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AN INTEGRAL IN TOPOLOGICAL SPACES II

W. F. PFEFFER

0. Introduction.

In the first part of this paper (see [12]) a Perron-like integral was defined in an arbitrary topological space and its basic properties were established. This part is mainly concerned with the connection between the Lebesgue integral and the integral defined in [12, 3.3].

Since the Lebesgue integral is absolutely convergent, it is natural to begin with a study of absolutely integrable functions. Some basic properties of these functions are proved in section 2. In section 3 we restrict ourselves to a locally compact Hausdorff space and describe there a rather large family of absolutely integrable functions. Using this family we define a certain measure, the Lebesgue integral with respect to which coincides with our integral on all non-negative functions (see sections 3 and 4). Another measure, closely related to the previous one, is introduced in section 4. This new measure is regular and has no regular extension. It is shown that the Lebesgue integral with respect to it is a restriction of our integral whenever the basic additive set function G which was used for the definition of our integral (see [12, 2]) can be extended to a regular measure. Section 5 is devoted to examples.

1. Preliminaries.

In this section we shall recall some definitions from [12] and establish our notation.

By E we denote the set of extended real numbers. Besides the usual addition and multiplication in E (see, for example, [3, (6.1), b), p. 54]) we define the division as follows: $a/0=+\infty$ for $a \geqq 0$, $a/0=-\infty$ for $a<0$, $a/(\pm\infty)=0$, and $a/b=a\cdot(1/b)$ for $a,b \in E$, $b \neq 0$, $b \neq \pm\infty$. If $a,b \in E$, then $a \vee b=\max(a,b)$ and $a \wedge b=\min(a,b)$.

For an arbitrary set A, $\mathfrak{F}(A)$ denotes the *class* (see [5, p. 251]) of all extended real-valued functions with domain *containing* A. If δ is a collection of sets we denote by $\mathfrak{F}_s(\delta)$ or $\mathfrak{F}_a(\delta)$ the family of all super-additive or additive functions on δ, respectively (see [12, 1.2]).

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Throughout P is a topological space and $P^{\sim}=P\cup(\infty)$ is a one-point compactification of P. If $A\subset P$, A^{-} and $A^{\cdot}$ denote the closure and the boundary of A in P, respectively; if $A\subset P^{\sim}$, $A^{\sim}$ and A^{0} denote the closure and the interior of A in $P^{\sim}$, respectively. For every $x\in P^{\sim}$ we choose once and for all a neighborhood base Γ_x at x in $P^{\sim}$ such that if $x\in P$, then $U\subset P$ for all $U\in\Gamma_x$.

Let σ be a pre-algebra (see [12, 1.1]) of subsets of P such that $\Gamma_x\subset\sigma$ for every $x\in P$. We shall assume that there is a fixed integer $p\geqq 1$ with the following property: for each $U\in\Gamma_\infty$ there are disjoint sets $U_{1,\infty},\ldots,U_{p,\infty}$ from σ for which $\bigcup_{i=1}^{p}U_{i,\infty}=U\cap P$. By λ we denote the system of all sets $A\in\sigma$ such that $A\subset\bigcup_{i=1}^{n}U_i$ where

$$U_i\in\bigcup\{\Gamma_x : x\in P\}\,,\qquad i=1,2,\ldots,n\,,$$

and we choose a non-negative function $G\in\mathfrak{F}_a(\sigma)$ which is finite on λ.

If $\delta\subset\sigma$ and $A\subset P^{\sim}$ we let $\delta_A=\{B\in\delta : B\subset A\}$. A system $\delta\subset\sigma$ is said to be *semihereditary* if and only if $\sigma_0\cap\delta\neq\emptyset$ for every finite disjoint collection $\sigma_0\subset\sigma$ whose union belongs to δ. A system $\delta\subset\sigma$ is said to be *stable* if and only if $\emptyset\notin\delta$ and for every $A\in\delta$ and every $x\in P^{\sim}$ there is a $U\in\Gamma_x$ such that $\delta_{A-U}\neq\emptyset$.

With every point $x\in P^{\sim}$ we associate a certain family $\varkappa_x$ of nets

$$\{B_\alpha,\alpha\in D,\succ\}\subset\sigma\,,$$

where $(D,\succ)$ is isotonically isomorphic to a cofinal subset of $(\Gamma_x,\subset)$. The collection $\{\varkappa_x : x\in P^{\sim}\}$ is called a *convergence*. For $x\in P^{\sim}$ and $\delta\subset\sigma$,

$$\varkappa_x(\delta)=\big\{\{B_\alpha\}\in\varkappa_x : \{B_\alpha\}\subset\delta\big\}\quad\text{ and }\quad\delta^{*}=\{x\in P^{\sim} : \varkappa_x(\delta)\neq\emptyset\}\,.$$

If $(D,\succ)$ is a directed set and $\alpha\in D$, we let

$$D(\alpha)=\{\beta\in D : \beta\succ\alpha\}\,.$$

A net $\{B_{\alpha_\beta},\beta\in D',\succ\}$ is said to be a *subnet* of a net $\{B_\alpha,\alpha\in D,\succ\}$ if and only if D' is a cofinal subset of D and $\alpha_\beta\succ\beta$ for all $\beta\in D'$.

Throughout we shall assume the convergence $\varkappa$ to satisfy the following conditions:

$\mathscr{K}_1$. For every $x\in P$, $\{U,U\in\Gamma_x,\subset\}\in\varkappa_x$ and for every integer i, $1\leqq i\leqq p$, $\{U_{i,\infty},U\in\Gamma_\infty,\subset\}\in\varkappa_\infty$.

$\mathscr{K}_2$. If $x\in P^{\sim}$ and $\{B_\alpha,\alpha\in D,\succ\}\in\varkappa_x$, then for every $U\in\Gamma_x$ there is an $\alpha_U\in D$ such that $B_\alpha\subset U$ for all $\alpha\in D(\alpha_U)$.

$\mathscr{K}_3$. If $x\in P^{\sim}$ and $\{B_\alpha\}\in\varkappa_x$, then every subnet of $\{B_\alpha\}$ belongs to $\varkappa_x$.

$\mathscr{K}_4$. If $x\in P^{\sim}$, $\{B_\alpha\}\in\varkappa_x$, and $A\in\sigma$, then also $\{B_\alpha\cap A\}\in\varkappa_x$.

$\mathscr{K}_5$. If $\delta \subset \sigma$ is a non-empty semihereditary system, then δ^* is non-empty.

$\mathscr{K}_6$. If $\delta \subset \sigma$ is a non-empty semihereditary, stable system, then δ^* is uncountable.

Let $x \in P^{\sim}$, $A \subset P$, and let F be a function on σ_A. We call the number

$$_{\sharp}F(x,A) = \inf\{\liminf F(B_\alpha) : \{B_\alpha\} \in \varkappa_x(\sigma_A)\}$$

the *lower limit* of F at x relative to A and the number

$$_{*}F(x,A) = {}_{\sharp}(F/G)(x,A)$$

the *lower derivate* of F at x relative to A.

Let $A \in \sigma$ and let f be a function on A^-. A superadditive function M on σ_A is said to be a *majorant* of f on A if and only if there is a countable set $Z_M \subset A^-$ such that

$$_{\sharp}(-G)(x,A) \geqq 0 \quad \text{for all } x \in Z_M ,$$

$$_{\sharp}M(x,A) \geqq 0 \quad \text{for all } x \in Z_M \cup (\infty) ,$$

and

$$-\infty \neq {}_{*}M(x,A) \geqq f(x) \quad \text{for all } x \in A^- - Z_M .$$

The number $I_u(f,A) = \inf M(A)$, where the infimum is taken over all majorants of f on A, is called the *upper integral* of f over A. If

$$I_u(f,A) = -I_u(-f,A) \neq \pm\infty ,$$

this common value is called the *integral* of f over A.

If $A \in \sigma$ and $f \in \mathfrak{F}(A^-)$ we denote by $\mathfrak{M}(f,A)$ the family of all majorants of f on A. The family of all functions integrable over $A \in \sigma$ is denoted by $\mathfrak{P}(A)$.

2. Absolutely integrable functions.

Let $A \in \sigma$. A function $f \in \mathfrak{P}(A)$ for which also $|f| \in \mathfrak{P}(A)$ is said to be *absolutely integrable* over A. The family of all functions absolutely integrable over A is denoted by $\mathfrak{P}_0(A)$.

2.1. Lemma. *Let $A \in \sigma$, $f \in \mathfrak{P}(A)$ and let $g \in \mathfrak{F}(A^-)$ have locally a narrow primitive function in A^-, that is, for every $x \in A^-$ let there be a $U \in \Gamma_x$ such that g has a narrow primitive function in $A \cap U$* (see [12, 6.3]). *Then*

$$I_u(f\vee g,A) = -I_u(-[f\vee g],A) \quad \text{and} \quad I_u(f\wedge g,A) = -I_u(-[f\wedge g],A) .$$

PROOF. If $I_u(-[f\vee g],A)=-\infty$, then by [12, 6.1],

$$I_u(f\vee g,A) = -I_u(-[f\vee g],A).$$

If $I_u(-[f\vee g],A)>-\infty$, then $I_u(-[f\vee g])$ is finite; for $I_u(-[f\vee g])\leqq -I(f)$. Given $\varepsilon>0$, choose $M\in\mathfrak{M}(f)$ such that $M(A)<I(f,A)+\varepsilon$. The function

$$N = M-I(f)-I_u(-[f\vee g])$$

is superadditive and finite on σ_A. Since $I_u(-[f\vee g])\leqq -I(f)$ and $I(f)\leqq M$,

$$N \geqq M \quad\text{and}\quad N \geqq -I_u(-[f\vee g]) \geqq -I_u(-g).$$

Because g has locally a narrow primitive function, it follows from $\mathscr{K}_2$ that $N\in\mathfrak{M}(f\vee g)$. Hence

$$I_u(f\vee g,A) \leqq N(A) \leqq \varepsilon-I_u(-[f\vee g],A),$$

and the arbitrariness of ε gives

$$I_u(f\vee g,A) = -I_u(-[f\vee g],A).$$

Furthermore,

$$I_u(f\wedge g,A) = I_u(-[-f\vee -g],A) = -I_u(-f\vee -g,A) = -I_u(-[f\wedge g],A).$$

2.2. COROLLARY. *If $A\in\sigma$ and $f\in\mathfrak{P}(A)$, then*

$$I_u(f^+,A) = -I_u(-f^+,A), \quad I_u(f^-,A) = -I_u(-f^-,A),$$
$$I_u(|f|,A) = -I_u(-|f|,A).$$

2.3. THEOREM. *Let $A\in\sigma$, $f,f_i\in\mathfrak{F}(A^-)$, $c_i\in E$, $c_i\neq\pm\infty$, $i=1,2$, and let $f(x)=c_1f_1(x)+c_2f_2(x)$ for all $x\in A^-$ for which $c_1f_1(x)+c_2f_2(x)$ has meaning. Then $f\in\mathfrak{P}_0(A)$ whenever $f_1,f_2\in\mathfrak{P}_0(A)$.*

PROOF. If $f_1,f_2\in\mathfrak{P}_0(A)$, then by [12, 6.5], $f\in\mathfrak{P}(A)$ and by [12, 6.4],

$$I_u(|f|,A) \leqq |c_1|I(|f_1|,A)+|c_2|I(|f_2|,A) < +\infty.$$

Now it suffices to apply 2.2.

