ON THE STRUCTURE OF THE SPACES $\mathcal{L}_k^{p,\lambda}$

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Introduction.

Under special conditions on the subset $\Omega$ in $\mathbb{R}^n$, S. Campanato [3] proved that the spaces $\mathcal{L}_k^{p,\lambda}(\Omega)$ are isomorphic to the Lipschitz spaces $C^{h,\alpha}(\Omega)$ (see definitions in section 1), where $h+\epsilon=(\lambda-n)/p > 0$ and $0<\epsilon<1$, $h$ integer $<k$.

With another method, based on the theory of interpolation spaces, we intend to prove that $\mathcal{L}_k^{p,\lambda}(\Omega)$ is equal to the Besov space $B^\alpha(\Omega)$, where $0<(\lambda-n)/p=\alpha<k$, even when $(\lambda-n)/p$ is an integer and with other conditions on $\Omega$.

The plan of this article is as follows. In section 1 we give the definition of $\mathcal{L}_k^{p,\lambda}(\Omega)$, $C^{h,\alpha}(\Omega)$ and $B^\alpha(\Omega)$. Section 2 contains alternative definitions of $B^\alpha(\Omega)$, when $\Omega=\mathbb{R}^n$. In section 3 we prove

$$B^\alpha(\Omega) = \mathcal{L}_k^{p,\lambda}(\Omega), \quad 0<\alpha<(\lambda-n)/p<k,$$

if $\Omega=\mathbb{R}^n$ (theorem 3.1). Section 4 treats the corresponding result for an open, bounded subset $\Omega$ of $\mathbb{R}^n$, subject to certain restrictions (theorem 4.1).

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1. Definition of $\mathcal{L}_k^{p,\lambda}(\Omega)$, $C^{h,\alpha}(\Omega)$ and $B^\alpha(\Omega)$.

Let $\Omega$ be an open subset of $\mathbb{R}^n$ and $p \geq 1$. Write

$$I_{x_0,r} = \{x \in \mathbb{R}^n | |x-x_0| \leq r\}$$

and $\Omega_{x_0,r} = \Omega \cap I_{x_0,r}$.

**Definition 1.1.** For $k$ integer $\geq 0$ and $\lambda \geq 0$ we say that $f \in \mathcal{L}_k^{p,\lambda}(\Omega)$ if $f \in L^p_{loc}(\Omega)$ and for every $r>0$ and $x_0 \in \overline{\Omega}$ there exists a polynomial $q_k(x)$

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of degree \(<k\), depending on \(x_0\), \(r\) and \(f\), and a constant \(C\), depending on \(f\), such that

\[
\left( \int_{Q_{x_0,r}} |f(x) - q_k(x)|^p \, dx \right)^{1/p} \leq C r^{1/p}.
\]

The infimum over all constants \(C\) in (1.1) is a semi-norm on the space \(\mathcal{L}^{p,\lambda}(\Omega)\), and it will be denoted \(|f|_{\mathcal{L}^{p,\lambda}(\Omega)}\). We decide to identify functions whose difference is a polynomial of degree \(<k\). Then we can use \(|f|_{\mathcal{L}^{p,\lambda}}\) as a norm and \(\mathcal{L}^{p,\lambda}(\Omega)\) is a Banach space.

**Remark 1.1.** The spaces \(\mathcal{L}^{p,\lambda}(\Omega)\) introduced in S. Campanato [3] are not quite the same as our spaces \(\mathcal{L}^{p,\lambda}(\Omega)\). Campanato works with the norm

\[
|f|_{\mathcal{L}^{p,\lambda}(\Omega)} = \left( |f|^p_{L^p(\Omega)} + \sup_{0<r<\text{diam}\Omega} r^{-\lambda} \inf_{q_k} \int_{Q_{x_0,r}} |f(x) - q_k(x)|^p \, dx \right)^{1/p}.
\]

Note also that Campanato uses the parameter \(k'=k-1\) in place of \(k\), so that our \(\mathcal{L}^{p,\lambda}\) is the space \(\mathcal{L}^{k,p,\lambda}\) in the sense of Campanato.

**Definition 1.2.** Let \(h\) be an integer \(\geq 0\) and let \(C^h(\Omega)\) be the space of all \(h\) times continuously differentiable functions in \(\Omega\).

Then \(C^h(\Omega)\) is a Banach space with the graph-norm

\[
|f|_{C^h(\Omega)} = \sum_{|\alpha| \leq h} \sup_{x \in \Omega} |D^\alpha f(x)|.
\]

Here \(l=(l_1, l_2, \ldots, l_n)\) is an \(n\)-tuple, \(|l|=l_1+l_2+\ldots+l_n\), and

\[
D^l f(x) = D_1^{l_1} D_2^{l_2} \ldots D_n^{l_n} f(x), \quad \text{where } D_\nu = \partial/\partial x_\nu.
\]

**Definition 1.3.** For \(0<\varepsilon \leq 1\) we say that \(f \in C^{h,\varepsilon}(\Omega)\) if \(f \in C^h(\Omega)\) and the derivatives of order \(h\) are Lipschitz continuous in \(\overline{\Omega}\) with exponent \(\varepsilon\). Take as a norm in \(C^{h,\varepsilon}(\Omega)\)

\[
|f|_{C^{h,\varepsilon}(\Omega)} = |f|_{C^h(\Omega)} + \sup_{|\alpha|=h} \sup_{x, y \in \Omega} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^\varepsilon}.
\]

Campanato ([2] and [3]) has given the following characterizations of \(\mathcal{L}^{p,\lambda}(\Omega)\), \(1 \leq p < \infty\), for \(\Omega\) open, bounded, and of "type \(\mathcal{A}\)" (see Campanato [3, p. 138]).

