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PARTIAL HYPOELLIPTICITY AND GEVREY CLASSES

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1. Statement of results.

By definition a linear partial differential operator $P(D), D=\partial/i\partial x$, with constant complex coefficients is called hypoelliptic if all distribution solutions u of $P(D)u=0$ in an open set Ω of R^n are infinitely differentiable functions. Hörmander showed in his thesis [7] that $P(D)$ is hypoelliptic if and only if there exist constants $C>0$ and $b \geqq 1$ such that

$$(1) \qquad |\mathrm{Re}\,\zeta|^{1/b} \leqq C(1+|\mathrm{Im}\,\zeta|), \quad \zeta \in C^n,\ P(\zeta)=0 .$$

A corresponding characterization of partially hypoelliptic operators was later given by Gårding and Malgrange [5]. Hörmander also noticed that $P(D)$ satisfies (1) for a given $b \geqq 1$ if and only if all solutions of $P(D)u=0$ are of Gevrey class b. This was generalized by Friberg [3] to the partial hypoelliptic case. We shall deal with the same classes of operators as Friberg in this paper. But instead of considering distribution solutions we shall see that one also can characterize these operators by C^∞ solutions. This will be proved by a method which is a generalization of chapter 4.4 in Hörmander's book [6]. By regularization we then obtain another proof of Friberg's results.

We shall use the notation in [6] to a large extent and we also need the following modification of Definition 4.4.2 in [6].

Definition 1. Let Ω be an open set in R^n and set $a=(a_1,\ldots,a_n)$ with $0 \leqq a_j \leqq \infty$. By $\Gamma^a(\Omega)$ we denote the set of functions $u \in C^\infty(\Omega)$ such that for every compact set $K \subset \Omega$ there is a constant C for which the inequality

$$(2) \qquad |D^\alpha u, K|_\infty = \sup\nolimits_K |D^\alpha u| \leqq C^{|\alpha|+1}\, \alpha^{a\alpha}$$

is valid for every multi-index α. Here

$$\alpha^{a\alpha} = \alpha_1^{a_1\alpha_1}\ldots\alpha_n^{a_n\alpha_n}$$

and we use the convention that

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$$\alpha_j^{a_j\alpha_j} = \begin{cases} 1 & \text{if} \quad \alpha_j = 0 , \\ \infty & \text{if} \quad \alpha_j > 0 \text{ and } a_j = \infty . \end{cases}$$

It follows from Heine–Borel's theorem that $u \in \Gamma^a(\Omega)$ if every point $x \in \Omega$ has an open neighbourhood V_x such that the restriction of u to V_x belongs to $\Gamma^a(V_x)$. Observe that, when $a_j = \infty$, the inequality (2) puts no restriction on $D^\alpha u$ with $\alpha_j > 0$. In particular, $\Gamma^a(\Omega) = C^\infty(\Omega)$ if all $a_j = \infty$. If all $a_j < \infty$, then we write $a < \infty$ and $a \geqq 1$ has a similar meaning. When some $a_j = \infty$ we shall often choose the coordinates such that $a_j = \infty$ if $j > n'$. Then we write $a = (a', \infty)$, where $a' = (a_1, \ldots, a_{n'})$ and split the coordinates $x = (x_1, \ldots, x_n)$ into $x = (x', x'')$, writing

$$x' = (x_1, \ldots, x_{n'}) \in \mathsf{R}^{n'}, \quad x'' = (x_{n'+1}, \ldots, x_n) = \mathsf{R}^{n''}$$

and $n' + n'' = n$. We denote the closed ball $|x| \leqq \sigma$ in R^n by B_σ and we use the norm $|x| = \max |x_j|$ so that $B_\sigma = B_\sigma' \times B_\sigma''$, where B_σ' and B_σ'' are the corresponding balls in $\mathsf{R}^{n'}$ and $\mathsf{R}^{n''}$ respectively.

If $a_j < \infty$, then $u \in \Gamma^a(\Omega)$ implies $D_j u \in \Gamma^a(\Omega)$, since

$$(\alpha_j + 1)^{a_j(\alpha_j+1)} \leqq C^{\alpha_j} \alpha_j^{a_j\alpha_j} \quad \text{for some constant } C .$$

Hence $\Gamma^a(\Omega)$ is invariant under derivation if $a < \infty$. On the other hand, if some component of a is infinite, we have the following

Theorem 1. *If $a = (a', \infty)$, then $\Gamma^a(\Omega)$ is invariant under derivation if and only if $a' \geqq 1$.*

Denote by Ω_σ the open set of points in Ω with distance $> \sigma > 0$ to the boundary of Ω. If $u \in \mathscr{D}'(\Omega)$ and $\varphi \in C_0^\infty(B_\sigma)$, then $u * \varphi \in C^\infty(\Omega_\sigma)$.

