ON FINITELY GENERATED FLAT MODULES

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0. Introduction.

In this note we shall consider rings \( A \) with the property that any flat and finitely generated \( A \)-module is projective, and we will prove that this class of rings is rather big. If we require that any flat left \( A \)-module is projective, we get the class of left perfect rings (cf. [1]), which is a small class of rings. For instance, a (commutative) integral domain \( D \) is left perfect if and only if \( D \) is a field, but any finitely generated flat module over an integral domain is projective (cf. [5]).

1. General remarks.

In this section \( A \) denotes a ring with an identity, and all modules considered are unitary left modules.

**Definition.** We say that a ring \( A \) has property \( P \) (we write \( A \in P \)), if every finitely generated flat \( A \)-module is projective.

A very useful tool in the study of flat modules is the following lemma.

**Lemma 1.1.** Let

\[
0 \to K \to F \to M \to 0
\]

be an exact sequence of \( A \)-modules, where \( F \) is \( A \)-free, then the following statements are equivalent:

i) \( M \) is \( A \)-flat.

ii) Given any \( k \in K \), there exists a homomorphism \( u_k : F \to K \), such that \( u_k(k) = k \).

iii) Given any \( (k_i)_{1 \leq i \leq n} \), there exists a homomorphism \( u_k : F \to K \), such that \( u_k(k_i) = k_i \) for every \( i \), \( 1 \leq i \leq n \).

**Proof.** See S. U. Chase [3, prop. 2.2].

As applications of this lemma we get the following two propositions.

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Proposition 1.2. Let \( A \) and \( B \) be any two rings, and let \( \varphi : A \to B \) \((\varphi(1_A) = 1_B)\) be a ring-homomorphism. If \( B \) viewed as a left \( A \)-module is flat and finitely generated, then \( A \in P \) implies \( B \in P \).

Proof. If \( N \) is any flat and finitely generated \( B \)-module, we have to prove that \( N \) is \( B \)-projective. If we consider \( N \) as an \( A \)-module, \( N \) is finitely generated and flat (cf. N. Bourbaki [2, Chap. 1, § 2, no. 7, prop. 8, cor. 3]), and hence \( N \) is \( A \)-projective.

We have an exact sequence
\[
0 \to L \to B^r \to N \to 0
\]
(1)

of \( B \)-modules, which is also an exact sequence of \( A \)-modules, and therefore it is split exact over the ring \( A \). Since \( B^r \) is a finitely generated \( A \)-module, \( L \) is a finitely generated \( A \)-module too, in particular finitely generated as a \( B \)-module. From this we get that (1) is split exact over \( B \) (lemma 1.1). Hence \( N \) is \( B \)-projective.

Corollary 1.3. If \( A \in P \) and \( G \) is a finite group, then the group ring \( A[G] \in P \).

Proposition 1.4. Let \( A \) and \( B \) be rings and \( \varphi : A \to B \) a ring-homomorphism. If \( B \in P \) and \( B \) is a faithfully flat right \( A \)-module, then \( A \in P \).

Proof. Let \( M \) be any flat and finitely generated \( A \)-module, and let
\[
0 \to K \to F \to M \to 0
\]
(2)

be an exact sequence, where \( F \) is free and finitely generated. From (2) we derive the exact sequence
\[
0 = \text{Tor}_1^A(B, M) \to B \otimes_A K \to B \otimes_A F \to B \otimes_A M \to 0
\]
(3)

of \( B \)-modules. \( B \otimes_A M \) is \( B \)-flat [2, Chap. 1, § 2, no. 7, prop. 8, cor. 2] and \( B \)-finitely generated. Hence \( B \otimes_A M \) is \( B \)-projective, and we get that \( B \otimes_A K \) is a finitely generated \( B \)-module. Since \( B \) is faithfully flat, it is readily checked that \( K \) is a finitely generated \( A \)-module, and proposition 1.4 follows from lemma 1.1

Corollary 1.5. If \( A \) is a semilocal (commutative) ring, then \( A \in P \).

Proof. If \( B \) is a quasilocal ring (that is, a commutative ring with a unique maximal ideal), then \( B \in P \). This follows from [2, Chap. 1, § 2, exerc. 23] or from [5]. If \( A \) is semilocal, then \( \prod_{m \in \mathcal{D}} A_m \in P \) (see [2,
Chap. 2, § 3, no. 3) for notation). The corollary follows now from proposition 1.4. and [2, Chap. 2, § 3, no. 3, prop. 10].