\(\mathcal{L}^{p,\lambda}(\Omega) = L^p(\Omega)\) if \(\lambda = 0\);

\(\mathcal{L}^{p,\lambda}(\Omega) = L^{p,\lambda}(\Omega)\), Morrey space, if \(0 \leq \lambda < n\) (for definition of \(L^{p,\lambda}(\Omega)\) see Campanato [3, p. 157]);
\[ \mathcal{L}^{p,\lambda}_k(\Omega) = C^{h\varepsilon}(\Omega) \text{ if } n < \lambda \leq n + k \cdot p, \ h \text{ integer } \leq k - 1, \]
\[ (\lambda - n)/p = h + \varepsilon, \ 0 < \varepsilon < 1 \text{ and } \Omega \text{ convex}; \]
\[ \mathcal{L}^{p,\lambda}_k(\Omega) = \mathcal{S}^{p}_{h}(\Omega) = \mathcal{L}^{1/n + h}_h(\Omega) \text{ if } (\lambda - n)/p = h \text{ integer } \leq k - 1 \]
\[ \text{and } \Omega \text{ convex}; \]
\[ \mathcal{L}^{p,\lambda}_k(\Omega) = P_k(\Omega) = \{\text{polynomials of degree } \leq k - 1\} \text{ if } \lambda > n + kp. \]

**Remark 1.2.** From the definition of \( \mathcal{L}^{p,\lambda}_k(\Omega) \) it follows that \( \mathcal{L}^{p,\lambda}_j(\Omega) \subset \mathcal{L}^{p,\lambda}_k(\Omega), \ j \leq k. \)

We are now going to give another characterization of \( \mathcal{L}^{p,\lambda}_k(\Omega) \) for \( 0 < (\lambda - n)/p < k \) with aid of interpolation spaces (see J. Peetre [6]).

Let \( A_0 \) and \( A_1 \) be Banach spaces with norms \( |\cdot|_{A_0} \) and \( |\cdot|_{A_1} \), respectively. Put
\[ K_s(a) = \inf_{a=a_0+a_1} (|a_0|_{A_0} + m^s |a_1|_{A_1}), \]
\[ J_s(a) = \max (|a|_{A_0}, m^s |a|_{A_1}), \ m \neq 1. \]

The interpolation space \( (A_0, A_1)_{\theta, q}, 0 < \theta < 1, 1 \leq q \leq \infty, \) is then defined by each one of the equivalent norms
\[ (1.2) \quad \left( \sum_{r=-\infty}^{\infty} (m^{-r\theta} K_s(a))^q \right)^{1/q}, \]
\[ (1.3) \quad \inf \left( \sum_{r=-\infty}^{\infty} (m^{-r\theta} J_s(u_r))^q \right)^{1/q}, \]
where infimum is to be taken over all \( u_r \) such that \( a = \sum_{r=\infty}^\infty u_r \) in \( A_0 + A_1 \).

(If \( q = \infty \) we take as usual the supremum norm.)

We will work with the space \( C^h(\Omega) \), but not with the same norm as Campanato used. We identify functions whose difference is a polynomial of degree \( < h \) and take as a norm
\[ |f|_{C^h(\Omega)} = \sup_{x \in \Omega, |x| = h} |Df(x)|. \]

From now on the notation \( C^h(\Omega) \) will refer to this definition.

**Definition 1.4.** The Besov space \( B^\alpha(\Omega) \) is defined by
\[ B^\alpha(\Omega) = (C^0(\Omega), C^k(\Omega))_{\alpha/k, \infty}, \text{ where } 0 < \alpha < k. \]
(In the sequel we let \( k \) be the same integer as we used in the definition of \( \mathcal{L}^{p,\lambda}_k(\Omega) \).)

The norm in \( B^\alpha(\Omega) \) is any one of the above mentioned interpolation norms. Also in \( B^\alpha(\Omega) \) we identify functions whose difference is a polynomial of degree \( < k \).

We need the following interpolation theorem.
THEOREM 1.1. Let $A_0, A_1, B_0, B_1$ be Banach spaces and $T$ a linear operator such that

$$T: A_0 \to B_0, \quad T: A_1 \to B_1.$$ 

Then

$$T: (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q} \quad \text{for} \quad 0 < \theta < 1 \quad \text{and} \quad q \geq 1.$$ 

For the corresponding operator norms $M_0, M_1$ and $M$, respectively, we have

$$M \leq M_0^\theta M_1^{1-\theta}.$$ 

The sign $\to$ stands for linear continuous mapping.