Definition 2. By $\tilde{\Gamma}^a(\Omega)$ we denote the set of distributions $u \in \mathscr{D}'(\Omega)$ such that $u * \varphi \in \Gamma^a(\Omega_\sigma)$ for every $\varphi \in C_0^\infty(B_\sigma)$ and $\sigma > 0$.

Schwartz [8] has shown that $\tilde{\Gamma}^a(\Omega) = \Gamma^a(\Omega)$ if $a < \infty$. If $a = (a', \infty)$ we get a definition of distributions which belongs to $\Gamma^{a'}(\Omega)$ in the variable x'. The next theorem gives a condition under which this definition is equivalent with those of Friberg [3] and Gorin [4] (IV respectively II below).

Theorem 2. *Let $1 \leqq a' < \infty$. Then the following conditions on a distribution u in an open set Ω are equivalent.*

I. $u \in \tilde{\Gamma}^{(a', \infty)}(\Omega)$.

II. *To each open bounded ball ω with $\bar{\omega} \subset \Omega$ there exists a function v,* *such that* $u = D^{\beta''}v$ *for some* β'' *and with all derivatives* $D^{\alpha'}v$ *continous in* ω *and satisfying for some constant* C

$$(3) \qquad |D^{\alpha}v, \omega|_{\infty} \leqq C^{|\alpha|+1}\, \alpha^{a\alpha}, \quad \alpha'' = 0\,.$$

III. *For every* $\sigma > 0$,

$$u*(\delta \otimes \varphi) \in \Gamma^{(a',\infty)}(\Omega_{\sigma}) \quad \text{if } \varphi \in C_0^{\infty}(B_{\sigma}'')\,.$$

Here δ *is the Dirac measure at* 0 *in* $\mathsf{R}^{n'}$.

IV. *For all open sets* $\Omega' \subset \mathsf{R}^{n'}$ *and* $\Omega'' \subset \mathsf{R}^{n''}$ *with* $\Omega' \times \Omega'' \subset \Omega$ *and every* $\varphi \in C_0^{\infty}(\Omega'')$ *the distribution in* x'

$$u_{\varphi}(x') = \int u(x', x'')\, \varphi(x'')\, dx''$$

belongs to $\Gamma^{a'}(\Omega')$.

Corollary. $\tilde{\Gamma}^{a}(\Omega) = \Gamma^{a}(\Omega)$ *if and only if* $a < \infty$. *When* $a = (a', \infty)$, *then*

$$\tilde{\Gamma}^{a}(\Omega) \cap C^{\infty}(\Omega) = \Gamma^{a}(\Omega)$$

if and only if $a' \geqq 1$.

Now we have the following extension of Friberg's result.

Theorem 3. *Let* $a = (a_1, \ldots, a_n)$ *and* $b = (b_1, \ldots, b_n)$ *with* a_j *and* b_j *rational numbers* $\geqq 1$ *if not infinite. Then the following conditions on* $P(D)$ *are equivalent.*

1) *For some open nonempty subset* Ω *of* R^n,

$$P(D)u = 0,\ u \in \Gamma^{a}(\Omega) \Rightarrow u \in \Gamma^{b}(\Omega)\,.$$

2) *There exists a positive constant* C *such that for all* $\zeta \in \mathsf{C}^n$ *with* $P(\zeta) = 0$,

$$|\zeta|_b = \sum_1^n |\zeta_j|^{1/b_j} \leqq C(1 + |\mathrm{Im}\, \zeta| + |\zeta|_a)\,.$$

3) *There exists a positive constant* C *such that for all* $\xi \in \mathsf{R}^n$,

$$|\xi|_b \leqq C\big(1 + d(\xi) + |\xi|_a\big)\,.$$

Here $d(\xi)$ *is the distance from* ξ *to the surface* $P(\zeta) = 0$ *in* C^n.

