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ON THE IMPOSSIBILITY OF REPRESENTING CERTAIN FUNCTIONS BY CONVOLUTIONS

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1. Introduction.

Let G be an arbitrary locally compact group and let p, q, r be real numbers satisfying $1 \leqq p < \infty$, $1 \leqq q < \infty$, $1/r = 1/p + 1/q - 1 > 0$. Then the inclusion

$$L_p(G) * (L_q(G) \cap L_q^{\star}(G)) \subseteqq L_r(G) \tag{i}$$

holds and is known as Young's inequality, see [2, (20.14) and (20.18)]. Recently E. Hewitt [1] has proved that in the case $p=1$ (hence $q=r$), (i) can be improved to read

$$L_1(G) * L_r(G) = L_r(G) . \tag{ii}$$

This raises the question as whether or not equality also holds in (i) for $p > 1$, $q > 1$. The purpose of this paper is to answer this question negatively. More precisely, we prove the following Theorem (1.1), which is the main result of this paper.

(1.1) THEOREM. *Let G be an infinite locally compact group and let p, q, r be as in Young's inequality with $p > 1$, $q > 1$. Then the subspace spanned by $L_p(G) * (L_q(G) \cap L_q^{\star}(G))$ is a dense subspace of the first category in $L_r(G)$, and the functions in $L_r(G)$ which cannot be factored in $L_p(G) * (L_q(G) \cap L_q^{\star}(G))$ comprise a dense subset of the second category in $L_r(G)$.*

Our proof of this theorem is elementary and, unlike many proofs of results of this nature, we actually construct a function in $L_r(G)$ which can not be written as a convolution product of the desired type. We use the theory of $L(p,q)$ spaces, and it is suggested by O'Neil's interesting paper [5]. A reader of [5] and the present note will be aware of our debt to Professor O'Neil.

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All terms and notation not explained in this note are as in Hewitt and Ross [2].

2. $L(p,q)$ spaces and proof of the main result.

For the convenience of the reader, we first reproduce a number of definitions and record some known facts about rearrangements of functions and $L(p,q)$ spaces [see (2.1) below], which will be needed in the sequel.

(2.1) DEFINITIONS. Let f be a complex-valued measurable function defined on a measure space (X,μ). For $y \geqq 0$, we define

$$m(f,y) = \mu\{x \in X : |f(x)| > y\}.$$

Note that $m(f,\cdot)$ is a non-increasing, right-continuous function. For $x \geqq 0$, we define

$$f^*(x) = \inf\{y : y > 0 \text{ and } m(f,y) \leqq x\} = \sup\{y : y > 0 \text{ and } m(f,y) > x\},$$

with the conventions $\inf \emptyset = \infty$ and $\sup \emptyset = 0$. We note that f^* is a non-increasing, right-continuous function and it is called the *non-increasing rearrangement of f onto the non-negative real numbers.* For $x > 0$, we define

$$f^{**}(x) = x^{-1} \int_0^x f^*(t)\, dt.$$

It is known that if f,g are measurable functions, then

$$\text{(i)} \qquad (f+g)^{**} \leqq f^{**} + g^{**}.$$

(An elementary, detailed proof of this fact is given in [6].) For a measurable function f defined on a measure space (X,μ), we define

$$\|f\|_{(p,q)} = \left\{\int_0^\infty [x^{1/p} f^{**}(x)]^q\, x^{-1} dx\right\}^{1/q}$$

for $1 < p < \infty$, $1 \leqq q < \infty$. We say that $f \in L(p,q) = L(p,q)(X) = L(p,q)(X,\mu)$ if $\|f\|_{(p,q)} < \infty$.

It is well known that $L(p,p) = L_p$ and $\|f\|_p \leqq \|f\|_{(p,p)} \leqq p'\|f\|_p$, where $1/p + 1/p' = 1$ (see for example [5] for proofs of these facts). A theorem of Hardy (see [8, p. 20]) tells us that

(ii) $$\|f\|_{(p,q)} \leqq p' \left\{ \int_0^\infty [x^{1/p} f^*(x)]^q \, x^{-1} dx \right\}^{1/q} \leqq p' \|f\|_{(p,q)} .$$

It is also easy to show (use inequality (i) above) that $L(p,q)$ is a Banach space with norm $\|\cdot\|_{(p,q)}$.

We now state a theorem about $L(p,q)$ spaces which includes Young's inequality as a special case. Even though a much stronger result is known [7], we give here a completely elementary proof for the case we need in this note. This theorem is stated in O'Neil [5] for unimodular groups and the last part of our proof is borrowed from the same paper. We have benefited from conversations with Professor O'Neil regarding this proof.