We also need

THEOREM 1.2 (S. Spanne [9]). Let $0 < \theta < 1$. Then

$$(\mathcal{L}^{p,\lambda_0}(\Omega), \mathcal{L}^{p,\lambda_1}(\Omega))_{\theta,\infty} \subset \mathcal{L}^{p,\lambda}(\Omega) \quad \text{with} \quad \lambda = (1-\theta)\lambda_0 + \theta \lambda_1.$$ 

2. Alternative definitions of $B^\alpha(\mathbb{R}^n)$.

We shall now give two alternative definitions of the space $B^\alpha(\Omega)$ in the case $\Omega = \mathbb{R}^n$. The first one characterizes $B^\alpha(\Omega)$ by means of the modulus of continuity. Write

$$A_{ty}f(x) = f(x+ty) - f(x),$$

$$A_{ty}^l f(x) = A_{ty}(A_{ty}^{l-1} f(x)), \quad l = 2, 3, \ldots.$$ 

Let $k$ be the integer in the definition of $\mathcal{L}^{p,\lambda}$ and suppose $\alpha < k$. We consider the norm

$$(2.1) \quad \sup_{0 < t < \infty, |y| \leq 1} t^{-\alpha} |A_{ty}^k f|_{L^\infty(\mathbb{R}^n)}.$$ 

(Again we identify functions whose difference is a polynomial of degree less than $k$.) We shall prove that (2.1) is then an equivalent norm on $B^\alpha(\mathbb{R}^n)$.

Our second alternative definition of $B^\alpha(\mathbb{R}^n)$ is the following one. Let $\varphi$ be a function in the Schwartz class $S$ and write

$$\varphi_\nu(x) = 2^{-\nu n} \varphi(2^{-\nu} x), \quad \nu \text{ integer}.$$ 

Let $\hat{\varphi}$ be the Fourier transform of $\varphi$ and suppose that $\hat{\varphi}(\xi)$ is not zero on the annulus $2^{-1} < |\xi| < 2$ and vanishes outside it. We then consider the norm

$$(2.2) \quad \sup_{\nu} 2^{-\nu \alpha} |\varphi_\nu * f|_{L^\infty(\mathbb{R}^n)}.$$ 

We identify functions whose difference is a polynomial of degree less than $k$, and we exclude all polynomials of degree higher than or equal $k$. 


This norm will depend on the function \( \varphi \), but two different \( \varphi \) will give rise to equivalent norms. See J. Peetre [8] and J. L"ofstr"om [5].

**Lemma 2.1.** The norms (2.1) and (2.2) are equivalent on \( B^s(\mathbb{R}^n) \).

**Proof.** We let \( \varphi \in S \) and \( \hat{\varphi}(\xi) \equiv 0 \) in \( \frac{1}{2} < |\xi| < 2 \), \( \text{supp} \hat{\varphi} = \{ \xi \mid \frac{1}{4} \leq |\xi| \leq 2 \} \), \( \varphi_r(x) = 2^{-rn} \varphi(2^{-r}x) \). We can take \( \varphi \) such that

\[
\sum_{r = -\infty}^{\infty} \hat{\varphi}_r(\xi) = 1 \quad \text{for } \xi \neq 0.
\]

See H"ormander [4, p. 121]. Now take \( f \) such that

\[
|\varphi_r * f|_{L^\infty} \leq C 2^{rn}.
\]

It suffices to show that

\[
|\Delta_{t \xi_1}^k f|_{L^\infty} \leq Ct^s.
\]

Form the function

\[
\hat{\varphi}_r(\xi) = (e^{it\xi_1} - 1)^k \hat{\varphi}_r(\xi).
\]

We get at once

\[
|\varphi_r * f|_{L^\infty} \leq 2^k |\varphi_r * f|_{L^\infty} \leq 2^k C 2^{rn}.
\]

Further we have

\[
\hat{\varphi}_r(\xi) = (e^{it\xi_1} - 1)^k \hat{\varphi}(2^r \xi) = (t2^{-r})^k \left( \frac{e^{it\xi_1} - 1}{\xi_1} \right)^k (\xi_1 2^r)^k \hat{\omega}(2^r \xi) \hat{\varphi}(2^r \xi).
\]

if \( \hat{\omega} \in S \) and \( \hat{\omega}(\xi) = 1 \), when \( \hat{\varphi}(\xi) \neq 0 \).

Let \( M_n \) be the space of Fourier transforms of bounded measures on \( \mathbb{R}^n \), normed by

\[
|\hat{\mu}|_{M_n} = \int_{\mathbb{R}^n} |d\mu|.
\]

Then it is easy to see that

\[
\left| \left( \frac{e^{it\xi_1} - 1}{\xi_1} \right)^k \right|_{M_n} \leq C \quad \text{and} \quad |(t\xi)^k \hat{\omega}(t\xi)|_{M_n} \leq C
\]

for \( 0 < t < \infty \) (see L. H"ormander [4]). Thus we get the estimate

\[
|\varphi_r * f|_{L^\infty} \leq C (t2^{-r})^k |\varphi_r * f|_{L^\infty} \leq (t2^{-r})^k C 2^{rn}.
\]

From (2.3) and (2.4) we get

\[
|\varphi_r * f|_{L^\infty} \leq C \min(1,(t2^{-r})^k)^{2rn}.
\]