Corollary 1.5. is well known (cf. S. Endo [5]).

**Corollary 1.6.** Let $A$ be any ring and $G$ any group. If the group ring $A[G] \in P$, then $A \in P$.

**Theorem 1.7.** Let $A$ be any ring. Then $A \in P$ if and only if $A[[x]] \in P$.

**Proof.** We need the following ideas.

i) Let $B$ be any ring and $x$ a central non-unit non-zero-divisor in $B$. If the $B$-module $M$ is $B$-flat, then $M/xM$ is $B/xB$-flat.

For a proof see [2, Chap. 1, § 2, prop. 8].

ii) Let $M$ be any $B$-modul, then there exists a natural $B$-isomorphism between $B[[x]] \otimes_B M/x[[x]] \otimes_B M$ and $M$.

The proof is trivial.

iii) Let $M$ be any finitely generated flat $B[[x]]$-module. If $M/xM$ is $B$-free, then $M$ is $B[[x]]$-free.

(This might be well known, but I have not been able to find a complete proof in the literature.)

The statement may be proved as follows. If $(\bar{m}_i)_{i \in I}$ is a finite base for the $B$-module $M/xM$, and $m_i$ denotes a representative in $M$ for $\bar{m}_i$, then $(m_i)_{i \in I}$ generate the $B[[x]]$-module $M$ [2, Chap. 2, § 3, no. 2, prop. 4, cor. 2].

From the exact sequence

$$0 \to K \to F \xrightarrow{\varphi} M \to 0$$

of $B[[x]]$-modules, where $F$ is free with base $(e_i)_{i \in I}$ and $\varphi(e_i) = m_i$ for every $i$, we derive the exact sequence of $B$-modules

$$0 = \text{Tor}_1^B[[x]](B[[x]], M) \to K/xK \to F/xF \xrightarrow{\varphi} M/xM \to 0.$$ 

$\varphi$ is a $B$-isomorphism so $K = xK$, and hence $K = 0$, that is, $M$ is $B[[x]]$-free.

Let us return to the proof of theorem 1.8. We assume $A \in P$ and have to prove that $A[[x]] \in P$. Let $M$ be any flat and finitely generated $A[[x]]$-module, then $M/xM$ is finitely generated and flat (cf. i)) viewed as an $A$-module, hence there exists a finitely generated projective $A$-
module $N$ such that $M/xM \oplus N$ is $A$-free with a finite base. Since $A[[x]] \otimes_A N$ is a finitely generated projective $A[[x]]$-module, $(A[[x]] \otimes_A N) \oplus M$ is a finitely generated flat $A[[x]]$-module. From the isomorphisms

$$(A[[x]] \otimes_A N) \oplus M/x(A[[x]] \otimes_A N) \oplus M) \approx A[[x]] \otimes_A N/x[[x]] \otimes_A N) \oplus M/xM \approx N \oplus M/xM$$

(cf. ii)) we infer that $(A[[x]] \otimes_A N) \oplus M$ is $A[[x]]$-free (cf. iii)), and hence $M$ is $A[[x]]$-projective.

Conversely, assume that $A[[x]] \in P$. If $M$ is any flat and finitely generated $A$-module, then $A[[x]] \otimes_A M$ is a flat and finitely generated $A[[x]]$-module [2, Chap. 1, § 2, no. 7, prop. 8, cor. 2], so $A[[x]] \otimes_A M$ is $A[[x]]$-projective. If $N$ is any $A[[x]]$-module, and $x$ is a non-zero-divisor in $N$, then it is well known that $\hld_A N/xN \leq \hld_A[x] N$. From this remark we infer that $A[[x]] \otimes_A M/x(A[[x]] \otimes_A M)$ is $A$-projective, hence $M$ is $A$-projective (cf. ii)).

For later purposes we need the following proposition, which is due to I. I. Sahaev.

PROPOSITION 1.8. If every cyclic flat left $A$-module is projective, then $A$ has no infinite set of orthogonal idempotents.

A proof may be found in [9].

For a commutative ring $A$ proposition 1.8. is due to Endo [6].

