FULL SETS OF STATES ON A C*-ALGEBRA

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§ 1. Introduction.

This note contains a discussion of necessary and sufficient conditions on a subset $S_0$ of the set $S$ of states on a $C^*$-algebra $A$, for its convex hull $\langle S_0 \rangle$ to be $w^*$-dense in $S$. In the case where $A$ has an identity, results to this effect have been known for some time [1], [2], [3]. If $A$ does not have an identity, the situation is a little more complicated, and will be dealt with on the following pages.

The set of states on a $C^*$-algebra $A$ is the family of all positive linear functionals on $A$ of norm equal to one. If $A$ has an identity, $S$ is convex and compact in the $w^*$-topology as a subset of $A^*$. If $A$ has no identity, $S$ is still convex, which may be proved by means of an approximate identity for $A$, but $S$ is no longer compact. In this case it is therefore often convenient to introduce the set $N$ of all positive linear functionals on $A$ with norm $\leq 1$. Then $N$ is compact, and $S$ is a subset of $N$ which is a face, i.e., if $\varrho_1, \varrho_2 \in N$ and $\varrho = \lambda \varrho_1 + (1 - \lambda) \varrho_2 \in S$, $0 < \lambda < 1$, then $\varrho_1$ and $\varrho_2$ both belong to $S$. Let $\tilde{A}$ denote the $C^*$-algebra obtained by adjoining an identity to $A$, and let $\tilde{S}$ be the set of states on $\tilde{A}$. For $\varrho \in N$ let $\tilde{\varrho}(a + \lambda 1) = \varrho(a) + \lambda$, $a \in A$, $\lambda \in \mathbb{C}$. The map $\varrho \to \tilde{\varrho}$ is a bicontinuous, affine isomorphism of $N$ onto $\tilde{S}$, and we have $\tilde{\varrho} | A = \varrho$.

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§ 2. Full sets of states.

For the case where $A$ has an identity, the facts are recorded in Theorem 1 below. The essential condition, introduced by Kadison [3], is the following: A family $S_0$ of positive linear functionals on a $C^*$-algebra is full if, for $a \in A$, $\varrho(a) \geq 0$ for all $\varrho \in S$ implies that $a \geq 0$.

In Theorem 1, the equivalence of conditions (i), (ii) and (iii) is known,
see [1] or [3]. That condition (iv) is equivalent to the others may also be known, but this is not to the author’s knowledge.

**Theorem 1.** Let $A$ have an identity. The following conditions on a subset $S_0$ of $S$ are equivalent:

(i) $S_0$ is full.

(ii) $S_0 \supseteq \partial_e S$ (the set of extreme points of $S$).

(iii) $\langle S_0 \rangle = S$.

(iv) $\|a\| = \sup_{\varrho \in S_0} \varrho(a)$ for any $a \geq 0$, $a \in A$.

**Proof.** For (i) $\iff$ (ii) $\iff$ (iii) we refer to [1, 2.6.2 and 3.4.1]. If $a \geq 0$, then $\|a\| = \varrho(a)$ for some $\varrho \in \partial_e S$, so (ii) $\implies$ (iv). To finish the proof, we show that (iv) $\implies$ (i). We do this through two lemmas.

**Lemma 1.** Let $A$ be a $C^*$-algebra. Suppose $S_0 \subseteq N$ and that $S_0$ satisfies (iv) above. Let $a \in A_h$, $a^+ \neq 0$, be given. Then, for any $\varepsilon > 0$, there is $\varrho \in S_0$ such that

$$\varrho(a^+) \geq \|a^+\| - \varepsilon, \quad \varrho(a^-) \leq \varepsilon.$$

**Proof.** Let $\varepsilon > 0$ be given. We have $a = a^+ - a^-$, and $\|a\| = \max(\|a^+\|, \|a^-\|)$. Suppose first $\|a\| = \|a^+\|$, and choose $\varrho \in S_0$ such that $\varrho(a^+) \geq \|a^+\| - \varepsilon$. Then

$$\|a^+\| = \|a^+ + a^-\| \geq \varrho(a^+ + a^-) = \varrho(a^+) + \varrho(a^-) \geq \|a^+\| - \varepsilon + \varrho(a^-).$$