However \( \Delta_{t \xi_1}^k f(x) = \sum_{r = -\infty}^{\infty} \varphi_r * f(x) \), so we have

\[
|\Delta_{t \xi_1}^k f|_{L^\infty} \leq C \sum_{r = -\infty}^{\infty} \min(1,(t2^{-r})^k)^{2rn} \leq C \int_0^\infty \min(1,(t2^{-r})^k) \frac{dx}{x} \leq Ct^s.
\]
From the above calculation we obtain the desired norm inequality.
For the other part of the proof we take \( f \) such that \(|\Delta_{\ell_1}^k f|_{L^\infty} \leq Ct^a\). Take also \( \Phi \in S(\mathbb{R}) \) such that \( \hat{\Phi}(\xi) = 0 \) exactly for
\[
\frac{1}{2 + c(n)} < |\xi| < 3, \quad c(n) > 0
\]
and set \( \Phi_\nu(x) = \Phi(x2^{-\nu}) \cdot 2^{-\nu} \). It is easy to prove that
\[
\left| \frac{\hat{\Phi}_\nu(\xi)}{(e^{it\xi} - 1)k} \right|_{L^1} \leq C \quad \text{for } t = 2^\nu.
\]
Now let \( \hat{\psi} \in S(\mathbb{R}^n) \) be such that \( \hat{\psi} = \hat{\Phi}(\xi_1) \cdot \hat{g}(\xi_2) \), where \( \hat{\Phi} \) is as above and \( \hat{g}(\xi) = 0 \) for exactly \(|\xi| < 3, \xi = (\xi_2, \xi_3, \ldots, \xi_n)\). Then
\[
\left| \frac{\hat{\psi}(\xi)}{(e^{it\xi} - 1)k} \right|_{L^1} \leq C \quad \text{for } t = 2^\nu
\]
and we conclude
\[
|\Delta_{\ell_1}^k f|_{L^\infty} = \left| \left( \frac{\hat{\psi}(\xi)}{(e^{it\xi} - 1)k} \right) \ast \Delta_{\ell_1}^k f \right|_{L^\infty} \leq C t^a \leq C 2^{ra}.
\]
Here \( \psi^p \) denotes the inverse Fourier transform of \( g \). Repeat the construction for each one of the coordinate axes and add the functions to get \( \hat{\psi} = \sum_{k=1}^n \hat{\psi}_k \). Now take \( \varphi \) as in (2.2). The function \( \hat{\psi}(\xi) \) can be chosen such that \( \hat{\psi}(\xi) = 1 \) for \( \frac{1}{2} \leq |\xi| \leq 2 \). Then \( \hat{\psi}(\xi) \hat{\varphi}(\xi) = \hat{\varphi}(\xi) \) (because \( \text{supp} \hat{\varphi} = \{ |\xi| : \frac{1}{2} \leq |\xi| \leq 2 \} \)). So we have
\[
|\varphi \ast f|_{L^\infty} = |\varphi \ast \varphi \ast f|_{L^\infty} \leq |\varphi|_{L^1} |\varphi \ast f|_{L^\infty} \leq C 2^{ra}.
\]
We also get the desired norm inequality.

**Theorem 2.1.** The norms (2.1) and (2.2) are equivalent norms on \( B^a(\mathbb{R}^n) \).

**Proof.** In view of lemma 2.1 it suffices to show that (2.2) is equivalent to the norm on \( B^a(\mathbb{R}^n) \). We take \( \varphi \) as in (2.2) and \( f \) such that
\[
|\varphi \ast f(x)| \leq C 2^{ra}.
\]
As before we can choose \( \varphi \) such that \( \sum_{k=1}^\infty \hat{\varphi}(\xi) = 1, \xi = 0 \). Let \( f_\nu(x) = \varphi \ast f(x) \). Then
\[ f(x) = \sum_{r=-\infty}^{\infty} f_r(x) \quad \text{(modulo polynomials of degree < } k), \]

where \( f_r(x) \) and its derivatives are continuous functions. For any integer \( s \geq 0 \),

\[
|f_s|_{C^s(\mathbb{R}^n)} = \sup_{|l| = s} |D^l f_s|_{C^0(\mathbb{R}^n)} = \sup_{|l| = s} |(D^l \varphi_s) \ast f|_{C^0(\mathbb{R}^n)} \\
\leq 2^{-sv} \sup_{|l| = s} |(D^l \varphi_s) \ast f|_{C^0(\mathbb{R}^n)} \leq C 2^{s(a-s)},
\]

because \( D^l \varphi \) is a function with essentially the same properties as \( \varphi \).

Let \( k \) be the usual integer \( > \alpha \). We have shown that

\[
|f_s|_{C^\infty(\mathbb{R}^n)} \leq C 2^{r\alpha}, \quad |f_s|_{C^k(\mathbb{R}^n)} \leq C 2^{r(a-k)}.
\]

We get (with \( m = 2^k \) in (1.3)) that \( (2^{r\alpha}-s/k) J_s(f_s) \leq \text{const.} \), that is, \( f \in (C^0(\mathbb{R}^n), C^k(\mathbb{R}^n))_{a/k, \infty} \).

Although the other part of the proof follows from section 3, we give a direct proof here. We take \( f \in B^\alpha(\mathbb{R}^n) = (C^0(\mathbb{R}^n), C^k(\mathbb{R}^n))_{a/k, \infty} \), \( k \) integer \( > \alpha \). Equivalently this means that

\[
m^{-\alpha} K_\varphi(f) < \text{const.},
\]

where

\[
K_\varphi(f) = \inf_{f = f_0 + f_1} (|f_0|_{C^0(\mathbb{R}^n)} + m^r |f_1|_{C^k(\mathbb{R}^n)}).
\]