2. On a generalization of a theorem of S. Endo.

In this section $A$ denotes a commutative ring with an identity.

The following theorem which might be known is essential for this section.

THEOREM 2.1. For a commutative ring $A$ the following properties are equivalent:

i) Every cyclic flat $A$-module is projective.

ii) $A \in P$.

PROOF. D. Lazard has proved that i) implies that every $D$-closed subset of $X = \text{Spec}(A)$ is open (cf. [8] for a proof and definitions), and if this condition is satisfied, then $A \in P$. The last statement follows immediately from [8, corollary 5.2] and [2, Chap. 2, § 5, no. 2, théorème 1].
Non-commutative rings for which condition i) or condition ii) holds have been studied by I. I. Sahaev [9].

**Theorem 2.2.** Let $A$ be a subring of $B$ ($B$ not necessarily commutative), and suppose $A$ is contained in the center of $B$. If $B \in P$, then $A \in P$.

**Proof.** Let $A/a$ be a flat $A$-module. Consider the exact sequence

$$(0) \to a \to A \to A/a \to (0)$$

of $A$-modules, and we have to prove that $a$ is finitely generated. From (5) we derive the exact sequence

$$(0) \to Ba \to B \to B/Ba \to (0)$$

of $B$-modules. $B/Ba$ is $B$-flat [2, Chap. 1, § 2, no. 7, prop. 8, cor. 2]. Since $B \in P$, we have $Ba=Be$, where $e$ is an idempotent in $B$. Let $e=b_1a_1 + \ldots + b_sa_s$. Since $A/a$ is flat, there exists an element $a' \in a$ such that $a_ia' = a_i$ for every $i \in \{1, \ldots, s\}$, so we conclude that

$$(6) \hspace{1cm} ea' = b_1a_1 + \ldots + b_s a_s = e.$$

Since $a' \in a$, we have $a' = be$ for a suitable $b \in B$, and therefore $a'e = a'$. This together with (6) implies that $e = a'$, that is, $e \in a$. For any $a \in a$, we have $a = ae$, hence $a = Ae$ and (5) must be split exact, that is, $A/a$ is $A$-projective.

**Corollary 2.3.** (cf. [6]). A finitely generated flat module over an integral domain is projective.

**Corollary 2.4.** (S. Endo, cf. [6]). Let $A$ be any commutative ring for which there exists a multiplicatively closed set $S$ consisting of non-zero divisors, such that $A_S$ is semilocal or $A_S \in P$. Then $A \in P$.

**Corollary 2.5.** $A \in P$ if and only if $A[x] \in P$.

**Proof.** Assume $A \in P$, then $A[[x]] \in P$ (theorem 1.7), so $A[x] \in P$ (theorem 2.2).

Conversely, if $A[x] \in P$, proposition 1.4 implies that $A \in P$.

3. **Examples and some remarks.**

**Lemma 3.1.** The ring $A$ has property $P$ if and only if any flat, countably related, finitely generated left $A$-module is projective.
PROOF. "only if" is obvious.

"if". Suppose $A \notin P$, and let $M$ be a finitely generated, not finitely related flat left $A$-module. Consider the exact sequence

$$0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0,$$

where $F$ is a free left $A$-module with base $(e_1, \ldots, e_n)$, and $K = \ker \varphi$. If $(k_{ij}^0)_{1 \leq i \leq n}$ is any set of $n$ elements of $K$, then $\sum_{1 \leq i \leq n} Ak_{ij}^0 = K_0 \subset K$.

(Here $\subset$ means "is a proper subset of"). Choose $k_0 \in K$, $k_0 \notin K_0$ and $\theta_1 : F \rightarrow K$ such that $\theta_1(k_{ij}^0) = k_{ij}^0$, $1 \leq j \leq n$, $\theta_1(k_0) = k_0$ (cf. lemma 1.1). If $\theta_1(e_j) = k_{ij}^1$, $1 \leq j \leq n$, then $K_0 \subset \sum_{1 \leq j \leq n} k_{ij}^1 \subset K$. If we continue this process, we get modules $(K_i)_{i \in \mathbb{N}_0}$ such that

$$K_0 \subset K_1 \subset \ldots \subset K_n \subset \ldots .$$

Let $K^*$ be equal to $\bigcup_{i=1}^\infty K_i$. Then $F/K^*$ is flat (lemma 1.1), countably related, but not finitely related, and the lemma is proved.

