\nu+1)\lambda_1^{-1} \iint u_t v^2 q (\log q)^{\nu}\, dx\, dt .
\end{aligned}
$$

We are again fortunate and the second and the last terms cancel against each other. Hence we get

$$
(4.2)\qquad I_{\nu+1} \leqq \nu(\nu+1)(\nu-1)^{-1}\lambda^{-1} I_\nu + \nu(\nu-1)^{-1}\lambda_1^{-1} \int_\Omega u_0^2 (\log q_0)^{\nu+1}\, dx .
$$

The procedure just described does not work for $\nu=0$ and $\nu=1$, instead we proceed as follows.

For $\nu=1$ the first step is almost the same, giving

$$
I_2 \leqq \tfrac{1}{2} I_2 + 2 \iint u_x^2 v^2 q^2\, dx\, dt - \lambda_1^{-1} \iint v v_t (\log q)^2\, dx\, dt .
$$

Further

$$\iint u_x{}^2 v^2 q^2\,dx\,dt \leqq \lambda_1{}^{-1}\iint a^{ij}u_i u_j v^2 q^2\,dx\,dt$$
$$= \lambda_1{}^{-1}\iint u_t v^2 q\,dx\,dt + 2\lambda_1{}^{-1}\iint a^{ij}u_i v_j v q\,dx\,dt$$
$$\leqq \lambda_1{}^{-1}\iint u_t v^2 q\,dx\,dt + \tfrac{1}{2}\iint u_x{}^2 v^2 q^2\,dx\,dt + {}$$
$$+ 2\lambda_2\lambda_1{}^{-2}\iint v_x{}^2\,dx\,dt\,.$$

Partial integration with respect to t gives

$$\iint u_t v^2 q\,dx\,dt \leqq \int_\Omega u_0{}^2(\log q)\,dx + 2\lambda_1\iint v\Delta v(\log q)\,dx\,dt$$
$$= \int_\Omega u_0{}^2\log q\,dx - 2\lambda_1\iint v_x{}^2\log q\,dx\,dt + 2\lambda_1\iint v_i u_i v q\,dx\,dt$$
$$\leqq \int_\Omega u_0{}^2\log q_0\,dx + \tfrac{1}{4}\lambda_1\iint u_x{}^2 v^2 q^2\,dx\,dt + 4\lambda_1\iint v_x{}^2\,dx\,dt\,.$$

This implies

$$\iint u_x{}^2 v^2 q^2\,dx\,dt \leqq 4\int_\Omega u_0{}^2\log q_0\,dx + 8(1+\lambda_2\lambda_1{}^{-2})\iint v_x{}^2\,dx\,dt\,. \tag{4.3}$$

On the other hand,

$$-\lambda_1{}^{-1}\iint v v_t(\log q)^2\,dx\,dt \leqq \tfrac{1}{2}\lambda_1{}^{-1}\int_\Omega u_0{}^2(\log q)^2 - \lambda_1{}^{-1}\iint u_t v^2 q\log q\,dx\,dt\,.$$

By (4.1) with $\nu=1$ we get

$$-\lambda_1{}^{-1}\iint u_t v^2 q\log q\,dx\,dt \leqq \lambda\iint v_x{}^2\log q\,dx\,dt\,.$$

Combining these inequalities we get

$$I_2 \leqq (16+\lambda_1{}^{-1})\int_\Omega u_0{}^2\log q\,dx + 32(1+\lambda_2\lambda_1{}^{-2})\iint v_x{}^2\,dx\,dt + 2\lambda I_1\,. \tag{4.4}$$

Finally

$$I_1 = \iint v_x{}^2\log q\,dx\,dt = 2\iint u_i v_i v q\,dx\,dt - \lambda_1{}^{-1}\iint v v_t\log q\,dx\,dt$$
$$\leqq 2\iint v_x{}^2\,dx\,dt + \lambda_1{}^{-1}\int_\Omega u_0{}^2\log q_0\,dx + \tfrac{1}{2}\lambda_1{}^{-1}\iint a^{ij}u_i u_j v^2 q^2\,dx\,dt -$$
$$- \lambda_1{}^{-1}\iint u_t v^2 q\,dx\,dt\,.$$