Now take \( \varphi \) as in (2.2). We let \( f = f_0 + f_1 \), where \( f_0 \in C^0(\mathbb{R}^n) \) and \( f_1 \in C^k(\mathbb{R}^n) \). We want to estimate \( |\varphi \ast f(x)| \). Putting \( D^k \Phi(y) = \varphi(y) \), we obtain

\[
2^{-\alpha} \left| \int \varphi(y) f(x-y) \, dy \right| = 2^{-\alpha} \left| \int \varphi(y) (f_0(x-y) + f_1(x-y)) \, dy \right| \\
\leq 2^{-\alpha} \left( \int |\varphi(y)| \, dy \, |f_0|_{C^0(\mathbb{R}^n)} + \int |\varphi(y)| \, f_1(x-y) \, dy \right) \\
\leq 2^{-\alpha} \left( C |f_0|_{C^0(\mathbb{R}^n)} + \int \Phi(y) \, D^k f_1(x-y) \, dy \right) \\
\leq 2^{-\alpha} C (|f_0|_{C^0(\mathbb{R}^n)} + 2^{rk} |f_1|_{C^k(\mathbb{R}^n)})
\]

(we omit the partition of \( \varphi \), see lemma 2.1.) and thus the estimation

\[
2^{-\alpha} |\varphi \ast f(x)| \leq C 2^{-\alpha} \inf_{f = f_0 + f_1} (|f_0|_{C^0} + 2^{rk}|f_1|_{C^k}) \\
\leq C (2^{rk})^{-s/k} K_\varphi(f) \leq \text{const.}
\]

The corresponding norm inequality follows from the calculations.
3. The case $\Omega = \mathbb{R}^n$.

**Theorem 3.1.** $B^\alpha(\mathbb{R}^n) = \mathcal{L}_{k}^{p,\lambda}(\mathbb{R}^n)$ for $0 < \alpha = (\lambda - n)/p < k$.

**Proof.** First we prove $B^\alpha(\mathbb{R}^n) \subset \mathcal{L}_{k}^{p,\lambda}(\mathbb{R}^n)$. By theorem 1.2 we get

$$(\mathcal{L}_{0}^{p,n}(\mathbb{R}^n), \mathcal{L}_{k}^{p,n+kp}(\mathbb{R}^n))_{\alpha/k, \infty} \subset \mathcal{L}_{k}^{p,\lambda}(\mathbb{R}^n)$$

with

$$\lambda = (1 - \alpha k^{-1})n + \alpha k^{-1}(n + kp) = n + \alpha p,$$

that is, $\alpha = (\lambda - n)/p$. (Here we use $\mathcal{L}_{0}^{p,n}(\mathbb{R}^n) \subset \mathcal{L}_{k}^{p,n}(\mathbb{R}^n)$, see remark 1.2.) Now it suffices to prove

$$B^\alpha(\mathbb{R}^n) \subset (\mathcal{L}_{0}^{p,n}(\mathbb{R}^n), \mathcal{L}_{k}^{p,n+kp}(\mathbb{R}^n))_{\alpha/k, \infty}$$

with $\alpha = (\lambda - n)/p$ and $0 < \alpha/k < 1$. Let $I$ be the identity mapping. We will show that

(3.1) $I: C^0(\mathbb{R}^n) \to \mathcal{L}_{0}^{p,n}(\mathbb{R}^n)$

(3.2) $I: C^k(\mathbb{R}^n) \to \mathcal{L}_{k}^{p,n+kp}(\mathbb{R}^n)$

To prove (3.1) let us take $f \in C^0(\mathbb{R}^n)$. Then

$$\left( \int_{|x-x_0| \leq r} |f(x)|^p \, dx \right)^{1/p} \leq |f|_{C^0(\mathbb{R}^n)} \left( \int_{|x-x_0| \leq r} 1 \, dx \right)^{1/p} = |f|_{C(\mathbb{R}^n)}^p C,$$

which means $f \in \mathcal{L}_{0}^{p,n}(\mathbb{R}^n)$. Next we prove (3.2). Let us take $f \in C^k(\mathbb{R}^n)$. Then from Taylor's formula we get

$$f(x) - (\text{polynomial of degree } < k) = (k!)^{-1} \Sigma_{|l| = k} (D^l f)(x_0 + \theta(x-x_0))(x-x_0)^l,$$

where

$$(x-x_0)^l = (x_1-x_{01})^{l_1} \ldots (x_n-x_{0n})^{l_n}.$$

It follows immediately that

$$|f(x) - q_k(x)| \leq C \sup_{|l| = k} |(x-x_0)^l(D^l f)(x_0 + \theta(x-x_0))|.$$

From this we get

$$\left( \int_{|x-x_0| \leq r} |f(x) - q_k(x)|^p \, dx \right)^{1/p} \leq C r^k \sup_{|l| = k} |D^l f| \left( \int_{|x-x_0| \leq r} 1 \, dx \right)^{1/p} \leq C r^{(n+p k)/p} |f|_{C^k(\mathbb{R}^n)}.$$

The desired norm inequalities also follow from the above.