2.4. COROLLARY. *Let $A\in\sigma$. If f,g belong to $\mathfrak{P}_0(A)$, then so do f^+, f^-, $|f|$, $f\vee g$, and $f\wedge g$. If $f,g\in\mathfrak{P}_0(A)$, $h\in\mathfrak{P}(A)$, and $f\leqq h\leqq g$, then also $h\in\mathfrak{P}_0(A)$.*

2.5. DEFINITION. Let $A\in\sigma$ and $f\in\mathfrak{F}(A^-)$. A function $M\in\mathfrak{M}(f,A)$ for which $Z_M=\emptyset$ is called a *narrow majorant* of f on A.

Using narrow majorants instead of majorants in definition [12, 3.3], we can define the *narrow integral* $I^{\wedge}(f,A)$. We shall introduce the symbols $\mathfrak{M}^{\wedge}$, $I_u^{\wedge}$, $\mathfrak{P}^{\wedge}$ and $\mathfrak{P}_0^{\wedge}$ the meaning of which is obvious.

The close examination of proofs given in [12, section 6], will show that all theorems of that section hold also for the narrow integral $I^{\wedge}$. It turns out that this is true even if the convergence $\varkappa$ satisfies only axioms $\mathscr{K}_1-\mathscr{K}_5$. We note that the narrow integral $I^{\wedge}$ and the integral defined in [8], [9] are closely related.

If $A \in \sigma$, then clearly

$$-I_u^{\wedge}(-f,A) \leqq -I_u(-f,A) \leqq I_u(f,A) \leqq I_u^{\wedge}(f,A)$$

for every $f \in \mathfrak{F}(A^-)$. Hence $\mathfrak{P}^{\wedge}(A) \subset \mathfrak{P}(A)$ and $I^{\wedge}(f,A)=I(f,A)$ for every $f \in \mathfrak{P}^{\wedge}(A)$. Examples 5.2 and [12, 8.6] show that the inclusions

$$\mathfrak{P}_0^{\wedge}(A) \subset \mathfrak{P}^{\wedge}(A) \subset \mathfrak{P}(A)$$

can be proper. However, we shall see that always $\mathfrak{P}_0^{\wedge}(A)=\mathfrak{P}_0(A)$.

2.6. Proposition. *Let $A \in \sigma, f \in \mathfrak{F}(A^-)$ and let $I_u(f,A) \neq \pm\infty$. If $f \geqq 0$ or $f \leqq 0$, then $I_u(f,A)=I_u^{\wedge}(f,A)$.*

Proof. Given $\varepsilon > 0$, choose $M \in \mathfrak{M}(f,A)$ such that $M(A) < I_u(f,A)+\varepsilon/2$ and let $Z_M=\{x_1,x_2,\ldots\}$.

Suppose $f \geqq 0$. Since ${}_{\sharp}(-G)(x_n,A) \geqq 0$, there is a decreasing sequence $\{U_k^n\}_{k=1}^{\infty} \subset \Gamma_{x_n}$ such that

$$G(U_k^n \cap A) \leqq \varepsilon/2^{n+k+1}, \quad k=1,2,\ldots \text{ (see } \mathscr{K}_1, \mathscr{K}_4) .$$

Letting

$$F_n(B) = \textstyle\sum_{k=1}^{\infty} G(U_k^n \cap B) \quad \text{for every } B \in \sigma_A ,$$

we have defined a function $F_n \in \mathfrak{F}_a(\sigma_A)$ for which $0 \leqq F_n \leqq \varepsilon/2^{n+1}$. If

$$\{B_\alpha, \alpha \in D, \succ\} \in \varkappa_{x_n}(\sigma_A)$$

and $k \geqq 1$ is an integer, then there is an $\alpha_k \in D$ such that $B_\alpha \subset U_k^n$ for all $\alpha \in D(\alpha_k)$ (see $\mathscr{K}_2$). Hence, $F_n(B_\alpha)/G(B_\alpha) \geqq k$ for all $\alpha \in D(\alpha_k)$ and it follows that ${}_{*}F_n(x_n)=+\infty$. Because $M \geqq 0$, $M^{\wedge}=M+\sum_n F_n$ belongs to $\mathfrak{M}^{\wedge}(f,A)$ and

$$I_u^{\wedge}(f,A) \leqq M^{\wedge}(A) \leqq M(A)+\tfrac{1}{2}\varepsilon < I_u(f,A)+\varepsilon . \tag{1}$$

Suppose $f \leqq 0$. Since ${}_{\sharp}I_u(f)(x_n,A)=0$ (see [12, 6.1]), there is $U_n \in \Gamma_{x_n}$ such that

$$I_u(f,U_n \cap A) \geqq -\varepsilon/2^{n+1}$$

(see $\mathscr{K}_1, \mathscr{K}_4$). Letting

$$F(B) = \sum_n I_u(f, U_n \cap B) \quad \text{for every } B \in \sigma_A$$

we have defined a function $F \in \mathfrak{F}_a(\sigma_A)$ (see [12, 6.6]) for which $-\frac{1}{2}\varepsilon \leqq F \leqq 0$. Because for every $B \in \sigma_{U_k \cap A}$,

$$M(B) - F(B) = M(B) - I_u(f, B) - \sum_{n \neq k} I_u(f, U_n \cap B) \geqq 0,$$

it follows from $\mathscr{K}_2$ that $_*(M-F)(x_k, A) \geqq 0 \geqq f(x_k)$. Hence $M^\wedge = M - F$ belongs to $\mathfrak{M}^\wedge(f, A)$ and again (1) is valid.

The inequality $I_u^\wedge(f, A) \leqq I_u(f, A)$ now follows from the arbitrariness of ε. Since $I_u(f, A) \leqq I_u^\wedge(f, A)$ for every $f \in \mathfrak{F}(A^-)$, the proof is completed.

2.7. Corollary. *For every* $A \in \sigma$, $\mathfrak{P}_0^\wedge(A) = \mathfrak{P}_0(A)$.

Proof. Obviously $\mathfrak{P}_0^\wedge(A) \subset \mathfrak{P}_0(A)$. If f belongs to $\mathfrak{P}_0(A)$, by 2.3 so do f^+ and f^-. Now according to 2.6, f^+ and f^- belong to $\mathfrak{P}_0^\wedge(A)$, and by an analogue of 2.3 for the narrow integral, so does f.

2.8. Definition. Let $A \in \sigma$ and $F \in \mathfrak{F}(\sigma_A)$. The functions $_0F \in \mathfrak{F}(\sigma_A)$ and $^0F \in \mathfrak{F}(\sigma_A)$ defined by the rules

$$_0F(B) = \inf \sum_{i=1}^n F(B_i) \quad \text{and} \quad {}^0F(B) = \sup \sum_{i=1}^n F(B_i)$$

for all $B \in \sigma_A$ are called the *lower* and *upper variation* of F on A, respectively; here the infimum and supremum are taken over all finite disjoint families $\{B_i\}_{i=1}^n \subset \sigma_B$ for which $\sum_{i=1}^n F(B_i)$ has meaning.

Obviously, $_0F \leqq F \leqq {}^0F$, $_0F \leqq F(\emptyset) \leqq {}^0F$, and $_0(-F) = -{}^0F$. Also $_0F(B) \geqq {}_0F(A)$ and ${}^0F(B) \leqq {}^0F(A)$ for every $B \in \sigma_A$.

2.9. Lemma. *Let $A \in \sigma$ and let $F \in \mathfrak{F}(\sigma_A)$ be finite. If F is superadditive, so is ^0F and $F - {}_0F \geqq {}^0F$. If F is additive, so are ^0F and $_0F$, and $F - {}_0F = {}^0F$.*

Proof. Let $B_1, \ldots, B_n$ be disjoint sets from σ_A with union B for which $\sum_{i=1}^n {}^0F(B_i)$ has meaning. Further, let $C \in \sigma_A$ and let $C_1, \ldots, C_r$ be disjoint sets from σ_C. Then by [10, (1.2)], there are disjoint sets $D_1, \ldots, D_s$ from σ_C such that

$$C - \bigcup_{i=1}^r C_i = \bigcup_{j=1}^s D_j.$$

Suppose F is superadditive. Given disjoint families $\{B_i^j\}_{j=1}^{k_i} \subset \sigma_{B_i}$, $= 1, 2, \ldots, n$,

$$^0F(B) \geqq F\left(\bigcup_{i=1}^n \bigcup_{j=1}^{k_i} B_i^j\right) \geqq \sum_{i=1}^n \sum_{j=1}^{k_i} F(B_i^j),$$

and we have ${}^0F(B) \geqq \sum_{i=1}^n {}^0F(B_i)$. Furthermore,

$$F(C) - {}_0F(C) \geqq F(C) - \sum_{j=1}^s F(D_j) \geqq \sum_{i=1}^r F(C_i),$$

and hence $F(C) - {}_0F(C) \geqq {}^0F(C)$.

Suppose F is additive. Given a disjoint family $\{B^j\}_{j=1}^k \subset \sigma_B$, we obtain

$$\sum_{j=1}^k F(B^j) = \sum_{j=1}^k \sum_{i=1}^n F(B^j \cap B_i) \leqq \sum_{i=1}^n {}^0F(B_i),$$

and so ${}^0F(B) \leqq \sum_{i=1}^n {}^0F(B_i)$. Also

$$F(C) - {}^0F(C) \leqq F(C) - \sum_{j=1}^s F(D_j) = \sum_{i=1}^r F(C_i),$$

and thus

$$F(C) - {}^0F(C) \leqq {}_0F(C).$$

Since ${}^0F$ is additive, so is ${}_0F = F - {}^0F$, and the proof is completed.