By putting $\varphi = v^2(u+\varepsilon)^{-1}$ in (2.4) and using Young's inequality we find that the sum of the last two terms is $\leqq 2\lambda_2 \iint v_x{}^2 dx dt$, hence

$$I_1 \leqq KI_0 + \lambda_1{}^{-1} \int_\Omega u_0{}^2 \log q_0 \, dx \tag{4.5}$$

By (4.2), (4.4), and (4.5)

$$I \leqq K\nu!\lambda^{-\nu} I_0 + K \sum_{\mu=0}^{\nu} \frac{\nu!}{(\nu-\mu)!} \lambda^{-\mu} \int_\Omega u_0{}^2 (\log q_0)^{\nu-\mu} \, dx \, .$$

Thus with $\alpha < \lambda$

$$\sum_{\nu=0}^{\infty} \frac{\alpha^\nu I_\nu}{\nu!} \leqq K \iint_{\Omega_T} v_x{}^2 \, dx dt \sum_{\nu=0}^{\infty} \left(\frac{\alpha}{\lambda}\right)^\nu + K \sum_{\nu=0}^{\infty} \frac{\alpha^\nu}{\nu!} \sum_{\mu=0}^{\nu} \frac{\nu!}{(\nu-\mu)!} \lambda^{-\mu} \int_\Omega u_0{}^2 q_0{}^2 \, dx$$

$$\leqq \frac{K\lambda}{\lambda-\alpha} \iint_{\Omega_T} v_x{}^2 \, dx dt + K \int_\Omega u_0{}^2 q_0{}^\alpha \sum_{\mu=0}^{\infty} \left(\frac{\alpha}{\lambda}\right)^\mu .$$

A simple integration by parts shows that

$$\iint_{\Omega_T} v_x{}^2 \, dx \, dt \leqq \tfrac{1}{2} \int_\Omega u_0{}^2 \, dx \leqq \tfrac{1}{2} \int_\Omega u_0{}^{2-\alpha} \, dx \, ,$$

the latter inequality being valid since $|u_0| \leqq 1$. Hence

$$\iint_{\Omega_T} v_x{}^2 q^\alpha \, dx dt \leqq \frac{K\lambda}{\lambda-\alpha} \int_\Omega u_0{}^{2-\alpha} \, dx \, . \tag{4.6}$$

This is an excellent opportunity for letting the approximation parameter m tend to infinity. The convergence in the integrals in (4.6) causes no difficulties. It should perhaps be remarked that $\lambda = \lambda^{(m)}$ will depend on m also, but since we have chosen α fixed $< \lambda$ and $\lambda^{(m)} \to \lambda$ this is no problem for m large enough.

The theorem now follows by letting ε tend to zero while employing Fatou's lemma.

5. The lower bound II.

To be able to derive a lower bound for u valid out to the boundary of Ω we have been forced to restrict the class of equations. Hence we make the assumption that the coefficients of (11) do not depend on t. By Fourier's method the solution u can be written

$$u = \sum_{k=1}^{\infty} c_k \psi_k \, e^{-\gamma_k t} , \tag{5.1}$$

where γ_k and ψ_k are, respectively, the eigenvalues and eigenfunctions of the operator $\partial/\partial x_i(a^{ij}\partial/\partial x_j)$ in L_2, and c_k is the I scalar product of $\cdot_k$ and u_0. The series (5.1) converges in $\mathring{V}_2{}^1(\Omega_T)$ for all T, and in $W_2{}^{1,1}(\Omega^{t_0}{}_T)$ for all $T > t_0 > 0$. It might be expected that a good lower bound could be derived using the representation (5.1), but this is rather doubtful. Anyhow, although a reasonable lower bound for ψ_1 is known (see [10]), good upper bounds for ψ_k, $k \geqq 2$, are conspicuously lacking. Hence we have been forced to use another, more direct method, which is an extension of the one used in [10].

We first state the main theorem of this section.