By means of (3.1) and (3.2) we conclude, using also theorem 1.1., that

$$I: B^\alpha(\mathbb{R}^n) = (C^0(\mathbb{R}^n), C^k(\mathbb{R}^n))_{\alpha/k, \infty} \to (\mathcal{L}_{0}^{p,n}(\mathbb{R}^n), \mathcal{L}_{k}^{p,n+kp}(\mathbb{R}^n))_{\alpha/k, \infty}.$$

Now we show that $\mathcal{L}_{p,\lambda}^{r}(\mathbb{R}^n) \subset B^{0}(\mathbb{R}^n)$, $\alpha = (\lambda - n)/p$, by proving that $f \in \mathcal{L}_{p,\lambda}^{r}(\mathbb{R}^n)$ implies (see (2.2))
\[ |f_{*}f|_{L^{\infty}} \leq C 2^{r_{\alpha}}. \]
We take a function $X \in C_{0}^{\infty}(\mathbb{R}^n)$ with support in a neighbourhood of the origin and such that
\[ \int q_{k}(x) X(x) \, dx = 0 \]
for any polynomial $q_{k}$ of degree $< k$ and
\[ \hat{X}(\xi) \neq 0 \quad \text{for } \frac{1}{2} < |\xi| < 2. \]
Let $X_{*}(x) = 2^{-m} X(2^{-m} x)$. Such a function $X$ exists, see lemma 3.1 below. Using Hölder’s inequality we get
\[ |X_{*}f(x)|_{L^{\infty}} = \left| \int X_{*}(x-y) f(y) \, dy \right|_{L^{\infty}} \]
\[ = \left| \int_{|x-y| \leq C 2^{\rho}} X_{*}(x-y)(f(y) - q_{k}(y)) \, dy \right|_{L^{\infty}} \]
\[ \leq |X_{*}|_{L^{p'}} |f - q_{k}|_{L^{p}(I_{x, C 2^{\rho}})} \leq C |X_{*}|_{L^{p'}} (2^{r})^{1/p}. \]
But
\[ \left( \int |X_{*}(x)|^{p'} \, dx \right)^{1/p'} = \left( \int |X(x2^{-m})|^{p'} \, dx \right)^{1/p'} 2^{-m} \]
\[ = \left( \int |X(x)|^{p'} \, dx \right)^{1/p'} 2^{m/p} 2^{-m} = 2^{-m/p} |X|_{L^{p'}}. \]
We get
\[ |X_{*}f|_{L^{\infty}} \leq C |X|_{L^{p'}} (2^{r})^{(\lambda - n)/p}. \]
Now we take $\varphi \in S$ such that $\hat{\varphi}(\xi) / 0$ for $\frac{1}{2} < |\xi| < 2$
\[ \text{and } \hat{X}(\xi) \neq 0 \text{ for } \frac{1}{2} < |\xi| < 2 \]
and $X$ as above. Then we get
\[ |\varphi_{*}X_{*}f|_{L^{\infty}} \leq |\varphi|_{L^{1}} |X_{*}f|_{L^{\infty}} \leq |\varphi|_{L^{1}} C 2^{r_{\alpha}} \leq \text{const.} 2^{r_{\alpha}}, \]
where $\alpha = (\lambda - n)/p$. But $\varphi = \psi \ast X \in S$ is a function such as $\varphi$ in (2.2). So we have proved that $f$ in the norm (2.2) is bounded.

**Lemma 3.1.** There exists a function $X \in C_{0}^{\infty}(\mathbb{R}^n)$ with support in a neighbourhood of the origin, $\int q_{k}(x) X(x) \, dx = 0$ for all polynomials $q_{k}$ of degree $< k$ and $\hat{X}(\xi) / 0$ for $\frac{1}{2} < |\xi| < 2$. 


Proof. We take a function $\theta(x) \in C_0^\infty(\mathbb{R})$ with support in $|x| \leq C$. Let $g(x) = D^k \theta(x)$. Then $g(x) \in C_0^\infty(\mathbb{R})$ with support in $|x| \leq C$. Of course we have

$$\int g(x) \, dx = 0, \quad \int x g(x) \, dx = 0, \ldots, \quad \int x^{k-1} g(x) \, dx = 0.$$

Further let $\psi(x) = g(x_1)g(x_2)\ldots g(x_n)$. Obviously $\psi(x) \in C_0^\infty(\mathbb{R}^n)$ and $\int q_k(x) \psi(x) \, dx = 0$ for all polynomials $q_k$ of degree $< k$. We have $\hat{\psi}(\xi) \in S$ and $\hat{\psi}(\xi)$ must be $\pm 0$ in some point $\xi_0$. Let us suppose that $|\xi_0| = 1$ (otherwise we can make a homothetic transformation that does not change the properties above of $\psi$). We may suppose that $\hat{\psi}(\xi) \geq 0$. Then $\hat{\psi}(\xi) > 0$ in a neighbourhood of $\xi_0$.

Consider $\{\hat{\psi}_B(\xi)\}$, where $\hat{\psi}_B(\xi) = \hat{\psi}(B^{-1}\xi)$ and $B$ an orthogonal matrix. $D^i \hat{\psi}_B(0) = 0$ because $D^i \hat{\psi}(0) = 0$. Further $\hat{\psi}_B(\xi) = \pm 0$ in $\xi_B = B\xi_0$. We have a set of functions $\{\hat{\psi}_B(\xi)\}$ such that $\hat{\psi}_B(\xi) > 0$ in a neighbourhood of a point on the unit sphere. Now we cover the unit sphere by a finite subset of such neighbourhoods corresponding to $\hat{\psi}_{B_1}, \hat{\psi}_{B_2}, \ldots, \hat{\psi}_{B_N}$. We get

$$\hat{\psi}_B(\xi) = \sum_{r=1}^N \hat{\psi}_{B_r}(\xi) \neq 0 \quad \text{on } |\xi| = 1$$

and in a neighbourhood of this set. We may suppose that this neighbourhood is $\frac{1}{2} < |\xi| < 2$ (otherwise we can repeat the covering argument above, now with $\hat{\psi}_B(\xi) = \hat{\psi}(t\xi)$, $t$ constant). The function $X(x)$ has the desired properties.