2.10. Proposition. *Let $A \in \sigma$ and $f \in \mathfrak{P}(A) - \mathfrak{P}_0(A)$. Then ${}_0I(f)(A) = -\infty$.*

Proof. Suppose ${}_0I(f)(A) > -\infty$. Since ${}_0I(f) \leqq I(f) < +\infty$, ${}_0I(f)$ is finite and so is ${}^0I(f) = I(f) - {}_0I(f)$ (see 2.9). Choose $M \in \mathfrak{M}(f, A)$ such that $M(A) < I(f, A) + 1$. Because ${}^0M \in \mathfrak{F}_s(\sigma_A)$ (see 2.9) and ${}^0M \geqq M^+$, it follows that ${}^0M \in \mathfrak{M}(f^+, A)$. Hence by [12, 6.4],

$$\begin{aligned} I_u(|f|, A) &\leqq 2I_u(f^+, A) - I(f, A) \\ &\leqq 2\,{}^0M(A) - I(f, A) \\ &\leqq 2\,{}^0I(f)(A) - I(f, A) + 2 < +\infty . \end{aligned}$$

Now according to 2.2, $|f| \in \mathfrak{P}(A)$ which is a contradiction.

2.11 Corollary. *Let $A \in \sigma$ and let σ_A be a σ-algebra. If $f \in \mathfrak{P}(A) - \mathfrak{P}_0(A)$, then $I(f)$ is an additive but not a σ-additive function on σ_A.*

The corollary follows from 2.10, [12, 6.6] and [14, (6.1), p. 10].

2.12. Theorem. *Let σ be an algebra and let $\varkappa = \varkappa^0$ be the natural convergence.* (See [12, 2].) *Then $\mathfrak{P}(A) = \mathfrak{P}^\wedge(A) = \mathfrak{P}_0(A)$ for every $A \in \sigma$.*

Proof. Let $A \in \sigma$, $f \in \mathfrak{P}(A)$ and let $\sigma^*_{-\infty}({}_0I(f), A) \neq \emptyset$ (see [12, 5.5]). Then there is an $x \in A^\sim$ and a net

$$\{B_\alpha\}_{\alpha \in D} \in \varkappa_x^0\left[\sigma_{-\infty}({}_0I(f), A)\right].$$

Choose an integer k. Since σ is an algebra, to every $\alpha \in D$ there is a $C_\alpha \in \sigma_{B_\alpha}$ such that $I(f, C_\alpha) < k$. Because $\{C_\alpha\}_{\alpha \in D} \in \varkappa_x{}^0(\sigma_A)$, it follows from the arbitrariness of k that ${}_{\sharp}I(f)(x) = -\infty$; a contradiction to [12, 6.15]. Thus $\sigma^*_{-\infty}({}_0I(f), A) = \emptyset$ and by [12, 4.1 and 5.6], also $\sigma_{-\infty}({}_0I(f), A) = \emptyset$. In particular, ${}_0I(f)(A) > -\infty$ and $f \in \mathfrak{P}_0(A)$ (see 2.10). We conclude

$$\mathfrak{P}_0{}^\wedge(A) \subset \mathfrak{P}^\wedge(A) \subset \mathfrak{P}(A) = \mathfrak{P}_0(A) = \mathfrak{P}_0{}^\wedge(A)\,,$$

and the theorem follows.

Examples 5.2 and [12, 8.6] show, that the previous theorem is not correct if σ is not an algebra.

2.13. Proposition. *Let $A \in \sigma$, $f \in \mathfrak{P}(A) - \mathfrak{P}_0(A)$ and let $M \in \mathfrak{M}(f, A)$. If $M(A) < +\infty$, then*

$$a = \inf\{{}_*M(x) : x \in A^-\} = -\infty\,.$$

Proof. Suppose $a > -\infty$ and let $b = a \wedge 0$. The function $N = M - bG$ is superadditive on σ_A, $N \geqq M$, and for all $x \in A^- - Z_M$,

$${}_*N(x) \geqq {}_*M(x) - b \geqq [{}_*M(x)]^+ \geqq f^+(x)\,.$$

Hence $N \in \mathfrak{M}(f^+, A)$ and by [12, 6.4],

$$I_u(|f|, A) \leqq 2I_u(f^+, A) - I(f, A) \leqq 2N(A) - I(f, A) < +\infty\,.$$

Now according to 2.2, $|f| \in \mathfrak{P}(A)$ which is a contradiction.

2.14. Lemma. *Suppose that $\varkappa = \varkappa^0$ is the natural convergence. Let $A \in \sigma$, $A^- = A^\sim$, and $F \in \mathfrak{F}(\sigma_A)$. If ${}_*F(x) > -\infty$ for all $x \in A^-$, then also*

$$a = \inf\{{}_*F(x) : x \in A^-\} > -\infty\,.$$

Proof. Let $a = -\infty$. Then there are $x_n \in A^-$ such that ${}_*F(x_n) < -n$, $n = 1, 2, \ldots$. Since A^- is compact the set $\{x_n\}_{n=1}^\infty$ has a cluster point $x_0 \in A^-$. Choose an integer k. Given $U \in \Gamma_{x_0}$, there is an integer $n_U \geqq -k$ for which $x_{n_U} \in U^0$. Furthermore, there is a set $B_U \in \sigma_{U \cap A}$ such that

$$F(B_U)/G(B_U) < -n_U \leqq k\,.$$

Since $\{B_U\}_{U \in \Gamma_{x_0}} \in \varkappa_x{}^0(\sigma_A)$, it follows from the arbitrariness of k that ${}_*F(x_0) = -\infty$; a contradiction.

2.15. COROLLARY. *Suppose that $\varkappa = \varkappa^0$ is the natural convergence. Then $\mathfrak{P}^\wedge(A) = \mathfrak{P}_0^\wedge(A)$ for every $A \in \sigma$ for which A^- is compact.*

Examples 5.2 and [12, 8.6], show that the corollary does not hold for all $A \in \sigma$.

2.16. REMARK. Theorem 2.12 and corollary 2.15 indicate that the natural convergence $\varkappa^0$ is usually too large to give us a conditionally convergent integral.

The next proposition in some sense characterizes the difference between the integrals I and $I^\wedge$. It will be illustrated by examples 5.2 and 5.3.

2.17. PROPOSITION. *Let $A \in \sigma$, $f \in \mathfrak{P}(A)$, $g \in \mathfrak{P}^\wedge(A)$, and let S and $S^\wedge$ be the sets of all points $x \in A^-$ for which there is a net*

$$\{B_\alpha, \alpha \in D, \succ\} \in \varkappa_x(\sigma_A)$$

of disjoint sets such that

$$\sum \{|I(f,B_\alpha)| : \alpha \in D(\beta)\} = +\infty \quad \text{and} \quad \sum \{|I^\wedge(g,B_\alpha)| : \alpha \in D(\beta)\} = +\infty$$

for every $\beta \in D$, respectively. Then S is countable and $S^\wedge$ is empty.

PROOF. We shall prove that S is countable. The proof that $S^\wedge$ is empty is similar and may be left to the reader. Let S be uncountable and let $F = M - I(f)$ where M is a finite majorant of f on A. Choose $x \in S - Z_M$ (such x exists, for Z_M is countable), $U \in \Gamma_x$, a finite $c < {}_*M(x) \wedge 0$, and a net

$$\{B_\alpha, \alpha \in D, \succ\} \in \varkappa_x(\sigma_A)$$

from the definition of the set S. Without loss of generality we may assume that

$$\sum \{I(f,B_\alpha) : \alpha \in D(\beta)\} = -\infty$$

for all $\beta \in D$ (see $\mathscr{K}_3$). According to $\mathscr{K}_2$ and [12, 5.2], there is an $\alpha_0 \in D$ such that

$$B_\alpha \subset A \cap U \quad \text{and} \quad M(B_\alpha) \geqq cG(B_\alpha) > -\infty$$

for all $\alpha \in D(\alpha_0)$. Select $\alpha_1, \ldots, \alpha_n \in D(\alpha_0)$ for which

$$\textstyle\sum_{i=1}^n I(f, B_{\alpha_i}) < cG(A \cap U) - F(A \cap U) .$$

Since F is a non-negative superadditive function,

$$\begin{aligned} F(A\cap U) \geqq \textstyle\sum_{i=1}^n F(B_{\alpha_i}) &= \textstyle\sum_{i=1}^n M(B_{\alpha_i}) - \sum_{i=1}^n I(f,B_{\alpha_i}) \\ &> c\textstyle\sum_{i=1}^n G(B_{\alpha_i}) - cG(A\cap U) + F(A\cap U) \\ &\geqq F(A\cap U); \end{aligned}$$

a contradiction.

We shall close this section by three almost-everywhere-type propositions which we shall list without proofs. Their proofs are identical with those of the corresponding propositions in [6, 34–37] and [7, III, 19, 20]. We begin with a definition.

2.18. Definition. For $A\subset P$, let χ_A denote the *characteristic function* of A in P. The set $A\subset P$ is said to be a *zero set* if and only if $I_u(\chi_A,P)=0$.

For $A\subset P$, functions $f,g\in\mathfrak{F}(A)$ are said to be equal *almost everywhere* on A, in notation $f\doteq g$, if and only if $\{x\in A: f(x)\neq g(x)\}$ is a zero set. The meanings of symbols $f\dot{<}g$ and $f\dot{\leqq}g$ are analogous.

Clearly, if $A\subset P$ is a zero set, then $\chi_A\in\mathfrak{P}_0(A)$ and $I(\chi_A,P)=0$. Using 2.3 and [12, 6.11] we can prove by the standard procedure that the system of all zero sets is a hereditary σ-ring (see [4, p. 41]).

Let us notice that in general there is no connection between a zero set and a set $A\in\sigma$ for which $G(A)=0$ (see 5.9 and 5.10).

2.19. Proposition. *Let $A\in\sigma$ and $f,g\in\mathfrak{F}(A^-)$. Then the following statements hold:*

(i) *if $I_u(f,A)<+\infty$ and $I_u(-f,A)<+\infty$, then $|f|\dot{<}+\infty$;*
(ii) *if $f\doteq g$, then $I_u(f,A)=I_u(g,A)$;*
(iii) *if $f,g\in\mathfrak{P}(A)$, $f\leqq g$ and $I(f,A)=I(g,A)$, then $f\doteq g$.*

2.20. Proposition. *Let $A\in\sigma$ and $f\in\mathfrak{P}(A)$. Then*

$$_*I(f)\ \dot{\leqq}\ f\ \dot{\leqq}\ -_*[-I(f)]\,.$$

2.21. Proposition. *Let $A\in\sigma$ and $f,g\in\mathfrak{P}(A)$. Then $I(f)=I(g)$ if and only if $f\doteq g$.*

3. The integral in a locally compact Hausdorff space.

So far given a set $A\in\sigma$, we know only very few functions which are integrable over A. From [12, (6.1)] it follows that finite constants belong to $\mathfrak{P}_0(A)$, provided A^- is compact. However, if A^- is not compact even this need not be true for non-zero constants (see 5.4). In this section we shall specialize the space P and then describe some important families of integrable functions. We shall also prove some theorems which will be used in section 4.