4) *For all open subsets* Ω *of* R^n,

$$P(D)u = 0,\ u \in \Gamma^{a}(\Omega) \Rightarrow u \in \Gamma^{b}(\Omega)\,.$$

5) *For all open subsets* Ω *of* R^n,

$$P(D)u = 0,\ u \in \tilde{\Gamma}^{a}(\Omega) \Rightarrow u \in \tilde{\Gamma}^{b}(\Omega)\,.$$

6) *For some open nonempty subset Ω of* R^n,

$$P(D)u=0,\ u \in \tilde{\Gamma}^a(\Omega) \ \Rightarrow\ u \in \tilde{\Gamma}^b(\Omega)\,.$$

Finally I want to thank Lars Gårding for suggesting the subject of this paper and for guiding me in the course of my work.

2. Proofs.

We need some preliminaries for the proof of theorem 1. Let K be a compact subset of Ω and choose $\varepsilon>0$ such that $K^{2\varepsilon}\subset\Omega$. Here $K^{2\varepsilon}$ is the compact set of points in R^n with distance $\leqq 2\varepsilon$ from K. Take a non-negative function

$$\varphi \in C_0^\infty(B_\varepsilon) \quad \text{with} \quad \int \varphi(x)\,dx = 1$$

and define for any natural number k the function φ_k by

$$\varphi_k(x) = k^n\,\varphi(kx)\,.$$

If χ is the characteristic function of K^ε, set

$$\chi_k = \varphi_k * \cdots * \varphi_k * \chi\,, \tag{4}$$

where φ_k occurs k times in the convolution.

Lemma 1. *For every k is the function χ_k defined by* (4) *in $C_0^\infty(K^{2\varepsilon})$ and is equal to* 1 *in K. Moreover, if*

$$\int |D_j\varphi(x)|\,dx \leqq C \quad \textit{for each } j\,,$$

then

$$|D^\alpha\chi_k(x)| \leqq (Ck)^{|\alpha|}, \quad |\alpha|\leqq k,\ x\in\mathsf{R}^n\,. \tag{5}$$

Proof. See Boman [1].

Lemma 2. *Let $a=(a',\infty)$ with $1\leqq a'<\infty$. Then, if $u\in\Gamma^a(\Omega)$, there exist a constant C, independent of k, such that for $k=|\alpha|$*

$$\|D^\alpha(\chi_k u)\|_2 = \left(\int |D^\alpha(\chi_k u)(x)|^2\,dx\right)^{\frac{1}{2}} \leqq C^{|\alpha|}\,\alpha^{a\alpha}, \tag{6}$$

$$\|D_n{}^2(\chi_k u)\|_2 \leqq C^{|\alpha|+1}\,.$$

Proof. It is enough to consider the case $\alpha''=0$. Since the support of χ_k is contained in the compact set $K^{2\varepsilon}\subset\Omega$, we get from the definition of $\Gamma^a(\Omega)$ and (5) that for $\beta\leqq\alpha$

$$\|D^{\alpha-\beta}\chi_k D^\beta u\|_2 \leqq (Ck)^{|\alpha-\beta|} C^{|\beta|+1} \beta^{a\beta} \leqq C^{|\alpha|+1} k^{a(\alpha-\beta)} |\alpha|^{a\beta} = C^{|\alpha|+1} |\alpha|^{a\alpha},$$

where $a\alpha = a_1\alpha_1 + \ldots + a_{n'}\alpha_{n'}$. But

$$|\alpha|^{a\alpha} \leqq (a\alpha)^{a\alpha} \leqq (a_1 + \ldots + a_{n'})^{a\alpha} \alpha^{a\alpha}$$

because $t\log t$ is a convex function of $t > 0$. Hence Leibniz's formula gives with a larger C

$$\|D^\alpha(\chi_k u)\|_2 \leqq C^{|\alpha|+1} \alpha^{a\alpha} \sum_{\beta\leqq\alpha} \binom{\alpha}{\beta} \leqq (nC)^{|\alpha|+1} \alpha^{a\alpha}.$$

In the same way we get

$$\|D_n^2(\chi_k u)\|_2 \leqq (1+Ck)^2 \max_{0\leqq j\leqq 2} \left(\int_{K^{2\varepsilon}} |D_n^j u|^2 \, dx \right)^{\frac{1}{2}},$$

which is less than $C^{|\alpha|+1}$ for another C, depending of course on u.