Hence $\varrho(a^-) \leq \varepsilon$. If $\|a\| = \|a^-\|$, let $b = a^+ - (\|a^+\|/\|a^-\|)a^-$, so $\|b\| = \|a^+\|$. Then choose $\varrho \in S_0$ such that

$$\varrho(b^+) = \varrho(a^+) \geq \|a^+\| - (\|a^+\|/\|a^-\|)\varepsilon \geq \|a^+\| - \varepsilon.$$

Then, by the first result

$$\varrho(b^-) = (\|a^+\|/\|a^-\|)\varrho(a^-) \leq (\|a^+\|/\|a^-\|)\varepsilon,$$

so $\varrho(a^-) \leq \varepsilon$. The proof is complete.

**Lemma 2.** Let $A$ be a $C^*$-algebra. If $S_0 \subseteq N$ and $\|a\| = \sup_{\varrho \in S_0} \varrho(a)$ for any $a \geq 0$ in $A$, then $S_0$ is full.

**Proof.** First, let $a \in A_h$ and suppose $\varrho(a) \geq 0$ for all $\varrho \in S_0$. Suppose $a^- \neq 0$, and apply Lemma 1 to $b = -a$. We can find $\varrho \in S_0$ such that

$$\varrho(a^-) \geq \frac{1}{2} \|a^-\|, \quad \varrho(a^+) \leq \frac{1}{4} \|a^-\|.$$
But then $\varrho(a) = \varrho(a^+) - \varrho(a^-) \leq \frac{1}{2}\|a^+\| - \frac{3}{2}\|a^-\| = -\frac{1}{2}\|a^-\| < 0$, a contradiction. Hence $a \geq 0$.

Now let $a \in A$ be arbitrary, with $\varrho(a) \geq 0$ for all $\varrho \in S_0$. Then $\varrho(a^*) = \overline{\varrho(a)} = \varrho(a) \geq 0$, so $\varrho(\text{Im}a) = 0$, and $\varrho(a) = \varrho(\text{Re}a) \geq 0$ for all $\varrho \in S_0$. Hence, by the first part of the proof $\text{Re}a \geq 0$ and both $\text{Im}a \geq 0$ and $-\text{Im}a \geq 0$ so $\text{Im}a = 0$. Consequently $a = \text{Re}a \geq 0$, and the proof of Lemma 2 and Theorem 1 is complete.

§ 3. Algebras without identity.

We now assume specifically that $A$ has no identity. Let $P$ be the pure states on $A$, that is, a state $\varrho$ on $A$ is in $P$ if $0 \leq \varrho \leq 1$, $\varrho \in N$, implies $\varrho = \lambda \varrho$ for some $\lambda$ between 0 and 1.

Lemma 3. (a) $P = \partial_e S$.

(b) $S \subseteq \overline{\langle P \rangle}$ (closure taken in $N$).

Proof. (a). We know that $\partial_e N = \{0\} \cup P$ [1, 2.5.5]. Hence each pure state of $A$ is an extreme point of $S$, since $P \subseteq S \subseteq N$. Conversely, each extreme point of $S$ is also an extreme point of $N$, since $S$ is a face of $N$. Since these extreme points have norm equal to one, they must belong to $P$ by the fact quoted above. Hence $P = \partial_e S$.

(b). $\langle\{0\}, P\rangle = \langle\partial_e N\rangle$ is dense in $N$ by the Krein–Milman theorem. Hence $\langle\{0\}, \overline{\langle P \rangle}\rangle$, which is compact, must be equal to $N$. Now $S$ consists of the elements of $N$ with norm equal to one, hence $S \subseteq \overline{\langle P \rangle}$.

Lemma 4. (Glimm [1, ex. 2.12.13]). $0 \in \overline{P}$.