4. The case bounded $\Omega \subset \mathbb{R}^n$.

We shall say that the open, bounded set $\Omega \subset \mathbb{R}^n$ satisfies assumption (H), if it has

1) the lifting property,
2) the cone property,

which properties we now define.

Definition 3.1. The set $\Omega \subset \mathbb{R}^n$ has the lifting property if there is a linear continuous mapping $L$ such that

$$L: C^j(\Omega) \rightarrow C^j(\mathbb{R}^n) \quad \text{for } j = 0, k$$

and $R \circ L$ is the identity mapping on $C^j(\Omega)$ if $R$ is the restriction to $\overline{\Omega}$ of a function defined in $\mathbb{R}^n$.

Definition 3.2. The set $\Omega \subset \mathbb{R}^n$ has the cone property if to every point $x$ in $\overline{\Omega}$ there exists a neighbourhood $O_x$ of $x$ and a corresponding bounded
cone \( C_x \) with vertex at the origin and the property \( y + C_x \subseteq \Omega \) for \( y \in \Omega \cap O_x \).

**Remark 4.1.** \( \Omega \) has the lifting property if the boundary of \( \Omega \) is of class \( C^k \). See S. Agmon [1, p. 128] and J. Peetre [8].

If \( \Omega \) is of class \( C^k \) it has the cone property. See Agmon [1, p. 129]. A convex set has the cone property.

**Theorem 4.1.** If \( \Omega \) satisfies the above assumption (H), then \( \mathcal{L}_k^{p,\lambda}(\Omega) = B^\alpha(\Omega) \) for \( 0 < \alpha = (\lambda - n)/p < k \).

**Proof.** We carry out the proof by showing that

\[
\mathcal{L}_k^{p,\lambda}(\Omega) \longrightarrow (C^0(\Omega), C^k(\Omega))_{a/k,\infty} \overset{L}{\longrightarrow} (C^0(\mathbb{R}^n), C^k(\mathbb{R}^n))_{a/k,\infty} \longrightarrow \mathcal{L}_k^{p,\lambda}(\mathbb{R}^n) \overset{R}{\longrightarrow} \mathcal{L}_k^{p,\lambda}(\Omega).
\]

Here steps two and four are immediate. Step three follows from section 1. Therefore only step one remains to be proved.

Choose a finite, open covering \( \{O_i\}_{i=1}^\infty \) of \( \overline{\Omega} \), such that to each \( O_i \) we can find a bounded cone \( C_i \) and \( x + C_i \subseteq \Omega \) for \( x \in \Omega \cap O_i \). This is possible, because \( \Omega \) is bounded and has the cone property (use the Heine–Borel theorem). Now take \( f \in \mathcal{L}_k^{p,\lambda}(\Omega) \). We shall consider \( \Omega \cap O_i \) and prove that \( f \in B^\alpha(\Omega \cap O_i) \). Let \( C_i \) be the cone corresponding to \( O_i \).

Choose a function \( X \in C_0^\infty(\mathbb{R}^n) \) with

a) the support in \( -C_i = \{y; -y \in C_i\} \) and
b) \( \int X(x) \, dx = 1 \) and
c) \( \int g(x)X(x) \, dx = 0 \) for all polynomials \( g \) of degree less than \( k \) and with no constant term.

**Lemma 4.1.** There exists a function \( X \in C_0^\infty(\mathbb{R}^n) \) with the properties a), b) and c) above.

**Proof.** Let us take \( \theta(x) \in C_0^\infty(\mathbb{R}) \) with the support in \([a, b]\) such that

\[
\int \frac{\theta(x)}{x^k} \, dx = \frac{1}{(k-1)!}.
\]

Then we have

\[
\int \frac{D^{k-1}\theta(x)}{x} \, dx = \ldots = (k-1)! \int \frac{\theta(x)}{x^k} \, dx = 1
\]

and
\[
\int x^{D^{k-1} \theta(x)} dx = 0, \quad \ldots, \quad \int x^{k-1} \frac{D^{k-1} \theta(x)}{x} dx = 0.
\]

Now set \( \Phi(x) = x^{-1} D^{k-1} \theta(x) \).

We want the support of \( X \) to be in \(-C_\varepsilon\). Choose a "cube" in \(-C_\varepsilon\) and construct \( X \) as a product of \( n \) functions \( \Phi(x) \) with their supports in the desired intervals. This completes the proof of the lemma.

Now let \( \psi_\varepsilon(x) = X(2^{\nu+1}x) 2^{(\nu+1)n} - X(2^\nu x) 2^n \). For \( \nu \geq 0 \), \( \psi_\varepsilon(x) \) has support in \(-C_\varepsilon\). Then we get

\[
\sum_{\nu=0}^{\infty} \psi_\varepsilon(x) + X(x) = \delta_0,
\]

for if \( g \in C_0^\infty(\mathbb{R}^n) \) we have, if \( N > N_0 \),

\[
\left| \int \left( \sum_{\nu=0}^{N} \psi_\varepsilon(x) + X(x) \right) g(x) dx - g(0) \right| = \left| \int X(2^{N+1}x) 2^{(N+1)n} g(x) dx - g(0) \right|
\]

\[
= \left| \int X(y) \left( g \left( \frac{y}{2^{(N+1)n}} \right) - g(0) \right) dy \right|
\]

\[
\leq \int |X(y)| \left| g \left( \frac{y}{2^{(N+1)n}} \right) - g(0) \right| dy < \varepsilon.
\]

Thus

\[
f(x) = \sum_{\nu=0}^{\infty} \psi_\varepsilon f(x) + X f(x) \quad \text{for} \quad x \in \overline{\Omega} \cap \Omega_\varepsilon.
\]

Note that the terms on the right side are well defined functions and they are continuously differentiable up to the order we want.