From now on we shall assume that P is a locally compact Hausdorff space and that U^- is compact for every $U \in \bigcup\{\Gamma_x : x \in P\}$. We note that the second assumption is of a purely technical nature and brings no loss of generality. It follows that the localization λ of σ (see [12, 2]) is precisely the system of those $A \in \sigma$ for which A^- is compact.

Let $\mathfrak{C}$ and $\mathfrak{U}$ be the families of all compact and all open subsets of P, respectively. By $\mathfrak{S}_0$ and $\mathfrak{S}$ we denote the σ-rings generated by $\mathfrak{C}$ and $\mathfrak{U}$, respectively. Obviously, $\mathfrak{S}$ is a σ-algebra which contains $\mathfrak{S}_0$.

For $A \subset P$ we denote by $\mathfrak{F}_c(A)$ the family of all functions $f \in \mathfrak{F}(A)$ which are continuous and finite on A.

3.1. Lemma. *Let $A \in \sigma$, $f \in \mathfrak{F}(A^-)$ and $F = I_u(f)$. If f is continuous and finite at $x \in A^-$, then either*

$$_*F(x) = -_*(-F)(x) = f(x) \quad \textit{or} \quad _*F(x) = {}_*(-F)(x) = +\infty .$$

Proof. Choose $\{B_\alpha, \alpha \in D, \succ\} \in \varkappa_x(\sigma_A)$. Since $[\pm F(B_\alpha)]/G(B_\alpha) = +\infty$, whenever $G(B_\alpha) = 0$ (see [12, (6.2)]), we may assume $G(B_\alpha) > 0$ for all $\alpha \in D$. Let ω_α and Ω_α denote the infimum and supremum of the set $\{f(y) : y \in B_\alpha^-\}$, respectively. By $\mathscr{K}_2$ there is $\alpha_0 \in D$ such that $B_\alpha \in \lambda$ and $-\infty < \omega_\alpha \leqq \Omega_\alpha < +\infty$ for all $\alpha \in D(\alpha_0)$. Therefore by [12, (3.1) and (6.1)],

$$\omega_\alpha G(B_\alpha) \leqq F(B_\alpha) \leqq \Omega_\alpha G(B_\alpha),$$

and hence

$$\omega_\alpha \leqq F(B_\alpha)/G(B_\alpha) \leqq \Omega_\alpha$$

for all $\alpha \in D(\alpha_0)$. According to [5, theorem 1(f), p. 86],

$$\lim \omega_\alpha = \lim \Omega_\alpha = f(x),$$

and the lemma follows.

3.2. Proposition. *Let $A \in \sigma$ and let $f \in \mathfrak{F}_c(A^-)$ be a non-negative function. Then*

$$I_u(f, A) = -I_u(-f, A).$$

Proof. The proposition holds trivially if $I_u(-f, A) = -\infty$. Let $I_u(-f, A) > -\infty$. Then by [12, 6.1], $_{\#}I_u(-f)(\infty) = 0$, and since $-I_u(-f) \geqq 0$, also $_{\#}[-I_u(-f)](\infty) = 0$. It follows from 3.1, that $-I_u(-f)$

is a narrow primitive function to f on A (see [12, 6.3]), and the proposition is proved.

3.3. Corollary. *Let* $A \in \sigma$ *and* $f \in \mathfrak{F}_c(A^-)$. *If* $I_u(|f|,A) < +\infty$, *then* $f \in \mathfrak{P}_0(A)$.

3.4. Corollary. *Let* $A \in \sigma$, $c \in E$, $c \neq \pm\infty$ *and* $f = c$ *on* A^-. *If* $G(A) < +\infty$, *then* $f \in \mathfrak{P}_0(A)$.

We note here that though $|c|G \in \mathfrak{M}(|f|,A)$, in general, $I(f,A) \neq cG(A)$ (see 5.9).

3.5. Corollary. *Let* $A \in \sigma$ *and* $f \in \mathfrak{F}_c(A^-)$. *Then* $f \in \mathfrak{P}_0(A)$ *whenever* f *vanishes outside some compact subset of* P.

Proof. There are disjoint sets $A_1, \ldots, A_n$ from λ such that

$$\{x \in A^- : f(x) \neq 0\}^- \subset \bigcup_{i=1}^n A_i^0 .$$

According to [10, (1.2)], there are disjoint sets $B_1, \ldots, B_m$ from σ for which

$$\bigcup_{j=1}^m B_j = P - \bigcup_{i=1}^n A_i .$$

Hence

$$\begin{aligned} I_u(|f|,A) &= \textstyle\sum_{i=1}^n I_u(|f|,A \cap A_i) + \sum_{j=1}^m I_u(|f|,A \cap B_j) \\ &= \textstyle\sum_{i=1}^n I_u(|f|,A \cap A_i) \\ &\leqq \sup\{|f(x)| : x \in A^-\} \textstyle\sum_{i=1}^n G(A \cap A_i) < +\infty . \end{aligned}$$

3.6. Corollary. *Let* $A \in \sigma$ *and* $C \in \mathfrak{C}$. *Then* $\chi_C \in \mathfrak{P}_0(A)$.

Since every integrable function $f \in \mathfrak{F}_c(A^-)$ has already a narrow primitive function (see 3.1), corollary 3.6 follows from 3.5 and [12, 6.14].

3.7. Proposition. *Let* $A \in \sigma$ *and* $f \in \mathfrak{F}_c(A^-)$. *If axiom* $\mathscr{K}_7$ (see [12, section 7]) *holds, then* $f \in \mathfrak{P}(A)$ *whenever* $I_u(f,A) \neq \pm\infty$.

Indeed, according to 3.1 and [12, 7.4], $I_u(f)$ is a narrow primitive function to f on A. We note, however, that axiom $\mathscr{K}_7$ plays the essential part in this proposition. A slight modification of example [12, 8.2] will provide a counterexample.

The next proposition is due to W. J. Wilbur.

3.8. Proposition. *Let* $A \in \sigma$, $f \in \mathfrak{F}(A^-)$, *and let* $I_u(|f|,A) < +\infty$. *Then*

$$_{\#}I_u(f)(\infty,A) = {}_{\#}[-I_u(f)](\infty,A) = 0 .$$

Proof. Assume first $f \geqq 0$ and choose a finite $M \in \mathfrak{M}(f,A)$. For $B \in \sigma_A$ we let

$$N(B) = \sup \textstyle\sum_{i=1}^n M(B_i) ,$$

where the supremum is taken over all finite disjoint families $\{B_i\}_{i=1}^n \subset \lambda_B$. Imitating the proof of 2.9, it is easy to show that the thus defined function $N \in \mathfrak{F}(\sigma_A)$ is superadditive. Since M is superadditive and nonnegative, $N \leqq M$ and $N(B) = M(B)$ whenever $B \in \lambda_A$. It follows that $N \in \mathfrak{M}(f,A)$. Suppose $_{\#}(-N)(\infty,A) = a < 0$. Then there is a net $\{B_\alpha\}_{\alpha \in D} \in \varkappa_\infty(\sigma_A)$ such that $N(B_\alpha) > -\frac{1}{2}a$ for all $\alpha \in D$. Choose $\alpha \in D$ and disjoint sets $B_\alpha^{\,1},\ldots,B_\alpha^{\,n}$ from λ_{B_α} for which

$$\textstyle\sum_{i=1}^n M(B_\alpha^{\,i}) > -\frac{1}{2}a .$$

Because $B_\alpha^{\,i-}$ are compact, there is $\beta \in D$ such that $B_\beta \cap \bigcup_{i=1}^n B_\alpha^{\,i} = \emptyset$ (see $\mathscr{K}_2$) and we can choose disjoint sets $B_\beta^{\,1},\ldots,B_\beta^{\,m}$ for which

$$\textstyle\sum_{i=1}^m M(B_\beta^{\,i}) > -\frac{1}{2}a .$$

Continuing this process by induction we construct a finite family $\{B_i\}_{i=1}^k \subset \lambda_A$ of disjoint sets such that

$$\textstyle\sum_{i=1}^k M(B_i) > M(A) .$$

Since this is impossible, we conclude

$$0 \leqq {}_{\#}(-N)(\infty,A) \leqq {}_{\#}[-I_u(f)](\infty,A) \leqq 0 .$$

If $f \in \mathfrak{F}(A^-)$ is arbitrary, then by [12, 5.1 and 6.4],

$$0 \geqq {}_{\#}[-I_u(f)](\infty,A) \geqq {}_{\#}[-I_u(f^+)](\infty,A) + {}_{\#}[-I_u(-f^-)](\infty,A) \geqq 0 .$$

Now the proposition follows from [12, 6.15].

3.9. Lemma. *Let* $A \in \sigma$ *and* $f \in \mathfrak{F}(A^-)$. *If* $f \geqq 0$ *and* $I_u(f,A) < +\infty$, *then*

$$I_u(f,A) = \sup\{I_u(f \cdot \chi_C, A) : C \in \mathfrak{C}_{A^-}\} .$$

Proof. Choose $\varepsilon > 0$. By $\mathscr{K}_1$, $\mathscr{K}_4$ and 3.8 there is a $U \in \Gamma_\infty$ such that

$$\textstyle\sum_{i=1}^p I_u(f, A \cap U_{i,\infty}) < \varepsilon$$

(see [12, section 2]). According to [10, (1.2)] there are disjoint sets $C_1,\ldots,C_n$ from σ for which

$$\bigcup_{j=1}^{n} C_j = A - U .$$

Letting $C = \bigcup_{j=1}^{n} C_j^-$ we have $C \in \mathfrak{C}_{A-}$ and

$$\begin{aligned} I_u(f,A) &< \sum_{j=1}^{n} I_u(f,C_j) + \varepsilon \\ &\leqq \sum_{j=1}^{n} I_u(f\chi_C, C_j) + \sum_{i=1}^{p} I_u(f\chi_C, A \cap U_{i,\infty}) + \varepsilon \\ &= I_u(f\chi_C, A) + \varepsilon . \end{aligned}$$

The lemma follows from the arbitrariness of ε.