Proof. There is no loss in generality by assuming that $A$ is faithfully represented on a Hilbert space $X$, and that $A$ operates non-degenerately on $X$. $A$ is an ideal in $\tilde{A}$, so if $0 \leq x \in A$, then $x$ is not invertible in $\tilde{A}$ since $1$, the identity operator on $X$, does not belong to $A$. Hence $x$ is not invertible in $\mathcal{L}(X)$, so

$$\inf_{\|\xi\|=1} \omega_{\xi}(x) = \inf_{\|\xi\|=1} (x\xi, \xi) = 0, \quad \xi \in X.$$ 

If $\{e_\nu\}$ is an approximate identity for $A$, then $\{e_\nu\}$ converges strongly to $1$. If $\|\omega_{\xi}\|$ denotes the norm of $\omega_{\xi}$ as a linear functional on $A$, we have $\|\omega_{\xi}\| = \lim_\nu \omega_{\xi}(e_\nu) = \|\xi\|^2$, so $\omega_{\xi} \in S$ when $\|\xi\| = 1$. This means that $0 \in \overline{S}$, since sets

$$U(x, \varepsilon) = \{\varrho \in A^* : \|\varrho(x)\| \leq \varepsilon ; \ 0 \leq x \in A, \ \varepsilon > 0\}$$
form a basis for the neighbourhoods of 0 in $A^\ast$. By Lemma 3 it then follows that $0 \in \langle P \rangle$, which clearly implies that $0 \in \overline{P}$. The proof is complete.

**Proposition 1.** $\overline{S} = \langle P \rangle = N$.

**Proof.** Since $\langle P \rangle \subseteq S$, the first equality follows by Lemma 3. $\overline{S}$ is closed, convex and contains $\partial_e N$ by Lemma 4, so $\overline{S} = N$.

**Lemma 5.** Let $S_0$ be a full subset of $N$. Let $a \in A_h$, $\lambda \in \mathbb{R}$, and suppose $0 \leq q(a) + \lambda$ for all $q \in S$. Then $\lambda \geq 0$.

**Proof.** Assume that $\lambda < 0$, and let $\{e_v\}$ be an approximate identity for $A$, $0 < e_v$, $\|e_v\| = 1$ for all $v$. Then

$$q(a + \lambda e_v) = q(a) + \lambda q(e_v) \geq q(a) + \lambda \geq 0$$

for all $q \in S_0$, so $a + \lambda e_v \geq 0$ for all $v$. Hence, for any $q \in P$,

$$q(a) + \lambda = \lim_v q(a) + \lambda q(e_v) \geq 0.$$ 

Since $0 \in \overline{P}$, it follows that $\lambda \geq 0$, a contradiction. Hence $\lambda \geq 0$ and the proof is complete.

**Corollary.** If $S_0$ is a full subset of $N$, then $0 \in \overline{S}_0$.

**Proof.** If $0 \notin \overline{S}_0$, then there is an element $0 \leq a \in A$ and a real number $\lambda > 0$ such that $\inf_{q \in S_0} q(a) \geq \lambda > 0$. But then $q(a) - \lambda \geq 0$ for all $q \in S_0$, which contradicts Lemma 5. The proof is complete.

It is fairly easy to see that if $S_0$ is a full subset of $N$, then $\langle S_0 \rangle$ need not be dense in $N$. However, with the strengthened assumption that $S_0 \subseteq S$ one may ask whether $\langle S_0 \rangle$ will be dense in $S$ (and then automatically in $N$, by Proposition 1). The following example, due to R. V. Kadison (oral communication), shows that this need not be so.

Let $A$ be the abelian $C^\ast$-algebra of all complex sequences $\{a_n\}$ such that $a_n \to 0$ as $n \to \infty$, equipped with the norm $\|a\| = \sup_n |a_n|$, $a = \{a_n\}$. For $n \neq m$ let $e_{n,m}(a) = \frac{1}{2}(a_n + a_m)$. Clearly $e_{n,m} \in S$, and we take $S_0 = \{e_{n,m} : n \neq m\}$. $S_0$ is full: Let $a \in A_h$ and suppose $a^+ \neq 0$. There is an $a_m < 0$, such that