THEOREM 5. *Let u be a non-negative solution of* (1.1) *where the a^{ij} do not depend on t. We also assume $\lambda > \frac{1}{2}$. Then if $\delta(x)$ denotes the distance from x to $\partial\Omega$,*

$$u(x,t) \geqq K\delta^r(x)\, e^{-r\lambda_1\mu_0 t}, \quad t \geqq t_0 > 0, \ x \in \Omega ,$$

for all $r = 2(2n+\beta+\beta^2)/\beta^2(\beta+\frac{1}{2})$ such that $\beta+\frac{1}{2} < \lambda$, $\beta > 0$.

Theorem 5 follows from Theorem 6 and Lemma 1.

THEOREM 6. *Let u be as in Theorem 4, and let v be a solution of the heat equation $v_t = \lambda_1 \Delta v$ in $\Omega^{t_0}_\infty$ with regular initial values $v|_{t_0} \leqq u|_{t_0}$. Then*

$$u \geqq K v^r, \quad x \in \Omega, \ t \geqq t_0 > 0 ,$$

for all $r = 2(2n+\beta+\beta^2)/\beta^2(\beta+\frac{1}{2})$ such that $\beta+\frac{1}{2} < \lambda$, $\beta > 0$.

REMARK 1. The regularity of $v|_{t_0}$ is needed in order to ensure that v_t is bounded.

REMARK 2. It is of course believed that the requirement $\lambda > \frac{1}{2}$ is due only to the method of proof and should not be there. Moreover, it is conjectured that r can be chosen considerably smaller, at least $r = n/\alpha$, $\alpha < \lambda$, which is the best known estimate for ψ_1, see [10].

The following lemma should be well known.

LEMMA 1. *Let v be a non-negative solution of the heat equation $v_t = \lambda_1 \Delta v$ in Ω_∞. Then*

$$v \geqq K\delta(x)\, e^{-\lambda_1\mu_0 t}, \quad x \in \Omega, \ t \geqq t_0 > 0 .$$

PROOF. One can e.g. proceed as follows. First consider the case when Ω is a sphere $|x| \leqq \sigma$ and $v|_{t=0}$ is the Dirac function with support at the origin. Then $\partial v/\partial n \geqq c > 0$ for $t = t_0$. This follows from symmetry and [6, p. 49]. The next step is to compare v with a constant times $\psi_0 e^{-\mu_0 \lambda_1 t}$ in $\Omega_\infty{}^{t_0}$, where ψ_0 is the non-negative solution of

$$\Delta u - \mu_0 u = 0, \quad u|_{\partial\Omega} = 0 .$$

Now by Harnack's inequality the lemma follows for arbitrary placement of the support of the Dirac function, and then of course for general initial values. For general Ω satisfying an interior sphere condition the lemma follows again by the maximum principle.

For the proof of Theorem 6 we need three more lemmas.

LEMMA 2. *If u is as in Theorem 5, then $u_t, u_{tt}, \ldots$ are all bounded in $\Omega^{t_0}{}_\infty$ for all $t_0 > 0$.*

PROOF. As we remarked above the series converges in $\mathring{W}_2^{1,1}(\Omega^{t_0}{}_T)$, and it is easy to see that the same is true for the differentiated series. Since the partial sums of the differentiated series are obviously solutions of (1.1), we can go to the limit in (2.4). Hence $u_t, u_{tt}, \ldots$ are solutions of (1.1), and the boundedness follows from Theorem 1.

LEMMA 3. *Let u and v be as in Theorem 4. Then*

$$\iint\limits_{\Omega^{t_0}\infty} |u_t| \frac{v^2}{u^{1+\beta}} \, dx dt < \infty, \quad \beta < \lambda - \tfrac{1}{2}, \; t_0 > 0 .$$