We find it convenient to use the term *S-function* to mean a function of the form $\sum_{i=1}^n \alpha_i \xi_{E_i}$, where $\alpha_i > 0$, $0 < \lambda(E_i) < \infty$, and ξ_A is the characteristic function of A.

(2.2) THEOREM. *Let G be an arbitrary locally compact group with left Haar measure λ and let p_1, p_2, q_1, q_2 be real numbers such that $1 < p_i < \infty$, $1 \leqq q_i < \infty$, $i = 1, 2$, and*

$$1/p_1 + 1/p_2 > 1 ,$$

and suppose

$$f \in L(p_1, q_1)(G, \lambda), \qquad g \in L(p_2, q_2)(G, \lambda) \cap (L(p_2, q_2)(G, \lambda))^\star .$$

Then $h = f * g \in L(r, s)(G, \lambda)$, where r is given by

$$1/r = 1/p_1 + 1/p_2 - 1$$

and $s \geqq 1$ is any number such that

$$1/q_1 + 1/q_2 \geqq 1/s .$$

PROOF. (The inclusion of this proof was suggested by the referee.) First note that we may assume without loss of generality that $f \geqq 0$, $g \geqq 0$, and $g = g^\star$ (otherwise we simply replace g by $g + g^\star$ throughout, and note that if $f * (g + g^\star)$ is in $L(r,s)$, so is $f * g$). For easy reference we divide the rest of the proof into three steps:

(I) If $(f_k)_{k=1}^\infty$ is a non-decreasing sequence of non-negative measurable functions defined on G such that $f_k(x) \uparrow f(x)$ for each $x \in G$, then we have (i) $m(f_k, y) \to m(f, y)$ for every $y \geqq 0$; (ii) $f_k^*(x) \to f^*(x)$ for every $x \geqq 0$; (iii) $f_k^{**}(x) \to f^{**}(x)$ for every $x > 0$. (We omit the simple proofs of these assertions.)

(II) We claim that for $t > 0$, we have

(*) $$h^{**}(t) \leqq \int_t^\infty f^{**}(u) \, g^{**}(u) \, du .$$

We prove this assertion in three stages:

(a) If f,g are S-functions with $f=\alpha\xi_E$, $g=\beta\xi_F$. We may suppose that $\alpha=1=\beta$. We verify the inequality (*) when $\lambda(E)\leqq\lambda(F)$ by considering two posibilities: If $t\geqq\lambda(F)$, then we have,

$$\int_t^\infty f^{**}(u)g^{**}(u)\,du = \int_t^\infty u^{-1}\lambda(E)\,u^{-1}\lambda(F)\,du = t^{-1}\lambda(E)\lambda(F)\,.$$

But

$$th^{**}(t) = \int_0^t h^*(u)\,du \leqq \|h\|_1 \leqq \|f\|_1\|g\|_1 = \lambda(E)\,\lambda(F)\,,$$

and therefore (*) follows in this case. If $t<\lambda(F)$, then

$$\int_t^\infty f^{**}(u)g^{**}(u)\,du \geqq \int_{\lambda(F)}^\infty f^{**}(u)g^{**}(u)\,du = \lambda(E)\,.$$

But

$$(\#) \qquad h^{**}(t) \leqq \|h\|_\infty \leqq \|f\|_1\|g\|_\infty = \lambda(E)\,,$$

and (*) follows in this case. (Note that the assumption $g=g^\star$ is used to obtain inequalities similar to (#) for the case $\lambda(E)>\lambda(F)$. See [2, (20.14.iv))].)

(b) If f,g are S-functions with $g=\beta\xi_F$. We write $f=\sum_{i=1}^n\alpha_i\xi_{E_i}$, $0<\alpha_1<\alpha_2<\ldots<\alpha_n$. Define $f_k(x)=0$ if $f(x)<\alpha_k$ and $f_k(x)=\alpha_k-\alpha_{k-1}$ if $f(x)\geqq\alpha_k$. Then we have

$$f = \textstyle\sum_{k=1}^n f_k\,, \quad f^* = \sum_{k=1}^n f_k^*, \quad f^{**} = \sum_{k=1}^n f_k^{**}\,,$$

and so

$$\begin{aligned} h^{**}(t) &\leqq \textstyle\sum_{k=1}^n (f_k*g)^{**}(t) \leqq \sum_{k=1}^n \int_t^\infty f_k^{**}(u)\,g^{**}(u)\,du \\ &= \int_t^\infty f^{**}(u)\,g^{**}(u)\,du\,. \end{aligned}$$

(c) If f,g are arbitrary S-functions. In this case we do the same thing as in (b) to g and then apply (b) to obtain (*). Finally (c) and (I) yield (II) easily.