$$\lim_n e_{n,m}(a) = \lim_n \frac{1}{2}(a_n + a_m) = \frac{1}{2}a_m < 0.$$ 

Hence $e_{n,m}(a) < 0$ for some $n$. Now let $q \in S$ be given by $q(a) = a_1$, and
let \( b = \{1/n\} \). Then \( \varrho_{n,m}(b) = \frac{1}{2}(1/n + 1/m) \leq \frac{3}{4} \) for all \( n \neq m \), so \( \varrho(b) \leq \frac{3}{4} \) for all \( \varrho \in \langle S_0 \rangle \). But \( \varrho(b) = 1 \), so \( |\varrho(b) - \varrho(b)| \geq \frac{1}{4} \) for all \( \varrho \in \langle S_0 \rangle \), which shows that \( \langle S_0 \rangle \) is not dense in \( S \).

If, however, \( S_0 \) is a full set of pure states on an abelian \( C^* \)-algebra \( A \), then \( \langle S_0 \rangle \) is dense in \( S \). Indeed, we have \( A = \mathcal{C}_0(X) = \) the family of continuous, complex functions vanishing at infinity, on a locally compact Hausdorff space \( X \). We may identify \( X \) with \( P \), and if \( S_0 \) is a full subset of \( X \), this means that a function \( f \in \mathcal{C}_0(X) \) is everywhere non-negative if it is non-negative on \( S_0 \). But then clearly \( S_0 \) is dense in \( X \), and consequently \( \langle S_0 \rangle \) is dense in \( \overline{\langle P \rangle} \supseteq S \).

It is interesting to note that this is not true in general. That is, even if \( S_0 \) is a full subset of \( P \), \( \langle S_0 \rangle \) need not be dense in \( S \). The following example is due to G. Kjærgård Pedersen (private communication).

Let \( A \) be the \( C^* \)-algebra of compact operators on an infinite dimensional Hilbert space \( X \). Take an arbitrary unit vector \( \eta_0 \in X \), and let

\[
K = \{ \xi \in X : \|\xi\| = 1, |(\xi, \eta_0)| \leq \frac{1}{2} \}.
\]

Now take \( S_0 \) to be the set of pure states \( \omega_\xi \) on \( A \) associated with vectors \( \xi \in K \). We claim that \( S_0 \) is full, but \( \langle S_0 \rangle \) not dense in \( S \). Indeed, let \( p \) be the one-dimensional projection determined by \( \eta_0 \), that is, \( p\xi = (\xi, \eta_0)\eta_0 \), \( \xi \in X \). Then, for \( \xi \in K \),

\[
\omega_\xi(p) = (p\xi, \xi) = |(\xi, \eta_0)|^2 \leq \frac{1}{4} < \|p\|,
\]

which shows that \( \langle S_0 \rangle \) is not dense in \( S \).

Now let \( x \in A_k \), and suppose that \( \omega_\xi(x) \geq 0 \) for all \( \xi \in K \). Let \( \epsilon > 0 \) be arbitrary, and let \( x_1 \) be the restriction of \( x \) to \( (1-p)X \). Then \( x_1 : (1-p)X \to X \) has no bounded inverse \( y \). For if such a \( y \) existed, then \( 1-x_1y = xy \), making \( 1 \) compact which is impossible since \( X \) is infinite dimensional. Hence there is a unit vector \( \xi_0 \in (1-p)X \) such that \( |x\xi_0| \leq \epsilon \). Now take any unit vector \( \eta \in X \) and define \( \xi = \frac{1}{\eta} + \alpha \xi_0 \) with \( \alpha \), \( |\alpha| \leq 1 \), chosen so as to make \( \|\xi\| = 1 \). Clearly \( \xi \in K \), so by assumption

\[
0 \leq \langle x\xi, \xi \rangle \leq \frac{1}{4} \langle x\eta, \eta \rangle + 2\|x\xi_0\| \leq \frac{1}{4} \langle x\eta, \eta \rangle + 2\epsilon.
\]

Hence \( \langle x\eta, \eta \rangle \geq -8\epsilon \) which, by the arbitrariness of \( \epsilon \) and \( \eta \), implies that \( x \geq 0 \). So \( S_0 \) is full and we have our example.