Proof of Theorem 1. Parseval's formula and the Cauchy inequality give

$$\|D^\alpha D_n v\|_2 \leqq \|D^{2\alpha} v\|_2^{\frac{1}{2}} \|D_n^2 v\|_2^{\frac{1}{2}}, \quad v \in C_0^\infty(\mathsf{R}^n).$$

If we apply this with $\alpha'' = 0$ to $v = \chi_{2k} u$ and use (6) we obtain

$$\|D^\alpha D_n(\chi_{2k} u)\|_2 \leqq C^{|\alpha|+\frac{1}{2}} (2\alpha)^{a\alpha} C^{|\alpha|+\frac{1}{2}},$$

which is less than $C^{|\alpha|+1}\alpha^{a\alpha}$ for another C. Since $\chi_{2k} = 1$ in K, we have proved that $u \in \Gamma^a(\Omega)$ implies $D_n u \in \Gamma^a(\Omega)$ if we use L^2-norms instead of supremum norms in Definition 2. In the same way we see that $\Gamma^a(\Omega)$ is invariant under D_j for $j > n'$ and we have already seen that this is valid also for $j \leqq n'$. Hence $\Gamma^a(\Omega)$ is invariant under derivation and then we can use Sobolev's lemma to pass from L^2-norms to supremum norms (cf. formula 4.4.7 in [6]).

In order to prove the converse assume $\Gamma^a(\Omega)$ invariant under derivation. Let F be the Fréchet space of all $u \in C^\infty(\Omega)$ such that the seminorms

$$\|u\|_{K,m} = \max_{|\alpha|\leqq m} |D^\alpha u, K|_\infty + \sup_\alpha \alpha^{-a\alpha} |D^\alpha u, K|_\infty$$

are finite for every compact set $K \subset \Omega$ and $m = 0, 1, 2, \ldots$. Take a compact set $K_0 \subset \Omega$. According to the assumption, the union for $r = 1, 2, \ldots$ of

$$F(r) = \{u \in F\,;\ |D^\alpha D_n u, K_0|_\infty \leqq r^{|\alpha|+1}\alpha^{a\alpha} \text{ for every } \alpha\}$$

is equal to F. Moreover, $F(r)$ is closed in F, because the mapping

$$F \ni u \to D^\alpha D_n u|_{K_0} \in C(K_0)$$

is continuous. It follows from Baire's theorem that $F(r)$ has an interior point for some r_0. Since $F(r_0)$ is convex and symmetric, the origin must be an interior point. Hence there is a constant $\delta > 0$, a compact set K and a natural number m such that if $||u|| = \delta$, then $u \in F(r_0)$, that is

$$(7) \qquad |D^\alpha D_n u, K_0|_\infty \leqq r_0^{|\alpha|+1} \alpha^{a\alpha} \delta^{-1} ||u||_{K,m}, \quad \alpha \geqq 0 .$$

This inequality is valid for all $u \in F$ since it is homogeneous with respect to u.

Now if not all components of a' are greater than or equal to 1, we shall deduce a contradiction from (7). Let us assume that $a_1 = d < 1$. Take then $K_0 = \{x^0\}$ and set

$$u(x) = (x_1 - x_1{}^0)^k \varphi((\log k)^k (x_n - x_n{}^0))$$

in (7) with $\alpha = (k, 0, \ldots, 0)$. We choose $\varphi \in C^\infty(R)$ such that all derivatives of φ are bounded and $\varphi'(0) = 1$. This gives with a large C

$$(k!)(\log k)^k \leqq C^{k+1} k^{kd} (C^m k^m (\log k)^{mk} + C^k \sup_{0 \leqq j \leqq k} j^{-jd} k^j) .$$

Since $d < 1$ and $j^{-jd} k^j$ is an increasing function of j when $0 \leqq j \leqq e^{-1} k^{1/d}$, we obtain for large k

$$(k!)(\log k)^k \leqq C^{k+1} (k^k + C^k k^k) .$$

This and the inequality $k! \geqq e^{-k} k^k$ gives the desired contradiction.

REMARK. It is possible to avoid Baire's theorem by the following explicit construction. Let $x \in R^2$ for notational simplicity and set for $d < 1$

$$u(x) = \sum_{k=0}^\infty k^{kd} \varphi((\log k)^k x_2) x_1{}^k / k! ,$$

where φ is chosen as above and such that $\varphi(x) = 1$ if $|x| \geqq 1$. Then one can show that $u \in \Gamma^{(d,\infty)}(R^2)$. But $D_2 u \notin \Gamma^{(d,\infty)}(R^2)$, since $D_1{}^k D_2 u(0,0) = k^{kd} (\log k)^k$.