3.10. COROLLARY. *Let $A \in \sigma$, $f \in \mathfrak{F}(A^-)$, $f \geqq 0$, and let $f\chi_C \in \mathfrak{P}_0(A)$ for every $C \in \mathfrak{C}_{A-}$. Then $f \in \mathfrak{P}_0(A)$ whenever $I_u(f,A) < +\infty$.*

3.11. NOTE. If in corollary 3.10 $I_u(f,A) = +\infty$, then, as example 5.5 shows, this need not imply $-I_u(-f,A) = I_u(f,A)$. On the other hand, from $-I_u(-f,A) = I_u(f,A)$ does not follow that $f\chi_C \in \mathfrak{P}_0(A)$ for every $C \in \mathfrak{C}_{A-}$ (see 5.6).

3.12. NOTATION. Let $\mathfrak{T}$ be the family of all sets $A \subset P$ such that $\chi_{A \cap C} \in \mathfrak{P}_0(P)$ for all $C \in \mathfrak{C}$. If $A \in \mathfrak{T}$ we let $\tau(A) = I_u(\chi_A, P)$.

3.13. PROPOSITION. *The triple $(P, \mathfrak{T}, \tau)$ is a complete measure space, $\mathfrak{S} \subset \mathfrak{T}$, and the measure τ is inner regular on $\mathfrak{U}$ and outer regular and finite on $\mathfrak{C}$.*

PROOF. The completeness of the measure τ is obvious (see [3, (11.20), p. 155]). By 3.7, $\chi_C \in \mathfrak{P}_0(P)$ whenever $C \in \mathfrak{C}$. According to this and [12, 6.11], given $C \in \mathfrak{C}$, the system

$$\mathfrak{T}_C = \{A \subset C : \chi_A \in \mathfrak{P}_0(P)\}$$

is a σ-algebra containing $\mathfrak{S}_C$. Thus $\mathfrak{T}$ is a σ-algebra containing $\mathfrak{S}$.

Let A_1, A_2 be disjoint sets from $\mathfrak{T}$. By [12, 6.4],

$$\tau(A_1 \cup A_2) \leqq \tau(A_1) + \tau(A_2) .$$

If $\tau(A_1)$ or $\tau(A_2)$ is infinite, so is $\tau(A_1 \cup A_2)$, and the equality holds. If $\tau(A_i) < +\infty$, $i = 1, 2$, then by 3.10, $\chi_{A_i} \in \mathfrak{P}_0(P)$, and according to [12, 6.5], the equality holds again. Hence τ is additive on $\mathfrak{T}$, and its σ-additivity follows from [12, 6.11].

Finally, both regularity properties (see [4, section 52, p. 224] of τ follow from [12, 6.14].

3.14. Corollary. *The measure τ is both outer and inner regular on $\mathfrak{S}_0$.*

This follows from [4, section 52, theorem F, p. 228].

3.15. Definition. Let $A \in \sigma$. A point $x \in A^-$ is said to be a *density* point of A if and only if $G(U \cap A) > 0$ for all $U \in \Gamma_x$.

The set of all density points of A is obviously closed and it is denoted by A_G. If $f, g \in \mathfrak{F}(A^-)$ and $f(x) = g(x)$ for all $x \in A_G$, then clearly $I_u(f, A) = I_u(g, A)$.

3.16. Proposition. *Let $A \in \sigma$, $f \in \mathfrak{F}(A^-)$, $f \geqq 0$, and let $I_u(f, A) < +\infty$. If A^- is paracompact, then the set $A_+ = \{x \in A_G : f(x) > 0\}$ is σ-bounded.*

Proof. We have to prove that A_+ is contained in a countable union of compact subsets of P (see [4, p. 4]). Suppose the proposition is not correct and choose a finite $M \in \mathfrak{M}^\wedge(f, A)$ (see 2.6). By $\mathscr{K}_1$ and [12, 5.2], for every $x \in A_+$ there is a $U_x \in \Gamma_x$ such that

$$M(U \cap A) \geqq \tfrac{1}{2} f(x)\, G(U \cap A) > 0$$

for all $U \in \Gamma_x(U_x)$. Using Zorn's lemma, we can find a maximal disjoint subfamily $\{V_x : x \in T\}$ of $\{U \in \Gamma_x(U_x) : x \in A_+\}$. If T is countable, then

$$B = [\bigcup \{V_x \cap A^- : x \in T\}]^-$$

is σ-bounded; for by [1, theorem 7.3, p. 241], A^- being paracompact is a disjoint union of open σ-compact subsets. Hence there is $y \in A_+ - B$ and $V_y \in \Gamma_y(U_y)$ such that $V_y \cap B = \emptyset$. Since this contradicts the maximality of $\{V_x : x \in T\}$, we conclude that T is uncountable. By a standard procedure we can find $c > 0$ such that $M(V_x \cap A) \geqq c$ for infinitely many $x \in T$. Choose an integer k such that $ck > M(A)$ and different $x_i \in T$ for which

$$M(V_{x_i} \cap A) \geqq c, \quad i = 1, 2, \ldots, k\,.$$

Since $M \geqq 0$, we obtain

$$M(A) \geqq \textstyle\sum_{i=1}^{k} M(V_{x_i} \cap A) \geqq ck > M(A)\,,$$

which is impossible.

3.17. Corollary. *Let $A \in \sigma$ and $f \in \mathfrak{P}_0(A)$. If A^- is paracompact, then the set $\{x \in A_G : f(x) \neq 0\}$ is σ-bounded.*

Example 5.7 shows that the corollary is not valid if we replace $\mathfrak{P}_0(A)$ by $\mathfrak{P}(A)$.

3.18. COROLLARY. *Let P be paracompact. Then $A \in \mathfrak{T}$ is σ-finite if and only if $A \cap P_G$ is σ-bounded. In particular, the measure τ is σ-finite if and only if P_G is σ-compact.*

3.19. REMARK. Example 5.8 shows that for a non-paracompact set A^-, or space P, statements 3.16–3.18 are generally false. On the other hand, from the proof of proposition 3.16 it follows that a weaker condition than the paracompactness of A^- will be sufficient. Namely, it suffices if the closure of any σ-bounded subset of A^- is also σ-bounded. Thus, for example, if A^- is a well ordered set with the order topology, proposition 3.16 still holds. However, this A^- is paracompact if and only if it contains a countable cofinal subset.

The following modest result holds generally.

3.20. PROPOSITION. *Let $A \in \sigma$, $f \in \mathfrak{F}(A^-)$, $f \geqq 0$, and let $I_u(f,A) < +\infty$. Then the set*

$$A_0 = \{x \in A^- : f(x) > 0 \textit{ and } {}_{\#}(-G)(x,A) < 0\}$$

is countable.

PROOF. Suppose A_0 is uncountable and choose a finite $M \in \mathfrak{M}^\wedge(f,A)$, (see 2.6). For every $x \in A_0$ there is $U_x \in \Gamma_x$ such that

$$M(U \cap A) \geqq \tfrac{1}{2}f(x)\, G(U \cap A) \geqq -\tfrac{1}{2}f(x)\, {}_{\#}(-G)(x,A) > 0$$

for all $U \in \Gamma_x(U_x)$. By a standard procedure we can find $c > 0$ and an infinite set $\{x_n\}_{n=1}^\infty \subset A_0$ such that

$$-\tfrac{1}{2}f(x_n)\, {}_{\#}(-G)(x_n,A) \geqq c \quad \text{for } n = 1,2,\ldots .$$

Choose an integer k for which $ck > M(A)$ and disjoint $U_n \in \Gamma_{x_n}(U_{x_n})$, $n = 1,2,\ldots,k$. Since $M \geqq 0$, we obtain

$$M(A) \geqq \textstyle\sum_{n=1}^k M(U_n) \geqq ck > M(A) ,$$

which is impossible.

3.21. COROLLARY. *Let $A \in \sigma$ and $f \in \mathfrak{P}_0(A)$. Then the set*

$$\{x \in A^- : f(x) \neq 0 \textit{ and } {}_{\#}(-G)(x,A) < 0\}$$

is countable.

Example 5.7 shows that even for a paracompact space P the corollary is not valid if we replace $\mathfrak{P}_0(A)$ by $\mathfrak{P}(A)$.

4. Connection with the Lebesgue integral.

If $(P, \mathfrak{A}, \mu)$ is a measure space and $A \in \mathfrak{A}$, then we denote by $\mathfrak{L}_\mu(A)$ the family of all functions $f \in \mathfrak{F}(A^-)$ whose restriction to A is $\mathfrak{A}$-measurable and for which the finite Lebesgue integral $\int_A f d\mu$ exists (see [3, (10.3), p. 126, (11.2), p. 149, and (12.2), p. 146]).

4.1. Lemma. *Let $A \in \sigma$ and $f \in \mathfrak{P}_0(A)$. Then also $f \wedge c \in \mathfrak{P}_0(A)$ for every $c \in E$, $c \geqq 0$.*

Proof. According to [12, (6.1)] and 2.1,

$$I_u(f \wedge c, A) = -I_u(-[f \wedge c], A) .$$

Since $-f^- \leqq f \wedge c \leqq f$, the lemma follows from 2.4.

4.2. Theorem. *Let τ be the measure from 3.12. Then $\mathfrak{P}_0(P) = \mathfrak{L}_\tau(P)$ and*

$$I(f,P) = \int_P f d\tau \quad \textit{for every } f \in \mathfrak{P}_0(P) .$$

Proof. Let $f \in \mathfrak{P}_0(P)$ and $A(f,c) = \{x \in P : f(x) > c\}$ for $c \in E$. If $c \geqq 0$, we set

$$f_n = 1 \wedge \big[n(f - [f \wedge c])\big], \quad n = 1, 2, \dots .$$

By 3.14,

$$\{f_n\}_{n=1}^\infty \subset \mathfrak{P}_0(P) ,$$

and since $f_n \nearrow \chi_{A(f,c)}$, it follows from [12, 6.11], that $A(f,c) \in \mathfrak{T}$. If $c < 0$, we have

$$A(f,c) = P - \bigcap_{n=1}^\infty A(-f, -c + cn^{-1}) .$$

Hence by 3.13 again $A(f,c) \in \mathfrak{T}$ and f is $\mathfrak{T}$-measurable.

Now let $f \in \mathfrak{F}(P)$ be a non-negative $\mathfrak{T}$-measurable function. Then there is an increasing sequence $\{s_n\}_{n=1}^\infty$ of non-negative $\mathfrak{T}$-simple functions which converges to f. If either $I_u(f,P)$ or $\int_P f d\tau$ is finite, then by 3.11 and 2.3,

$$\{s_n\}_{n=1}^\infty \subset \mathfrak{P}_0(P) \quad \text{and} \quad I(s_n, P) = \int_P s_n d\tau .$$

It follows from [12, 6.11] that $f \in \mathfrak{P}_0(P)$ and $I(f,P) = \int_P f d\tau$.