We now give the characterization of dense subsets of \( S \), similar to Theorem 1.

**Theorem 2.** Let \( A \) be a \( C^* \)-algebra without identity, and let \( S_0 \) be a subset of \( N \). The following statements are equivalent:

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(ii)' \( S_0 \subseteq P \).

(iii)' \( \langle S_0 \rangle \subseteq S \) (or \( \langle S_0 \rangle = N \)).

(iv)' \( \|a\| = \sup_{e \in S_0} \varphi(a) \) for any \( a \geq 0 \), \( a \in A \).

Each of these conditions implies

(i)' \( S_0 \) is full.

**Proof.** (iii)' \( \Rightarrow \) (ii)' by Proposition 1 and the partial converse of the Krein–Milman theorem. (ii)' \( \Rightarrow \) (iv)' as in the proof of Theorem 1. (iv)' \( \Rightarrow \) (i)' is covered by Lemma 2, so we are left to prove (iv)' \( \Rightarrow \) (iii)'.

Clearly (iii)' is equivalent to (iii) of Theorem 1 for \( \tilde{A} \) by the map \( N \to \tilde{S} \), and it is therefore, by Theorem 1, sufficient to show that \( S_0 \) is full as a subset of \( \tilde{S} \). Now \( S_0 \) is full as a subset of \( N \), so by Lemma 5 it suffices to show that if \( a \in A_h \), \( \lambda \geq 0 \), and \( \varphi(a) + \lambda \geq 0 \) for all \( \varphi \in S_0 \), then \( a + \lambda 1 \geq 0 \). If \( a \geq 0 \) there is nothing to prove, so we may assume that \( a^- \neq 0 \). Then, by Lemma 1, for any \( \varepsilon > 0 \) there is a \( \varphi \in S_0 \) such that \( \varphi(a^+) \leq \varepsilon \) and \( \varphi(a^-) \geq \|a^-\| - \varepsilon \).

Now

\[
0 \leq \varphi(a) + \lambda = \varphi(a^+) - \varphi(a^-) + \lambda ,
\]

so

\[
\|a^-\| - \varepsilon \leq \varphi(a^+) + \lambda \leq \varepsilon + \lambda ,
\]

so

\[
\|a^-\| \leq \lambda + 2 \varepsilon.
\]

Hence \( \|a^-\| \leq \lambda \) since \( \varepsilon > 0 \) was arbitrary, which implies that \( a + \lambda 1 \geq 0 \).

The proof is complete.

We will give another result in the same direction. A \( C^* \)-subalgebra \( B \) of a \( C^* \)-algebra \( A \) is called **facial** (order-related, see [4]) if \( B^+ \) is an order-ideal of \( A^+ \). For \( y \in A \), let \( A[y] \) denote the smallest facial \( C^* \)-subalgebra of \( A \) containing \( y \).

**Theorem 3.** Let \( S_0 \subseteq N \). Then \( \langle S_0 \rangle \) is dense in \( N \) if and only if for each \( 0 \leq x \in A \) and each \( \varepsilon > 0 \) there is \( \varphi \in S_0 \) such that

\[
\|\varphi | A[x] \| > 1 - \varepsilon .
\]

**Proof.** To prove the "if" part it is, by Theorem 2 (iv)', enough to show that \( \sup_{e \in S_0} \varphi(x) = 1 \) for an arbitrary \( x \in A^+ \), \( \|x\| = 1 \). So let \( \varepsilon > 0 \) together with such an \( x \) be given. Let \( f \) be the real continuous function on \([0,1]\) given by

\[
f(\lambda) = \begin{cases} 
0 & \text{for } 0 \leq \lambda \leq 1 - \varepsilon , \\
\varepsilon^{-1}(\lambda - 1) + 1 & \text{for } 1 - \varepsilon \leq \lambda \leq 1 .
\end{cases}
\]
Put \( y = f(x) \). By assumption there is a \( \varrho \in S_0 \) such that \( \| \varrho \|_{A[y]} > 1 - \varepsilon \).