PROOF THAT I IMPLIES II IN THEOREM 2. Let $y \in \Omega$ and choose a number $\sigma > 0$ and an open ball ω with center y entirely contained in Ω_σ. Then for each $\varphi \in C_0^\infty(B_\sigma)$ there exists a constant $C = C(\varphi)$ such that

$$\sup_\omega |D^\alpha (u * \varphi)| \leqq C^{|\alpha|+1} \alpha^{a\alpha} .$$

It follows that the Fréchet space $C_0^\infty(B_\sigma)$ is the union for $r = 1, 2, \ldots$ of all φ with $C(\varphi) \leqq r$. An application of Baire's theorem, as in the proof of theorem 1, shows that for some C and m

$$(8) \quad \sup_\omega |D^\alpha (u * \varphi)| \leqq C^{|\alpha|+1} \alpha^{a\alpha} \max_{|\beta| \leqq m} |D^\beta \varphi, B_\sigma|_\infty, \quad \varphi \in C_0^\infty(B_\sigma), \ \alpha \geqq 0 .$$

Hence the derivatives $D^{\alpha'}(u*\varphi)$ are continuous and $u*\varphi$ satisfies (3) for every $\varphi \in C_0^m(B_\sigma)$, since this space is dense in $C_0^\infty(B_\sigma)$. Now put

$$\psi(x) = \textstyle\prod_{i=1}^n x_i^{m+1}/(m+1)!$$

if all $x_i \geqq 0$ and $\psi(x)=0$ otherwise, and let $\varphi=\psi\gamma$, where $\gamma \in C_0^\infty(B_\sigma)$ equals 1 in $B_{\frac{1}{2}\sigma}$. Then $\varphi \in C_0^m(B_\sigma)$ and $D^{m+2}\varphi=\delta+\zeta$, where $D=D_1\ldots D_n$ and $\zeta \in C_0^\infty(B_\sigma)$. It follows that in ω

$$u = D^{m+2}(u*\varphi) - u*\zeta . \tag{9}$$

Now ω is a product of intervals $y_i^0 < x_i < y_i^1$ and hence

$$w(x) = \int \psi(x-z)(u*\zeta)(z)\,dz, \qquad z_i > y_i^0 \text{ for all } i ,$$

satisfies $D^{m+2}w = u*\zeta$ in ω and has the property (3). Hence, since derivation with respect to x' does not influence (3),

$$v = (D')^{m+2}(u*\varphi - w)$$

has the desired properties.

Proof that II implies III in Theorem 2. It suffices to show that every $y \in \Omega_\sigma$ is the center of a ball $\omega_0 \subset \Omega$ such that

$$\sup_{\omega_0} |D^\alpha u * (\delta \otimes \varphi)| \leqq C^{|\alpha|+1}\, \alpha^{a\alpha}$$

for some C and all α. Let $\omega \subset \Omega$ be a ball with center y and radius $> \frac{1}{2}\sigma$ such that (3) holds and choose ω_0 so that $\omega_0 + B_{\frac{1}{2}\sigma} \subset \omega$. The desired estimate is then obtained from (3) and the formula

$$(D^\alpha u * (\delta \otimes \varphi))(x) = \int_{B_\sigma''} (D^\alpha v)(x', x''-y'')\, D^{\beta''}\varphi(y'')\, dy'' .$$

Proof that III implies IV in Theorem 2. Fix a constant $c>0$ such that $\operatorname{supp}\varphi \in \Omega_c''$ and take a σ with $0<\sigma\leqq c$. By a partition of unity we can then find a finite number of functions $\varphi_k \in C_0^\infty(B_\sigma'')$ and an equal number of points $a_k'' \in \Omega_\sigma''$ such that

$$\varphi(x'') = \textstyle\sum \varphi_k(a_k''-x'') .$$

This gives

$$u_\varphi(x') = \textstyle\sum_k u*(\delta\otimes\varphi_k)(x', a_k'') .$$

Since $\Omega_\sigma' \times \Omega_\sigma'' \subset \Omega_\sigma$, it follows that $u_\varphi \in \Gamma^{a'}(\Omega_\sigma')$ for all σ with $0<\sigma\leqq c$. Therefore $u_\varphi \in \Gamma^{a'}(\Omega)$, because this is a local property of u_φ.