PROOF. Integration by parts gives

$$\iint |u_t|^2 v^2 u^{-1-\alpha} \, dx dt = \frac{1}{\alpha} \int\limits_{t_0} u_t v^2 u^{-\alpha} \, dx - \frac{1}{\alpha} \int\limits_{T} u_t v^2 u^{-\alpha} \, dx + $$
$$+ \iint u_{tt} v^2 u^{-\alpha} \, dx \, dt + 2 \iint u_t v_t v u^{-\alpha} \, dx dt ,$$

$$\iint u_t v_t v u^{-\alpha} \, dx dt = \frac{1}{1-\alpha} \int\limits_{T} u^{1-\alpha} v_t v \, dx - \frac{1}{1-\alpha} \int\limits_{t_0} u^{1-\alpha} v_t v \, dx -$$
$$- \iint u^{1-\alpha} \{v_t^2 + v_{tt} v\} \, dx dt .$$

By Theorem 3 and Lemma 2 all integrals involved are bounded independently of T, for $\alpha < \lambda$. Now by Young's inequality,

$$\iint |u_t| \frac{v^2}{u^{1+\beta}}\,dx\,dt \leqq \tfrac{1}{2}\iint |u_t|^2 \frac{v^2}{u^{1+\frac{1}{2}+\beta}}\,dx\,dt + \tfrac{1}{2}\iint \frac{v^2}{u^{\frac{1}{2}+\beta}}\,dx\,dt\,.$$

The two integrals on the right are bounded by the above and Theorem 4 for $\frac{1}{2}+\beta<\lambda$.

We shall also need the following Sobolev type lemma. (At this point the author profited from a discussion with prof. L. Nirenberg. See [9].)

LEMMA 4. *Let $f=f(x_1,x_2,\ldots,x_n,t)$ be a non-negative function belonging to $\mathring{W}_2^{1,1}(\Omega_T)$ for all $T>0$. Then for $0<b\leqq 1$,*

$$\left[\iint_{\Omega_T} |f|^{2+b/n}\,dx\,dt\right]^{n/(2n+b)} \leqq K\left[\iint_{\Omega_T} |f_x|^2\,dx\,dt\right]^{1/2} + \left[\iint_{\Omega_T} |f_t|\,|f|^{b-1}\,dx\,dt\right]^{1/b} + \left[\int_{\Omega} |f(x,T)|^b\,dx\right]^{1/b},$$

where K depends only on the diameter of Ω. If $b=1$, the non-negativity is superfluous.

PROOF. We need only consider the case when the right hand side is finite.

For almost every $(x,t)\equiv\Omega_T$,

$$|f(x,t)|^{2/n} \leqq \left[\int |f_i|\,dx_i\right]^{2/n} \leqq K\left[\int |f_i|^2 dx_i\right]^{1/n}, \quad i=1,\ldots,n\,,$$

$$|f^b(x,t)|^{1/n} \leqq \left[\int |f_t|\,|f|^{b-1}\,dt\right]^{1/n} + |f(x,T)|^{b/n}\,.$$

Multiplying we find

$$|f|^{2+b/n} \leqq K\left\{\left[\int |f_t|\,|f|^{b-1}\,dt\right]^{1/n} + |f(x,T)|^{b/n}\right\}\prod_{i=1}^n\left[\int |f_i|^2\,dx_i\right]^{1/n}.$$

Now integrate successively with respect to x_j, $j=1,\ldots,n$ and t, and use Hölder's inequality each time:

$$\int \prod_{i=1}^n g_i \leqq \prod_{i=1}^n\left[\int g_i^n\right]^{1/n}.$$

This gives us

$$\iint |f|^{2+b/n}\,dx\,dt \leqq K\prod_{i=1}^n\left[\iint f_i^2\,dx\,dt\right]\left\{\left[\iint |f_t|\,|f|^{b-1}\,dx\,dt\right]^{1/n} + \left[\int |f(x,T)|^b\,dx\right]^{1/n}\right\}.$$

Young's inequality completes the proof.

Proof of Theorem 6. We shall use Lemma 4 to derive an iterative inequality for the sequence of integrals

$$I_\nu = \iint \frac{v^{(k+2)p^\nu-k}}{(u+\varepsilon)^{\alpha p^\nu}}\,dx\,dt\,,$$

where $p=1+\frac{1}{2}b/n$, where b and k will be determined later, and where the integration is taken over $\Omega^{t_0}{}_{\infty}$.