Let \( \mu_\varrho \) be the measure \( \varrho \) assigns to \( \sigma(x) \subseteq [0, 1] \). Let \( E = \sigma(x) \cap (1 - \varepsilon, 1] \). Then clearly \( \mu_\varrho(E) = \lim_n \varrho(y^{1/n}) \).

On the other hand, \( y \) is strictly positive in \( A[y] \) in the sense of [5]. Indeed, if \( \gamma \) is a state on \( A[y] \) and \( \gamma(y) = 0 \), then \( \gamma \equiv 0 \) on the set \( \{ z \in A[y] : \exists r \in R^+, \ 0 \leq z \leq ry \} \). This set is dense in \( A[y]^+ \) by [4, Lemma 1.1], hence \( \gamma \equiv 0 \) on \( A[y] \), a contradiction. But then \( \{ y^{1/n} \} \) is an approximate identity for \( A[y] \) by [5, Theorem 1]. Hence

\[
\| \varrho \|_{A[y]} = \lim_n \varrho(y^{1/n}),
\]

so \( \mu_\varrho(E) > 1 - \varepsilon \). It follows that

\[
\varrho(x) = \int_{\varrho(x)} \lambda \, d\mu_\varrho(\lambda) \geq \int_E \lambda \, d\mu_\varrho(\lambda) \geq (1 - \varepsilon)\mu_\varrho(E) > (1 - \varepsilon)^2.
\]

Since \( \varepsilon > 0 \) was arbitrary, we get \( \sup_{\varrho \in S_0} \varrho(x) = 1 \). The proof of the converse is straightforward and is left to the reader.

In conclusion we wish to remark that the property of a set \( S_0 \) to be dense in the set of states \( S \) has an obvious probabilistic interpretation. Let \( a \) be a self-adjoint element of a \( C^* \)-algebra \( A \) with identity, and let \( \sigma(a) \) be the spectrum of \( a \). Each \( \varrho \in S \) induces a probability measure \( \mu_\varrho \) on \( \sigma(a) \) by

\[
\varrho(f(a)) = \int_{\sigma(a)} f(\lambda) \, d\mu_\varrho(\lambda), \quad \lambda \in \sigma(a),
\]

where \( f \to f(a) \) is the Gelfand-transformation of \( \sigma(\sigma(a)) \) onto the \( C^* \)-subalgebra \( A(a) \) generated by \( a \) and \( 1 \). Let \( \lambda_0 \) be an element of \( \sigma(a) \), and let \( \varepsilon > 0 \) be arbitrary. We claim that if \( S_0 \) is dense in \( S \), then for each \( \delta > 0 \) there is a \( \varrho_0 \in S_0 \) such that

\[
\mu_{\varrho_0}([\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]) > 1 - \delta.
\]

Indeed, let \( f : \sigma(a) \to [0, 1] \) be a continuous function such that \( f(\lambda_0) = 1 \), and \( f \equiv 0 \) outside the interval \( [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \). Then \( \|f(a)\| = 1 \) so there is a \( \varrho_0 \in S_0 \) such that \( \varrho_0(f(a)) > 1 - \delta \). Hence

\[
1 - \delta < \int_{\sigma(a)} f(\lambda) \, d\mu_{\varrho_0}(\lambda) = \int_{\lambda_0 - \varepsilon}^{\lambda_0 + \varepsilon} f(\lambda) \, d\mu_{\varrho_0}(\lambda) \leq \mu_{\varrho_0}([\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]).
\]

In the context of quantum mechanics, this may be formulated as follows. If \( S_0 \) is a dense family of states, then it is sufficiently rich in the following sense: If \( a \) is an observable, \( \lambda_0 \) a point in the spectrum of \( a \)
and $E$ an arbitrary open interval in $\sigma(a)$ containing $\lambda_0$, then there is a state $\rho_0 \in S_0$ such that the probability for a measurement of $a$ in the state $\rho_0$ to fall in $E$ can be made arbitrarily close to one.

REFERENCES


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