For a general function $f \in \mathfrak{T}(P)$ we apply our previous result to f^+ and f^-. This completes the proof.

Except for a few special cases (see [13]) we do not know whether also every function $f \in \mathfrak{P}(P)$ is $\mathfrak{T}$-measurable. The next proposition gives a conditional answer.

4.3. PROPOSITION. *If the lower derivate of every super-additive function on σ is $\mathfrak{T}$-measurable, then also every integrable function is $\mathfrak{T}$-measurable.*

PROOF. Given $f \in \mathfrak{P}(P)$, choose $M_n \in \mathfrak{M}(f)$ and $N_n \in \mathfrak{M}(-f)$ such that

$$M_n(P)+N_n(P) \leqq 1/n, \quad n=1,2,\dots .$$

Let $p_n(x)={}_*M_n(x)$ for $x \in P-Z_{M_n}$ and $p_n(x)=+\infty$ for $x \in Z_{M_n}$; similarly, let $q_n(x)=-{}_*N_n(x)$ for $x \in P-Z_{N_n}$ and $q_n(x)=-\infty$ for $x \in Z_{N_n}$. Because Z_{M_n}, Z_{N_n} are countable, p_n, q_n are $\mathfrak{T}$-measurable, and so are $p=\sup p_n$ and $q=\inf q_n$. For $x \in P$ we put $r(x)=p(x)-q(x)$ if this difference has meaning and $r(x)=0$ otherwise. Since $q \leqq f \leqq p$, $r \geqq 0$ and

$$\begin{aligned} 0 \leqq I_u(r,P) &\leqq \inf I_u(p_n-q_n,P) \\ &\leqq \inf[I_u(p_n,P)+I_u(-q_n,P)] \leqq \inf[M_n(P)+N_n(P)] = 0. \end{aligned}$$

By 2.19, (iii), $r \doteq 0$ which means $p \doteq q \doteq f$.

We note that in general there is no connection between $I(f,A)$ and $\int_A f\,d\tau$ for $A \in \sigma \cap \mathfrak{T}$ for which $A \neq P$ (see 5.10). However, we have the following proposition.

4.4. PROPOSITION. *The following two conditions are equivalent:*

(i) *$I_u(f,A)=I_u(g,A)$ for every $A \in \sigma$ and every $f,g \in \mathfrak{F}(A^-)$ for which $f=g$ on A.*

(ii) *$\sigma \subset \mathfrak{T}$, and if $A \in \sigma$, then $\mathfrak{P}_0(A)=\mathfrak{L}_\tau(A)$ and $I(f,A)=\int_A f\,d\tau$ for every $f \in \mathfrak{P}_0(A)$.*

PROOF. (i) $\Rightarrow$ (ii). If $A \subset P$ and $f \in \mathfrak{F}(A)$, let $f_A(x)=f(x)$ for $x \in A$ and $f_A(x)=0$ for $x \in P-A$. Let $A \in \sigma$. Then $f \in \mathfrak{P}_0(A)$ if and only if $f_A \in \mathfrak{P}_0(P)$; for

$$I_u(g_A,P) = I_u(g_A,A)+I_u(g_A,P-A) = I_u(g,A)$$

for every $g \in \mathfrak{F}(A^-)$. In particular, if $A \in \lambda$ and $f=\chi_{A^-}$, we obtain $A \in \mathfrak{T}$, (see 4.2, and [12, (6.1)]). Since $\mathfrak{T}$ is a σ-algebra and σ is a pre-algebra generated by $\lambda, \sigma \subset \mathfrak{T}$. On the other hand, if $A \in \mathfrak{T}$, then $f \in \mathfrak{L}_\tau(A)$ if and only if $f_A \in \mathfrak{L}_\tau(P)$. Thus by 4.2, $\mathfrak{P}_0(A)=\mathfrak{L}_\tau(A)$ for every $A \in \sigma$ and

$$I(f,A) = I(f_A,P) = \int_P f_A\,d\tau = \int_A f\,d\tau$$

for every $f \in \mathfrak{P}_0(A)$.

(ii) $\Rightarrow$ (i). Let $A \in \sigma$. By 3.12,

$$A^- - A \in \mathfrak{T} \quad \text{and} \quad I(\chi_{A^- - A},A) = \int_A \chi_{A^- - A}\,d\tau = 0.$$

Therefore $h = (+\infty)\chi_{A^- - A}$ belongs to $\mathfrak{P}_0(A)$ and

$$I(h,A) = \lim I(n\chi_{A^- - A}, A) = 0 .$$

Given $f,g \in \mathfrak{F}(A^-)$ such that $f = g$ on A, we have

$$f(x) \leqq g(x) + h(x) \quad \text{and} \quad g(x) \leqq f(x) + h(x)$$

for all $x \in A^-$ for which the right sides have meaning. An application of [12, 6.4] will complete the proof.

4.5. Definition. A measure μ defined on a σ-algebra $\mathfrak{A}$ of subsets of P is said to be *regular* if and only if $\mathfrak{U} \subset \mathfrak{A}$, μ is finite on $\mathfrak{C}$, outer regular on $\mathfrak{A}$, and inner regular on $\mathfrak{U}$.

This definition is taken from [3, (12.39), p. 177].

It follows from [3, (12.40), p. 187], that a regular measure μ is inner regular on every $A \in \mathfrak{A}$ which is $\mu - \sigma$-finite.

4.6. Notation. For $A \subset P$ let

$$\tau_0(A) = \inf\{\tau(U) : U \in \mathfrak{U} \text{ and } A \subset U\} .$$

It is easy to see that τ_0 is an outer measure in P (see [3], (10.2), p. 126) and we denote by $\mathfrak{T}_0$ the family of all τ_0-measurable subsets of P (see [3, (10.5), p. 127]).

4.7. Proposition. *The triple $(P, \mathfrak{T}_0, \tau_0)$ is a complete measure space and the measure τ_0 is regular. A set $A \subset P$ belongs to $\mathfrak{T}_0$ if and only if $A \cap C$ does for every $C \in \mathfrak{C}$. Furthermore, $\mathfrak{T}_0 \subset \mathfrak{T}$ and $\tau_0(A) = \tau(A)$ for every set $A \in \mathfrak{T}_0$ which is $\tau_0 - \sigma$-finite.*

Proof. By [3, (10.7) and (10.11), p. 128–9], $(P, \mathfrak{T}_0, \tau_0)$ is a complete measure space. Using 3.12, the verbatim repetition of the proof from [3, (9.32), p. 123] will show that $\mathfrak{U} \subset \mathfrak{T}_0$. Since $\tau_0(A) = \tau(A)$ for every $A \in \mathfrak{U} \cap \mathfrak{C}$, the measure τ_0 is regular (see 3.13).

The proof of the second statement is again the verbatim repetition of the proof given in [3, (10.31), p. 138].

Let $A \in \mathfrak{T}_0$ and $\tau_0(A) < +\infty$. By [3, (10.4), p. 139], there are monotone sequences $\{C_n\}_{n=1}^{\infty} \subset \mathfrak{C}$ and $\{U_n\}_{n=1}^{\infty} \subset \mathfrak{U}$ such that

$$\bigcup_{n=1}^{\infty} C_n = C \subset A \subset U = \bigcap_{n=1}^{\infty} U_n ,$$

$\tau_0(U_1) < +\infty$, and $\tau_0(U - C) = 0$. We have

$$\begin{aligned} 0 = \tau_0(U) - \tau_0(C) &= \lim \tau_0(U_n) - \lim \tau_0(C_n) \\ &= \lim \tau(U_n) - \lim \tau(C_n) = \tau(U) - \tau(C) = \tau(U - C) . \end{aligned}$$

Since τ is complete, $A \in \mathfrak{T}$ and $\tau_0(A) = \tau(A)$. If $A \in \mathfrak{T}_0$ is $\tau_0-\sigma$-finite, then there is an increasing sequence $\{A_n\}_{n=1}^{\infty} \subset \mathfrak{T}_0$ such that

$$A = \bigcup_{n=1}^{\infty} A_n \quad \text{and} \quad \tau_0(A_n) < +\infty$$

for $n = 1, 2, \ldots$. Thus

$$\tau_0(A) = \lim \tau_0(A_n) = \lim \tau(A_n) = \tau(A) .$$

Because $A \in \mathfrak{T}$ if and only if $A \cap C \in \mathfrak{T}$ for every $C \in \mathfrak{C}$, $\mathfrak{T}_0 \subset \mathfrak{T}$; and the proof is completed.

4.8. Corollary. *If $A \in \mathfrak{T}_0$, then $\mathfrak{L}_{\tau_0}(A) \subset \mathfrak{L}_{\tau}(A)$ and $\int_A f d\tau_0 = \int_A f d\tau$ for every $f \in \mathfrak{L}_{\tau_0}(A)$.*

Indeed, for if $f \in \mathfrak{L}_{\tau_0}(A)$, then the set $\{x \in A : f(x) \neq 0\}$ is $\tau_0 - \sigma$-finite.

4.9. Corollary. *If P is paracompact, then $\tau_0(A) = \tau(A)$ for every $A \in \mathfrak{T}_0$.*

Proof. Let $A \in \mathfrak{T}_0$. If $\tau(A)$ is infinite, so is $\tau_0(A)$. Let $\tau(A) < +\infty$. By 3.18, $A \cap P_G$ is σ-bounded and hence $\tau_0 - \sigma$-finite. Therefore,

$$\begin{aligned} \tau_0(A) \geqq \tau(A) &= \tau(A \cap P_G) = \tau_0(A \cap P_G) \\ &= \inf\{\tau(U) : U \in \mathfrak{U} \text{ and } A \cap P_G \subset U\} \\ &= \inf\{\tau[U \cup (P - P_G)] : U \in \mathfrak{U} \text{ and } A \cap P_G \subset U\} \geqq \tau_0(A) . \end{aligned}$$

4.10. Problem. There are two open questions:

(i) Is $\mathfrak{T}_0$ a proper part of $\mathfrak{T}$?

(ii) Does corollary 4.9 hold for an arbitrary locally compact Hausdorff space P?

We feel very strongly that the answer to (i) is negative, at least, in paracompact spaces. Concerning (ii), a remark similar to 3.18 applies here.

Some contribution to these problems will be given in [13].

In theorem 4.2 the integral I on $\mathfrak{P}_0(P)$ was represented as the Lebesgue integral with respect to the measure τ. However, the measure τ itself was defined by means of the integral I (see 3.12). Now we shall start with an apriori given measure μ which is equal to G on λ and we shall investigate the relationship between the integral I and the Lebesgue integral with respect to the measure μ.