Applying the lemma to the function

$$f = v^{(k+2)p^{\nu-1}2^{-1}-kp^{-1}2^{-1}}/(u+\varepsilon)^{\alpha p^{\nu-1}2^{-1}}$$

we get

$$I_\nu^{2p^{-1}} \leqq K_\nu\left[\iint v_x^2\,\frac{v^{(k+2)p^{\nu-1}-kp^{-1}-2}}{(u+\varepsilon)^{\alpha p^{\nu-1}}}\,dx\,dt + \iint u_x^2\,\frac{v^{(k+2)p^{\nu-1}-kp^{-1}}}{(u+\varepsilon)^{\alpha p^{\nu-1}+2}}\,dx\,dt\right]^{1/2} +$$
$$+\,K_\nu\left[\iint |u_t|\,\frac{v^{b((k+2)p^{\nu-1}2^{-1}-kp^{-1}2^{-1})}}{(u+\varepsilon)^{b\alpha p^{\nu-1}2^{-1}+1}}\,dx\,dt\right]^{1/b} +$$
$$+\,K_\nu\left[\iint |v_t|\,\frac{v^{b((k+2)p^{\nu-1}2^{-1}-kp^{-1}2^{-1})-1}}{(u+\varepsilon)^{\alpha p^{\nu-1}b2^{-1}}}\,dx\,dt\right]^{1/b}.$$

Note that the lemma is applied first in $\Omega^{t_0}{}_T$, then by letting T tend to infinity the boundary integral disappears since v tends to zero exponentially and the denominator is $\geqq\varepsilon>0$. We consider the third integral first and apply Hölder's inequality with exponent $2/b$, which gives us the following estimate

$$\left[\iint |u_t|^{2/(2-b)}\,\frac{v^{lb/(2-b)}}{(u+\varepsilon)^{2/(2-b)}}\,dx\,dt\right]^{\frac{1}{2}(2-b)/b}\left[\iint \frac{v^{(k+2)p^{\nu-1}-kp^{-1}-l}}{(u+\varepsilon)^{\alpha p^{\nu-1}}}\,dx\,dt\right]^{1/2}.$$

Hence if we choose b such that $2/(2-b)=1+\beta$, $\beta<\lambda-\frac{1}{2}$, and l such that $lb/(2-b)=2$, the first factor will be bounded independently of ε and ν, by Lemma 3, and if we then choose k such that $kp^{-1}+l=k$ the second factor is equal to $I_{\nu-1}$. This gives us

$$b = 2\beta/(1+\beta), \qquad l = 2/\beta, \qquad k = 2(2n+\beta)/\beta^2\,.$$

The fourth integral in the inequality above is found to be $\leqq KI_{\nu-1}$ after use of Hölder's inequality and the fact that v_t and v are bounded and decay exponentially as $t\to\infty$.

Now to estimate the second integral we put $\varphi=v^{(k+2)p^{\nu-1}-kp^{-1}}/u^{\alpha p^{\nu-1}+1}$ in (2.4) and use Young's inequality to get

$$\iint a^{ij} u_i u_j \frac{v^{(k+2)p^{\nu-1}-kp^{-1}}}{(u+\varepsilon)^{\alpha p^{\nu-1}+2}}\, dx dt$$

$$\leqq \iint v_x{}^2 \frac{v^{(k+2)p^{\nu-1}-kp^{-1}-2}}{(u+\varepsilon)^{\alpha p^{\nu-1}}}\, dx dt + K_\nu \iint u_t \frac{v^{(k+2)p^{\nu-1}kp^{-1}}}{(u+\varepsilon)^{\alpha p^{\nu-1}+1}}\, dx dt .$$