COROLLARY 3.2. If $1fPD(A) = 0$, then $A \in P$.

PROOF. Let $M$ be any finitely generated, countably related flat $A$-module. We conclude that $1hd_A M \leq 1$ (cf. C. U. Jensen [7, lemma 2]), hence $M$ is $A$-projective.

REMARK 1. In the special case $1fPD(A) = 0$, $A$ is left perfect (cf. H. Bass [1, theorem 6.3]), and the corollary follows from [1, theorem P].

From corollary 3.2 and section 1 (remark) we infer that if $1fPD(A) = 0$, then $A$ has no infinite set of orthogonal idempotents, so we have proved the following (cf. [1, theorems P and 6.3]):

PROPOSITION 3.3. If $1fPD(A) = 0$ and every nonzero right $A$-module has nonzero socle, then $1fPD(A) = 0$.

In general, $1fPD(A) = 0$ does not imply that $1fPD(A) = 0$. Example: $F[[x,y]]/(x^2, xy)$, where $F$ is commutative field.

EXAMPLE 1. Let $T$ be any infinite connected normal topological space (i.e. $T$ satisfies $(T_2)$ and $(T_4)$), then $A = C(T, +, \cdot, R)$ (the ring of continuous real-valued functions on $T$) is an example of a commutative indecomposable ring not having property $P$.

PROOF. An application of Urysohn’s lemma enables us to construct functions $(f_i)_{1 \leq i < \infty}$ such that $f_i \in A$, $f_i f_{i+1} = f_i$ and $Af_i \subset Af_{i+1}$. Let a be
the ideal generated by the $f_i$'s. $A/a$ is $A$-flat (lemma 1.1), but $A/a$ is not $A$-projective. This example is due to C. U. Jensen.

Example 2. The ring $A$ defined below is indecomposable, commutative, and coherent, but $A \notin P$.

Let $A$ be the subring of $C(R, +, \cdot, R)$ consisting of the functions $f(x)$ of the form

$$f(x) = \begin{cases} \frac{p_i(x)}{q_i(x)}, & x \leq -k_f, \ k_f \in \mathbb{N}, \\ \frac{p_i(x)}{q_i(x)}, & x \in [i, i+1], \ i \in \{-k_f, \ldots, k_f-1\}, \\ \frac{p(x)}{q(x)}, & x \geq k_f, \end{cases}$$

where $\bar{p}(x)$, $\bar{q}(x)$, $p_i(x)$, $q_i(x) \in R[x]$ for every $i \in \{-k_f, \ldots, k_f-1\}$ and $q_i(x) \neq 0$ for every $x \in [i, i+1]$, $\bar{q}(x) \neq 0$ for $x \leq -k_f$, $\bar{q}(x) \neq 0$ for $x \geq k_f$.

By a straight-forward, but tedious computation, it can be proved that this ring $A$ has the required properties.

If $A$ satisfies a certain extra condition, then $A \in P$.

Theorem. Let $A$ be a commutative ring. If $A$ has no infinite set of orthogonal idempotents, $A$ is coherent and $\text{whd}_A(Aa) < \infty$ for every $a \in A$, then $A \in P$.

Proof. $A$ is a finite direct sum of integral domains (cf. L. W. Small [10]), hence $A \in P$.

Remark 2. Professor P. M. Cohn has communicated to me an example of a non-commutative ring $A$, which is an integral domain, and for which $A \notin P$.

Let $A$ be the $K$-algebra on the generators $a_{ij}^{(\nu)}$, $i, j = 1, 2, \nu = 1, 2, \ldots$, and defining relations

$$\sum_j a_{ij}^{(\nu)} a_{jk}^{(\nu)} = \delta_{i, \nu} a_{ik}^{(\nu)}.$$  \hspace{1cm} (7)

$A$ is 1-fir (cf. P. M. Cohn [4]), thus $A$ is an integral domain, hence any cyclic flat left $A$-module is $A$-projective. The existence of the relations (7) implies that $A_2 \notin P$. From the Morita-equivalens between $A$ and $A_n$ we get that $A \in P$ if and only if $A_n \in P$ for every $n$. Therefore $A \notin P$.  


Thus the commutativity of the ring $A$ is essential for the validity of theorem 2.1.

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REFERENCES


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