(III) We have (following O'Neil [5])

$$\|h\|_{(r,s)}^s = \int_0^\infty [x^{1/r}\,h^{**}(x)]^s\,x^{-1}\,dx$$

(by step (II))

$$\leqq \int_0^\infty \left[x^{1/r}\int_x^\infty f^{**}(t)\,g^{**}(t)\,dt\right]^s x^{-1}\,dx$$

(by changing variables: $x=y^{-1}$, $t=u^{-1}$)

$$= \int_0^\infty \left[y^{-1/r} \int_0^y f^{**}(u^{-1}) g^{**}(u^{-1}) u^{-2}\, du \right]^s y^{-1} dy$$

$$(1/r + 1/r' = 1,\ \varphi(u) = f^{**}(u^{-1}) g^{**}(u^{-1}) u^{-2})$$

$$= \int_0^\infty \left[y^{1/r'} y^{-1} \int_0^y \varphi(u)\, du \right]^s y^{-1} dy$$

$$\left(F(y) = \int_0^y \varphi(u)\, du \right)$$

$$= \int_0^\infty [y^{-1} F(y)]^s y^{s/r'-1}\, dy$$

(by Hardy's theorem [8, p. 20])

$$= r^s \int_0^\infty [y^{1/r'} \varphi(y)]^s y^{-1}\, dy$$

$$= r^s \int_0^\infty [y^{1/r'} f^{**}(y^{-1}) g^{**}(y^{-1}) y^{-2}]^s y^{-1}\, dy$$

$$(y = x^{-1})$$

$$= r^s \int_0^\infty [x^{1+1/r} f^{**}(x) g^{**}(x)]^s x^{-1}\, dx\,.$$

There exist positive numbers m_1 and m_2 such that

$$1/m_1 + 1/m_2 = 1 \quad \text{and} \quad 1/m_1 \leqq s/q_1,\ 1/m_2 \leqq s/q_2\,.$$

Now apply Hölder's inequality with the complementary indices m_1, m_2 to the last integral above to obtain

$$\begin{aligned} ||h||^s_{(r,s)} &\leqq r^s \int_0^\infty x^{-1/m_1} [x^{1/p_1} f^{**}(x)]^s x^{-1/m_2} [x^{1/p_2} g^{**}(x)]^s\, dx \\ &\leqq r^s \left\{ \int_0^\infty [x^{1/p_1} f^{**}(x)]^{sm_1} x^{-1}\, dx \right\}^{1/m_1} \left\{ \int_0^\infty [x^{1/p_2} g^{**}(x)]^{sm_2} x^{-1}\, dx \right\}^{1/m_2} \\ &= r^s ||f||^s_{(p_1, sm_1)} ||g||^s_{(p_2, sm_2)} < \infty\,, \end{aligned}$$

the last inequality follows from a well-known theorem of A. P. Calderón (which states that $L(p,q) \subseteqq L(p, q+l)$ if $l \geqq 0$, see O'Neil [5, p. 137] for an easy proof) and $q_1 \leqq sm_1$, $q_2 \leqq sm_2$.

A special case of the preceding theorem is the following corollary, which we single out for easy reference.

(2.3) COROLLARY. *Let G and λ be as in* (2.2) *and let p, q, r be as in Young's inequality, with $p > 1$, $q > 1$, then*

$$L_p(G) * (L_q(G) \cap L_q^\star(G)) \subseteqq L(r,1)(G) .$$

We need the following factorization theorem of Hewitt [1] for later reference. We reproduce only what we need, the original theorem being considerably more refined.