Integrating in the last integral by parts and using that v is a solution of the heat equation we get

$$K_\nu \iint u_t \frac{v^{(k+2)p^{\nu-1}-kp^{-1}}}{(u+\varepsilon)^{\alpha p^{\nu-1}+1}}\, dx\, dt$$

$$\leqq K_\nu \int_{t_0} u^{(k+2-\alpha)p^{\nu-1}-kp^{-1}}\, dx + K_\nu \iint v_x{}^2 \frac{v^{(k+2)p^{\nu-1}-kp^{-1}-2}}{(u+\varepsilon)^{\alpha p^{\nu-1}}}\, dx\, dt +$$

$$+ \tfrac{1}{2} \iint a^{ij} u_i u_j \frac{v^{(k+2)p^{\nu-1}+kp^{-1}}}{(u+\varepsilon)^{\alpha p^{\nu-1}-kp^{-1}}}\, dx dt .$$

Note now that $v^{-2} \leqq K v^{-l}$ and $v_x{}^2 \leqq K$ to get

$$I_\nu^{p^{-1}} \leqq K_\nu I_{\nu-1} + K_\nu \int_{t_0} u^{(k+2-\alpha)p^{\nu-1}-kp^{-1}}\, dx . \tag{5.2}$$

We observe also that if $\liminf I_{\nu-1} J_\nu^{-1} = 0$ for all $\varepsilon > 0$, where J_ν is the integral for $t = t_0$ in (5.2), then

$$I_\nu^{p^{-\nu}} \leqq K \max_{t_0} u^{k+2-\alpha}$$

for infinitely many ν, which leads to

$$\operatorname{ess\,sup}_{\Omega t_0 \infty} v^{k+2}(u+\varepsilon)^{-\alpha} \leqq K < \infty , \tag{5.3}$$

where K is independent of ε, and this is the statement of the theorem. (In fact this case does not accur unless $v \equiv 0$, since we can choose K arbitrarily small).

In the opposite case we have that for at least one $\varepsilon > 0$

$$J_\nu \leqq K I_\nu, \quad \nu = 1, 2, \ldots , \tag{5.4}$$

but since when ε decreases I_ν increases and J_ν does not, (5.4) is valid for all sufficiently small ε. Using (5.4) in (5.2) we find

$$I_\nu^{1/p} \leqq K_\nu I_{\nu-1} .$$

An examination of the constants shows that $K_\nu \leqq K^\nu$, and after iteration the last inequality becomes

$$I_\nu^{p^{-\nu}} \leqq I_0 \prod_{\mu=0}^{\nu} K^{\mu p^{-\mu}} \leqq K I_0 .$$

But I_0 is bounded independently of ε by Theorem 4, and by letting ν tend to infinite we are again led to (5.3) and the proof of Theorem 6 is complete.

REFERENCES

1. S. Agmon and L. Nirenberg, *Properties of solutions of ordinary differential equations in Banach space*, Comm. Pure Appl. Math. 16 (1963), 121–239.
2. D. G. Aronson, *On the Green's function for second order parabolic differential equations with discontinuous coefficients*, Bull. Amer. Math. Soc. 69 (1963), 841–847.
3. D. G. Aronson, *Uniqueness of positive weak solutions of second order parabolic equations*, Ann. Polon. Math. 16 (1965), 285–303.
4. A. Friedman, *Convergence of solutions of parabolic equations to a steady state*, J. Math. Mech. 8 (1959), 57–76.
5. A. Friedman, *Asymptotic behavior of solutions of parabolic equations of any order*, Acta Math. 106 (1961), 1–43.
6. A. Friedman, *Partial differential equations of parabolic type*, Prentice–Hall, Englewood Cliffs, N. J., 1964.
7. O. A. Ladyženskaya, V. A. Solonnikov, i N. N. Ural'ceva, *Lineǐnye i kvazilineǐnye uravneniya paraboličeskogo tipa*, Izdatel'stvo „Nauka", Moskva, 1967.
8. J. Moser, *A Harnack inequality for parabolic equations*, Comm. Pure Appl. Math. 17 (1964), 101–134.
9. L. Nirenberg, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa 13 (1959), 115–162.
10. K.-O. Widman, *A quantitative form of the maximum principle for elliptic partial differential equations with coefficients in L_∞*, Comm. Pure Appl. Math. 21 (1968), 507–513.

UNIVERSITY OF UPPSALA, SWEDEN