4.11. PROPOSITION. *Suppose that $(P,\mathfrak{A},\mu)$ is a measure space, $\sigma\subset\mathfrak{A}$, and $G(A)=\mu(A)$ for every $A\in\lambda$. Let $A\in\sigma$ and suppose that $f\in\mathfrak{F}_c(A^-)$ vanishes outside of some compact set. Then $I(f,A)=\int_A f\,d\mu$.*

PROOF. Using the same method as in 3.1 it can be shown that the indefinite integral $\int f\,d\mu$ is a narrow primitive function to f on A.

4.12. PROPOSITION. *Suppose that $(P,\mathfrak{A},\mu)$ is a measure space with a regular measure $\mu,\sigma\subset\mathfrak{A}$, and $G(A)=\mu(A)$ for every $A\in\lambda$. Then $\mathfrak{A}\subset\mathfrak{T}_0$ and $\mu(A)=\tau_0(A)$ for all $A\in\mathfrak{A}$.*

PROOF. Because μ is regular, the Lebesgue integral with respect to μ is a Radon measure (see [3, (9.1), p. 114]). Hence by 4.11, 4.7, and [12, 6.14], $\mu(C)=\tau_0(C)$ for every $C\in\mathfrak{C}$. From the regularity of μ and τ_0 it follows that $\mu(A)=\tau_0(A)$ for every $A\in\mathfrak{S}$. Let $A\in\mathfrak{A}$ and $C\in\mathfrak{C}$. By [3, (10.34), p. 139], there are sets $B,D\in\mathfrak{S}$ such that

$$B\subset A\cap C\subset D \quad\text{and}\quad \tau_0(B-D) = \mu(B-D) = 0\,.$$

From the completeness of τ_0 it follows that $A\cap C\in\mathfrak{T}_0$ and thus by 4.7 also $A\in\mathfrak{T}_0$.

According to this proposition there is no loss of generality in assuming directly that $\sigma\subset\mathfrak{T}_0$ and that $G(A)=\tau_0(A)$ for all $A\in\lambda$.

4.13. LEMMA. *Suppose that $\sigma\subset\mathfrak{T}_0$ and that $G(A)=\tau_0(A)$ for all $A\in\lambda$. Let $A\in\sigma$ and let $B\in\mathfrak{T}_0$ be $\tau_0-\sigma$-finite. Then*

$$I_u(\chi_B,A) = -I_u(-\chi_B,A) = \tau_0(A\cap B)\,.$$

PROOF. Let $B\in\mathfrak{C}$. Then there is a non-increasing sequence $\{U_n\}\subset\mathfrak{U}$ such that

$$\{U_n^-\}\subset\mathfrak{C},\quad B\subset\textstyle\bigcap_{n=1}^\infty U_n,\quad \tau_0(\bigcap_{n=1}^\infty U_n-B) = 0\,.$$

Choose $\varphi_n\in\mathfrak{F}_c(P)$ such that $0\leqq\varphi_n\leqq 1$, $\varphi_n=1$ on B and $\varphi_n=0$ on $P-U_n$, and let

$$f_n = \textstyle\bigwedge_{i=1}^n\varphi_i,\quad n=1,2,\ldots\,.$$

Using 4.11, 2.19, (ii) and [12, 6.13] we obtain that $\chi_B\in\mathfrak{P}_0(A)$ and

$$I(\chi_B,A) = \lim I(f_n,A) = \lim\int_A f_n\,d\tau_0 = \int_A \chi_B\,d\tau_0 = \tau_0(A\cap B)\,.$$

Let $B \in \mathfrak{T}_0$ be an arbitrary $\tau_0 - \sigma$-finite set. Then there is a non-decreasing sequence $\{C_n\} \subset \mathfrak{C}$ such that

$$\bigcup_{n=1}^{\infty} C_n \subset B \quad \text{and} \quad \tau_0(B - \bigcup_{n=1}^{\infty} C_n) = 0 .$$

Using 2.19, (ii) and [12, 6.11] we obtain

$$I_u(\chi_B, A) = -I_u(-\chi_B, A) = \lim I(\chi_{C_n}, A) = \lim \tau_0(A \cap C_n) = \tau_0(A \cap B) .$$

If $B \in \mathfrak{T}_0$ is not $\tau_0 - \sigma$-finite the previous lemma is in general false (see 5.5).

4.14. Proposition. *If $\sigma \subset \mathfrak{T}_0$, then the following two conditions are equivalent:*

(i) *If $A^- - A$ is $\tau_0 - \sigma$-finite, then $I_u(f, A) = I_u(g, A)$ for every $f, g \in \mathfrak{F}(A^-)$ for which $f = g$ on A.*

(ii) *$G(A) = \tau_0(A)$ for every $A \in \lambda$.*

Proof. (i) $\Rightarrow$ (ii). If $A \in \lambda$, then A^- is compact, and hence $\tau_0(A^\cdot) = \tau_0[(P-A)^\cdot] < +\infty$. By 4.7 and [12, 6.8, (6.1)],

$$\tau_0(A) = \tau(A) = I(\chi_A, P) = I(\chi_A, A) + I(\chi_A, P - A) = I(\chi_{A^-}, A) = G(A).$$

(ii) $\Rightarrow$ (i). According to 4.13, $I(\chi_{A^- - A}, A) = 0$, and we can repeat verbatim the second part of the proof of proposition 4.4.

4.15. Corollary. *Suppose that $(P, \mathfrak{A}, \mu)$ is a measure space with a regular σ-finite measure $\mu, \sigma \subset \mathfrak{A}$ and $G(A) = \mu(A)$ for every $A \in \lambda$. Let $A \in \sigma$, $F \in \mathfrak{F}_s(\sigma_A)$ and let $Z \subset A^-$ be a countable set. If*

$$\begin{aligned} {}_{\#}F(x) &\geqq 0 && \text{for all } x \in Z \cup (\infty) , \\ {}_{*}F(x) &\geqq 0 && \text{for all } x \in A - Z , \\ {}_{*}F(x) &> -\infty && \text{for all } x \in A^- - (A \cup Z) , \end{aligned}$$

then $F \geqq 0$.

Proof. Since $F \in \mathfrak{M}({}_*F, A)$, the corollary follows from 4.12 and 4.14.

The reader should compare this corollary with [12, 5.9]. Examples 5.9 and 5.10 show that the regularity of the measure μ is essential here as well as in proposition 4.12.

4.16. Theorem. *Suppose that τ_0 is σ-finite, $\sigma \subset \mathfrak{T}_0$, and that $G(A) = \tau_0(A)$ for all $A \in \lambda$. Then $\mathfrak{L}_{\tau_0}(A) \subset \mathfrak{P}_0(A)$ for every $A \in \sigma$ and*

$$I(f,A) = \int_A f\, d\tau_0 \quad \textit{for every } f \in \mathfrak{L}_{\tau_0}(A)\,.$$

This theorem follows from 4.14, 4.4 and 4.8.

4.17. Proposition. *Suppose that τ_0 is σ-finite, $\sigma \subset \mathfrak{T}_0$ and that $G(A) = \tau_0(A)$ for all $A \in \lambda$. Let $A \in \sigma$ and $f \in \mathfrak{P}(A)$. If there are disjoint sets $A_1, \ldots, A_n$ from σ_A with union A and such that f does not change its sign on each A_i, $i = 1, 2, \ldots, n$, then $f \in \mathfrak{P}_0(A)$.*

Proof. We may assume that $f \geqq 0$ on $\bigcup_{i=1}^k A_i$ and $f \leqq 0$ on $\bigcup_{i=k+1}^n A_i$, where k is an integer, $0 \leqq k \leqq n$. By 2.2 and 4.4 we obtain

$$\begin{aligned} -I_u(-|f|,A) = I_u(|f|,A) &= \textstyle\sum_{i=1}^n I_u(|f|,A_i) \\ &= \textstyle\sum_{i=1}^k I(f,A_i) - \sum_{i=k+1}^n I(f,A_i) \neq \pm\infty. \end{aligned}$$

Loosely speaking, this proposition says that if we want to obtain a conditionally convergent integral, the system σ must not be too large.

5. Examples.

The majority of the examples given in this section serve as counterexamples. The rest are included for illustration purposes.

5.1. Example. Consider the situation from example [12, 8.5]. According to 2.15, $\mathfrak{P}^0(P) = \mathfrak{P}_0{}^0(P)$. Hence by [9, 90], $\mathfrak{P}^0(P)$ is just the family of all functions which are Lebesgue integrable over P and the integral I^0 is equal to the Lebesgue integral. This gives the affirmative answer to the problem from [9, 91].

5.2. Example. Let $P = E_1$, $\sigma = \mathfrak{R}_1$ (see [12, section 8]) and consider the natural convergence $\varkappa^0$. If $f(x) = x^{-1} \sin x^{-1}$ for $x \neq 0$ and $f(0) = 0$, then, using 3.2 and 2.17, it is easy to see that $f \in \mathfrak{P}([0,1))$, but $f \notin \mathfrak{P}^\wedge([0,1))$.

Using 2.15, it is not too hard to see that the integral $I^\wedge$ coincides with the one-dimensional Lebesgue integral on every set $A \in \sigma$ for which A^- is compact. However, for an arbitrary set $A \in \sigma$ this is not any more correct; for, for example, the function $x^{-1} \sin x$ belongs to $\mathfrak{P}^\wedge([1, +\infty))$.

From [12, 7.3] it follows that the integral I is closed with respect to the formation of improper integrals. Nevertheless, example 93 in [9] shows that I does not coincide with the classical Perron integral (see also example 8.4 in [12]).

5.3. EXAMPLE. Let $P=E_2$, $\sigma=\mathfrak{R}_2$, and consider the natural convergence $\varkappa^0$. For $(x,y)\in P$ set: $f(x,y)=x^{-1}\sin x^{-1}$ if $x\neq 0$ and $f(x,y)=0$ otherwise; $g(x,y)=x^{-1}\sin x$ if $x\neq 0$ and $g(x,y)=1$ otherwise. Then it follows from 3.3 and 2.17 that $f\notin\mathfrak{P}([0,1)\times[0,1))$, but $g\in\mathfrak{P}([0,+\infty)\times[0,1))$.

5.4. EXAMPLE. Let $P=[0,1)$ with the half-open-interval topology (see [5, chapter I, K, p. 59]), let σ be the system of all Borel subsets of P, and let G be the Lebesgue measure on σ (this has good meaning, since the Borel subsets of P coincide with the Borel subsets of $[0,1)$ in the ordinary topology). The space P is Hausdorff but not locally compact; for by [5, chapter I, K(d), p. 58], every compact subset of P is countable. We shall consider the natural convergence $\varkappa^0$ in P and show that the function $f=\chi_P$ is not integrable over P.