(2.4) THEOREM (E. Hewitt). *Let A be a complex Banach algebra with norm $\|\cdot\|$, and let L be a complex Banach space with norm $|||\cdot|||$. Suppose that there is a mapping of $A \times L$ into L, and we write the image of (μ,x) as $\mu \cdot x$ for all $\mu \in A$, $x \in L$. We further suppose that this mapping has the following properties*:

(i) $(\mu+\nu)\cdot x = (\mu\cdot x)+(\nu\cdot x) = \mu\cdot x+\nu\cdot x$;
$\mu\cdot(x+y) = (\mu\cdot x)+(\mu\cdot y) = \mu\cdot x+\mu\cdot y$;
(ii) $(t\mu)\cdot x = t(\mu\cdot x) = \mu\cdot(tx)$ *for any complex number* t;
(iii) $(\mu\nu)\cdot x = \mu\cdot(\nu\cdot x)$;
(iv) $|||\mu\cdot x||| \leqq c\,\|\mu\|\,|||x|||$, *where c is a real constant* $\geqq 1$;
(v) *for every finite set $\{\mu_1,\ldots,\mu_m\} \subseteqq A$, $x \in L$, and every positive real number a, we can choose a single element $\nu \in A$ such that $\|\nu\| \leqq d$ (d is a positive constant) and $\|\nu\mu_j-\mu_j\| < a$, $j = 1,2,\ldots,m$, and $|||\nu\cdot x - x||| < a$.*

Then $A\cdot L = L$.

Next we obtain two consequences of Hewitt's factorization theorem. The straight-forward proof of the first one is omitted.

(2.5) COROLLARY. *Let $A, L, \cdot$ be as in (2.4) and suppose that A_0 is a dense subset of A containing 0, and L_0 is a dense subset of L containing 0. Then $A_0\cdot L_0$ is dense in L.*

(2.6) COROLLARY. *Let G,p,q,r be as in Young's inequality, then $L_p(G) * (L_q(G) \cap L_q^\star(G))$ is dense in $L_r(G)$.*

PROOF. The preceding Corollary tells us that $C_{00}(G) * C_{00}(G)$ is dense in $L_r(G)$, where $C_{00}(G)$ denotes the set of all continuous functions on G with compact supports.

(2.7) THEOREM. *Let G be an* i n f i n i t e *locally compact group with left Haar measure λ, and let p, q_1, q_2 be real numbers such that $1 < p < \infty$ and $1 \leqq q_1 < q_2 < \infty$. Then $L(p,q_1)(G,\lambda)$ is properly contained in $L(p,q_2)(G,\lambda)$.*

PROOF. That $L(p,q_1) \subseteqq L(p,q_2)$ is the well-known theorem of A. P. Calderón already used in the proof of Theorem (2.2). See O'Neil [5] for a simple proof that if $f \in L(p,q_1)$, then

(i) $$\|f\|_{(p,q_2)} \leqq (q_1 p^{-1})^{1/q_1 - 1/q_2} \|f\|_{(p,q_1)} .$$

Now we proceed to show that there exists a function $F \in L(p,q_2)$, but $F \notin L(p,q_1)$. First we record three simple facts for later use.

(a) Since $1 \leqq q_1 < q_2 < \infty$, there exists $\beta > 0$ such that $q_1\beta < 1$ and $q_2\beta > 1$. Take $\beta = 1/q_2 + (1 - q_1/q_2)/2q_1$, for example.
(b) If we define $\varphi(x) = x^{1/p}(\log x)^{-\beta}$, then φ is a strictly increasing function on $(e^{p\beta}, \infty)$.
(c) $\sum_{n=n_0}^{\infty} n^{-1}(n+1)^{-1} = n_0^{-1}$.

Now we construct the desired function F by considering two cases:

CASE I. Suppose that G is non-discrete. Let n_0 be a positive integer such that $1/n_0 < \lambda(G)$ and $n_0 > e^{p\beta}$, where β is as in (a) above. Next we choose a sequence $(V_n)_{n=n_0}^{\infty}$ of pairwise disjoint λ-measurable subsets of G such that
$$\lambda(V_n) = n^{-1}(n+1)^{-1}, \quad n = n_0, n_0 + 1, \ldots .$$
Define
$$F = \textstyle\sum_{n=n_0}^{\infty} a_n \xi_{V_n} ,$$
where
$$a_n = n^{1/p}(\log n)^{-\beta}, \quad n = n_0, n_0 + 1, \ldots .$$