Let \( f_\varepsilon(x) = \psi_\varepsilon f(x) \) for \( \nu \geq 0 \) and \( f_{-1}(x) = X f(x) \). We have \( f(x) = \sum_{\nu} f_\varepsilon(x) \) for \( x \in \Omega \cap \Omega_\varepsilon \). We shall prove that

\[
2^\nu J(2^{-\nu} f_\varepsilon) \leq C |f|_{\mathcal{L}^p(\Omega)}, \quad \text{for all} \quad \nu.
\]

For \( \nu \geq 0 \) we have, taking supremum over all \( x \in \overline{\Omega} \cap \Omega_\varepsilon \),

\[
|f_\varepsilon|_{\mathcal{O}(\Omega \cap \Omega_\varepsilon)} = \sup \left| \int \psi_\varepsilon(x-y) f(y) dy \right|
\]

\[
= \sup \left| \int \psi_\varepsilon(x-y) (f(y) - q_k(y)) dy \right|
\]

\[
= \sup \left| 2^\nu \int \psi_0((x-y) 2^\nu) (f(y) - q_k(y)) dy \right|
\]

\[
\leq \sup 2^\nu \left( \int |\psi_0((x-y) 2^\nu)|^p dy \right)^{1/p} \left( \int \left( \frac{1}{\text{supp } \psi_0} |f(y) - q_k(y)|^p dy \right)^{1/p}
\]
\[ \leq \sup 2^n 2^{-n/p'} C \left( \int_{\Omega \cap I_{x,2^{-n}}} |f(y) - q_k(y)|^p \, dy \right)^{1/p} \]
\[ \leq (2^r)^{n/p} C |f|_{L_k^{p,\lambda}(\Omega)} C 2^{-n/p} = C(2^{-r})^{(\lambda-n)/p} |f|_{L_k^{p,\lambda}(\Omega)} . \]

Further we get
\[ |f|_{C_k(\Omega \cap O_i)} = \sup_{|\alpha| = k} |D^\alpha f|_{C_k(\Omega \cap O_i)} = \sup |D^\alpha \varphi_0 * f(x)| \]
\[ = \sup \left| \int D^\alpha \varphi_0 ((x-y) 2^r) 2^n f(y) \, dy \right| \]
\[ = \sup \left| 2^k \int \varphi_0 ((x-y) 2^r) 2^n f(y) \, dy \right| , \]

where the last three suprema are taken for \( x \in \overline{\Omega} \cap O_i \), \( |x| = k \). But \( \varphi_0 = D^\alpha \varphi_0 \) is a function with essentially the same properties as \( \varphi_0 \). Thus we can use the same estimate as above. We get
\[ |f|_{C_k(\Omega \cap O_i)} \leq 2^k C(2^{-r})^{(\lambda-n)/p} |f|_{L_k^{p,\lambda}(\Omega)} . \]

With similar methods we can treat \( f_{-1}(x) = X \ast f(x) \) and get analogous estimates.

Then we have for all \( r \)
\[ (2^k)^{\alpha/k} \max (|f|_{C_k(\Omega \cap O_i)}, 2^{-k} |f|_{C_k(\Omega \cap O_i)}) \leq C|f|_{L_k^{p,\lambda}(\Omega)} , \]
where \( \alpha = (\lambda-n)/p \), \( 0 < \alpha < k \).

We have proved that \( f \in B^\alpha(\Omega \cap O_i) \) for \( \alpha = (\lambda-n)/p \) and for an arbitrary set \( O_i \) in the construction above.

**Lemma 4.2.** If \( f(x) \in B^\alpha(\Omega \cap O) \) and \( \eta(x) \in C^\infty(\mathbb{R}^n) \) and supp\( \eta \subset O \), we have \( \eta(x)f(x) \in B^\alpha(\Omega) \).

**Proof.** It suffices to notice that the mapping
\[ F: f(x) \rightarrow \eta(x)f(x) \]
is such that
\[ F: C^0(\Omega \cap O) \rightarrow C^0(\Omega), \quad F: C^k(\Omega \cap O) \rightarrow C^k(\Omega) . \]
The statement then follows by the interpolation theorem.

Now we can conclude the proof of theorem 4.1. In fact we choose to the finite, open covering \( \{O_i\}_{i=1}^r \) of \( \overline{\Omega} \) a partition of unity, that is, functions \( \{\eta_i\}_{i=1}^r \) such that
\( \eta_1 \) has support in \( O_1 \) and \( \sum_{i=1}^r \eta_i(x) = 1, \) when \( x \in \bar{\Omega}. \)

We have shown that \( \eta_1 f \in B^\alpha(\Omega) \) (lemma 4.2). Thus we have also \( \sum_{i=1}^r \eta_i f \in B^\alpha(\Omega). \) But \( \sum_{i=1}^r \eta_i(x)f(x) = f(x) \) when \( x \in \bar{\Omega}. \) Thereby theorem 4.1 is proved.

REFERENCES


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