Let $M\in\mathfrak{M}(-f,P)$ and $\varepsilon>0$. If $A\in\sigma$ is countable, then $G(A)=0$ and hence by [12, (6.2)], $M(A)\geqq I_u(-f,A)=0$. Since ${}_{\#}M(\infty)\geqq 0$, there is a countable set $A\in\sigma$ such that

$$-\varepsilon \leqq M(P-A) \leqq M(P-A) + M(A) \leqq M(P)\,.$$

From the arbitrariness of ε it follows that $M(P)\geqq 0$ and thus $I_u(-f,P)=0$.

Suppose that there is $N\in\mathfrak{M}(f,P)$ such that $N(P)<\varepsilon$ for some $\varepsilon\in(0,1)$ and let

$$\delta = \{A\in\sigma : N(A)/G(A)<\varepsilon\}\,.$$

Then σ is a non-empty semihereditary system (see section 1); for $P\in\sigma$ and if A,B are disjoint sets from $\sigma-\delta$, then

$$M(A\cup B)/G(A\cup B) \geqq [M(A)+M(B)]/[G(A)+G(B)] \geqq \varepsilon\,,$$

and so $A\cup B\in\sigma-\delta$. The system δ is also stable (see section 1) in $[0,1]$ with the ordinary topology; for $\emptyset\notin\delta$ and if $A\in\delta$ and $x\in[0,1]$, then there is an ordinary spherical neighborhood U of x such that

$$\begin{aligned} N(A-U)/G(A-U) &\leqq [N(A)-N(A\cap U)]/[G(A)-G(A\cap U)] \\ &\leqq N(A)/[G(A)-G(A\cap U)] < \varepsilon\,, \end{aligned}$$

and so $A-U\in\delta$. (Notice that the system δ is not stable in $P^{\smile}$: consider ∞.) It follows from [10, 2.5] that δ^w (see [10, 1.5]) is uncountable. Hence by [5, chapter I, K(d), p. 58], the set of all accumulation points from the right of δ^w is uncountable. Since every accumulation point from the right of δ^w, belongs to δ^*, ${}_{*}N(x,P)\leqq\varepsilon<1$ for uncountably many $x\in P$. This contradiction shows that $I_u(f,P)=1$.

5.5. EXAMPLE. Let $P = E_d \times E_1$, where E_1 is the set of all real numbers with the usual topology and E_d is the same set with the discrete topology. Then P is a metrizable locally compact topological group and hence according to [2, (8.13), p. 76], it is paracompact. Let $f \in \mathfrak{F}_c(P)$ vanish outside some compact subset of P. Since the Riemann integrals $\int_{-\infty}^{+\infty} f(x,y)\, dy$ are finite for all $x \in E_d$ and equal zero for all but a finite number of $x \in E_d$, we can let

$$J(f) = \sum\left\{\int_{-\infty}^{+\infty} f(x,y)\, dy : x \in E_d\right\}$$

and denote by μ the regular measure in P induced by the function J (see [3, section 9, 10]); the space P and the measure μ are described in [2, (11.33), p. 127]). On the sets $(x) \times E_1, x \in E_d$, the measure μ obviously coincides with the one-dimensional Lebesgue measure μ_1.

Let σ be the pre-ring of all sets $A \subset P$ such that

$$A_x = \{y \in E_1 : (x,y) \in A\}$$

is an interval in E_1 for all $x \in E_d$ and $A_x = \emptyset$ or $A_x = E_1$ for all but a finite number of $x \in E_d$. For $z \in P^{\sim}$, let $\varkappa_z$ consist of all sequences $\{B_n\}_{n=1}^{\infty} \subset \sigma$ which satisfy axiom $\mathscr{K}_2$ and such that either $z \in \bigcap_{n=1}^{\infty} B_n^{\sim}$ or $B_n = \emptyset$ for all sufficiently large n. Using [10, 3.1, 3.2], it can be easily seen that the convergence $\varkappa = \{\varkappa_z : z \in P^{\sim}\}$ satisfies axioms $\mathscr{K}_1 - \mathscr{K}_6$. With $G = \mu$ on λ it is not too difficult to show that for $A \in \sigma$, $f \in \mathfrak{P}(A)$ if and only if the classical Perron integrals $\int_{A_x} f(x,y) dy$ exist for all $x \in E_d$ and the sum

$$\sum\left\{\int_{A_x} f(x,y)\, dy : x \in E_d\right\}$$

is convergent (see [5, chapter II, G, p. 77]); we have

$$I(f,A) = \sum\left\{\int_{A_x} f(x,y) dy : x \in E_d\right\}.$$

From 4.3, [9, 89] and [3, (10.31), p. 138], it follows that every integrable function is μ-measurable. Hence all the measures μ, τ and τ_0, (see 3.12 and 4.6) coincide and $\mathfrak{P}_0(A) = \mathfrak{L}_\mu(A)$ for every $A \in \sigma$ for which $A^- - A$ is $\mu - \sigma$-finite.

The closed set $B = E_d \times (0)$ is not σ-finite; for $\mu(C) = 0$ or $+\infty$ according to whether $C \subset B$ is countable or uncountable, respectively. Let $A = E_d \times (0, +\infty)$ and let $M \in \mathfrak{M}(-\chi_B, A)$. Then

$$M \in \mathfrak{M}(-\chi_B, (x) \times (0, +\infty)) \quad \text{for every } x \in E_d$$

and hence

$$_*M(z,A) \geqq 0 \quad \text{for all } z \in A^- .$$

Therefore $M \geqq 0$ which implies $I_u(-\chi_B, A) = 0$. However, since A^- is paracompact, it follows from 3.16 that $I_u(\chi_B, A) = +\infty$.

5.6. EXAMPLE. Consider the situation from [12, 8.4], with the natural convergence $\varkappa^0$. Let $B \subset [0,1]$ be the Lebesgue non-measurable set and let $f = \chi_B + \chi_{P-[0,1]}$. Then

$$-I_u(-f,P) = I_u(f,P) = +\infty ,$$

however,

$$f \wedge \chi_{[0,1]} \notin \mathfrak{P}_0(P);$$

for $f \wedge \chi_{[0,1]} = \chi_B$ and, according to [9, p. 90], every function from $\mathfrak{P}_0(P)$ is Lebesgue measurable.

5.7. EXAMPLE. In $P = A \times \{0,1\}$, where A is an uncountable set, we shall consider the discrete topology. Hence P is a locally compact and paracompact space. Let σ be the pre-ring consisting of all finite subsets of P and of all sets $(A-B) \times \{0,1\}$ where B is a finite subset of A. For $B \in \sigma$ put $G(B)$ equal to the number of elements of B, if B is finite, and put $G(B) = +\infty$ otherwise. For $z \in P$, $\varkappa_z$ consists of trivial nets $\{\emptyset\}$ and $\{(z)\}$; $\varkappa_\infty$ consists of the net $\{U, U \in \Gamma, \subset\}$ where Γ is the family of all sets $B \in \sigma$ for which $B \cup (\infty)$ is a neighborhood of ∞. Obviously, axioms $\mathscr{K}_1 - \mathscr{K}_6$ are satisfied here. For $(x,y) \in P$ let $f(x,y) = (-1)^y$, and for $B \in \sigma$ let

$$F(B) = \sum\{f(z) : z \in B\} ,$$

if B is finite, and $F(B) = 0$ otherwise. Then F is a narrow primitive function to f on P. Hence $f \in \mathfrak{P}(P)$ and $I(f,P) = 0$. However, according to 2.2 and 3.19,

$$-I_u(-f^+,P) = I_u(f^+,P) = +\infty .$$

5.8. EXAMPLE. Let $P = [0,1]^A - (\theta)$, where A is an uncountable set and θ is a zero function on A. With the relative topology from $[0,1]^A$ the space P is Hausdorff and locally compact but not σ-compact (see [5, p. 114]). Let σ be the pre-ring consisting of all sets $\times_{\alpha \in A} K_\alpha - (\theta)$, where $K_\alpha \subset [0,1]$ is an interval for all $\alpha \in A$ and $K_\alpha = [0,1]$ for all but a finite number of $\alpha \in A$. On σ we define the natural convergence $\varkappa^0$, and if

$$K = \times_{\alpha \in A} K_\alpha - (\theta)$$

belongs to σ we let

$$G(K) = \prod_{\alpha\in A}\mu_1(K_\alpha);$$

here μ_1 denotes the one-dimensional Lebesgue measure in $[0,1]$. Obviously, $\tau(P)=1$; however, since $P_G=P$ (see 3.14), the set $P\cap P_G$ is not σ-bounded (see 3.18).

5.9. EXAMPLE. For an ordinal α, $W(\alpha)$ denotes the set of all ordinals less than α with the order topology. Let $P=W(\Omega)$, where Ω is the first uncountable ordinal. For a Borel set $A\subset P$ we set $\mu(A)=1$ or 0 according to whether A does or does not contain an uncountable closed subset, respectively. By [4, section 53, problem 10, p. 231], the function μ is a non-regular measure on $\mathfrak{S}$. Letting $\sigma=\mathfrak{S}$, $G=\mu$, and using the natural convergence $\varkappa^0$, we obtain

$$\mathfrak{T} = \mathfrak{T}_0 = \exp P,$$

where $\exp P$ denotes the family of all subsets of P, and $\tau=\tau_0=0$.

5.10. EXAMPLE. Let $P=W(\Omega+1)$ and define μ on $\mathfrak{S}$ as in 5.9. Letting $\sigma=\mathfrak{S}$, $G=\mu$, and using the natural convergence $\varkappa^0$, we obtain

$$\mathfrak{T} = \mathfrak{T}_0 = \exp P,$$

and

$$\tau(A) = \tau_0(A) = \chi_A(\Omega) \quad \text{for all } A\subset P.$$

If $A=P-\{\Omega\}$, then, although $\chi_{P-A}(x)=\chi_\emptyset(x)=0$ for all $x\in A$,

$$I(\chi_{P-A},A) = G(A) = 1 \quad \text{while} \quad I(\chi_\emptyset,A) = 0;$$

also $\int_A \chi_{P-A}\,d\tau=0$.

If $F=-G$, then F is σ-additive on σ; then ${}_*F(x)=+\infty$ for $x\in P-\{\Omega\}$, and ${}_*F(\Omega)=-1$. Nevertheless, $F(P-\{\Omega\})=-1$.

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UNIVERSITY OF CALIFORNIA, BERKELEY, U.S.A.