We assert that $F \in L(p,q_2)$, but $F \notin L(p,q_1)$. Note that we have $a_n < a_{n+1}$, and for $y \geqq 0$,
$$\begin{aligned} m(F,y) &= 1/n_0 && \text{if } 0 \leqq y < a_{n_0} , \\ &= 1/(n+1) && \text{if } a_n \leqq y < a_{n+1} . \end{aligned}$$
Thus
$$F^*(x) = a_n \quad \text{if } 1/(n+1) \leqq x < 1/n .$$
Hence
$$\begin{aligned} \int_0^{\infty} x^{q_2/p-1}[F^*(x)]^{q_2}\,dx &= \int_0^{1/n_0} x^{q_2/p-1}[F^*(x)]^{q_2}\,dx \\ &= \sum_{n=n_0}^{\infty} a_n^{q_2} \int_{1/(n+1)}^{1/n} x^{q_2/p-1}\,dx \\ &= \frac{p}{q_2}\sum_{n=n_0}^{\infty} a_n^{q_2}\left(n^{-q_2/p} - (n+1)^{-q_2/p}\right) \\ &= \frac{p}{q_2}\sum_{n=n_0}^{\infty} \frac{n^{q_2/p}}{(\log n)^{\beta q_2}}\,\frac{q_2}{p(n+\zeta_n)^{q_2/p+1}}, \qquad 0 < \zeta_n < 1 , \\ &\leqq \textstyle\sum_{n=n_0}^{\infty} (\log n)^{-\beta q_2} n^{-1} < \infty . \end{aligned}$$

Therefore $F \in L(p,q_2)$, by virtue of the inequality (ii) of (2.1). A similar computation shows that $F \notin L(p,q_1)$.

Case II. Suppose that G is a discrete group. Since G is infinite, we can choose a sequence $(x_n)_{n=2}^{\infty}$ of distinct points in G. Define

$$\begin{aligned} F(x) &= a_n && \text{if } x=x_n,\ n=2,3,\ldots\,, \\ &= 0 && \text{otherwise}\,, \end{aligned}$$

where

$$a_n = n^{-1/p}(\log n)^{-\beta}, \quad n=2,3,\ldots\,.$$

We note that $a_n > a_{n+1}$,

$$\begin{aligned} m(F,y) &= 0 && \text{if } a_2 \leqq y\,, \\ &= n && \text{if } a_{n+2} \leqq y < a_{n+1}\,, \\ &= \infty && \text{if } y=0\,, \end{aligned}$$

and

$$F^*(x) = a_{n+2} \quad \text{if } n \leqq x < n+1\,.$$

Calculations like those for Case I show that $F \in L(p,q_2)$, but $F \notin L(p,q_1)$. Thus Theorem (2.7) is proved.

Proof of Theorem (1.1). Corollary (2.3) and Theorem (2.7) tell us that $L_p(G) * (L_q(G) \cap L_q{}^*(G))$ is a proper subset of $L_r(G)$, while (2.6) tells us it is dense in $L_r(G)$. The rest of the statements in the Theorem (1.1) follows from Calderón's theorem, the inequality (i) in the proof (2.7), and the following version of the open mapping theorem: Let T be a continuous linear map of a complete pseudometrizable linear topological space E into a (Hausdorff) linear topological space F. If the range of T is of the second category in F, then T maps E *onto* F (see [3, (11.4)]).

The main theorem has two interesting corollaries.

(2.8) Corollary. *Let G be an infinite compact group and let r be any real number such that $1 < r < \infty$. Then $L_r(G) * L_r(G)$ is a dense subset of the first category in $L_r(G)$.*

Proof. Define $p=q=2r/(1+r)>1$. Then we have $1/p+1/q=1+1/r$, $p<r$, and $L_r(G) \subseteqq L_p(G)$. Hence $L_r(G) * L_r(G) \subseteqq L_p(G) * L_q(G)$ which is properly contained in $L_r(G)$.

(2.9) Corollary. *Let G and r be as in (2.9). Then $L_r(G)$ is a Banach algebra without any bounded (in L_r-norm) approximate unit.*

PROOF. If $L_r(G)$ had a bounded approximate unit, then Hewitt's factorization theorem (2.4) would be applicable and one would have $L_r(G) * L_r(G) = L_r(G)$, which is impossible by Corollary (2.8).

3. Remarks and open problems.

(a) Corollary (2.8) is of course well-known for the case $r=2$. In fact, if G is Abelian, then $L_2(G) * L_2(G)$ is just the space of functions on G with absolutely convergent Fourier series. In the general (non-Abelian) case, M. G. Krein [4] has shown that $L_2(G) * L_2(G)$ is the space of all complex linear combinations of continuous positive-definite functions on G. It will be interesting to have an exact description of the set $L_r(G) * L_r(G)$ for $r \neq 2$.

(b) When is $L_p(G) * (L_q(G) \cap L_q^\star(G))$ a linear space?

Added in proof.

Dr. G. I. Gaudry has informed me that he has obtained a stronger version of theorem (1.1) when G is the n-dimensional Euclidean space.

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