PROOF THAT IV IMPLIES I IN THEOREM 2. Let $K' \subset \Omega'$ and $K'' \subset \Omega''$ be compact sets. In the same way as in the proof of theorem 1 we get

$$\sup_{K'} |D^{\alpha'} u| \leqq C^{|\alpha|+1} \alpha^{a\alpha} \max_{|\beta| \leq m} |D^{\beta''} \varphi, K''|_{\infty}, \quad \varphi \in C_0^\infty(K''), \ \alpha''=0 .$$

Take $\varphi \in C_0^\infty(B_\sigma)$ and set $\omega = K_\sigma' \times K_\sigma''$. This inequality and the formula

$$D^{\alpha'}(u * \varphi)(x) = \int \left(\int D^{\alpha'} u(y)\, \varphi(x-y)\, dy'' \right) dy'$$

gives with a larger C that

$$\sup_\omega |D^{\alpha'}(u * \varphi)| \leqq C^{|\alpha|+1} \alpha^{a\alpha} .$$

In fact, $D^{\beta''}\varphi$ is bounded on B_σ and $y'' \to \varphi(x-y)$ has its support in $\{x''\} + B_\sigma'' \subset K''$ if $x \in \omega$. It follows that $u * \varphi \in \Gamma^{(a',\infty)}(\Omega_\sigma)$ because every point of Ω_σ is contained in some open set $\omega = K_\sigma' \times K_\sigma''$.

PROOF OF THE COROLLARY. It is evident that if $u \in \Gamma^a(\Omega)$, then $u \in \tilde{\Gamma}^a(\Omega)$. Conversely, if $a < \infty$ and $u \in \tilde{\Gamma}(\Omega)$, then (9) shows that $u \in \Gamma^a(\Omega)$. In fact, $u * \varphi$ and $u * \zeta$ belong to Γ^α and Γ^α is invariant under derivation. Hence $\tilde{\Gamma}^a(\Omega) = \Gamma^a(\Omega)$ if $a < \infty$. Here $a < \infty$ is also a necessary condition, since when $a = (a', \infty)$, then $u \in \tilde{\Gamma}^a$ and $u \notin \Gamma^a$ for $u(x) = v(x')w(x'')$, where $v \in \Gamma^{a'}$ and w continuous but not C^∞.

The sufficiency of $a' \geqq 1$ for the second part of the corollary follows in the same way as above with the help of theorem 1. The converse also follows from this theorem, since $\tilde{\Gamma}^a \cap C^\infty$ obviously is invariant under derivation.

PROOF THAT 1) IMPLIES 2) IN THEOREM 3. We divide this proof into three steps.

Step 1. Assume that all a_j are finite.

Put for $u \in C^\infty(\Omega)$ and compact $K \subset \Omega$

$$||u||_K = \sup_{\alpha > 0} \alpha^{-a\alpha} |D^\alpha u, K|_\infty$$

and let N be the Fréchet space of all $u \in C^\infty(\Omega)$ such that $P(D)u = 0$ and $||u||_K < \infty$ for all $K \subset \Omega$. Take a compact set $K_0 \subset \Omega$. According to the assumption in (1), the union for $r = 1, 2, \ldots$ of

$$N(r) = \{u \in N \, ; \ |D^\beta u, K_0| \leqq r^{|\beta|+1} \beta^{b\beta} \text{ for every } \beta\}$$

is equal to N. It follows from Baire's theorem, as in the proof of theorem 1, that for some constant C and compact $K \subset \Omega$

$$(10) \qquad |D^\beta u, K_0|_\infty \leqq C^{|\beta|+1} \beta^{b\beta} \, ||u||_K, \quad u \in N, \ \beta \geqq 0 .$$

Take any $\zeta \in \mathbf{C}^n$ with $P(\zeta)=0$ and set $u(x)=e^{i\langle x,\zeta\rangle}$ in (10). If A is an upper bound for $|x|$ when $x \in K$, this gives with an $x_0 \in K$

$$|\zeta^\beta|\, e^{-\langle x_0, \operatorname{Im}\zeta\rangle} \leqq C^{|\beta|+1}\, \beta^{b\beta}\, e^{A|\operatorname{Im}\zeta|} \sup_\alpha \alpha^{-a\alpha}|\zeta^\alpha| .$$

Since

$$\alpha_j^{-a_j\alpha_j}\, |\zeta_j|^{\alpha_j} \leqq \exp(ae^{-1}|\zeta_j|^{1/a_j})$$

we get by putting $\beta=(0,\ldots,\beta_j,\ldots,0)$ that for some constants c and C

$$|\zeta_j|^{1/b_j} \leqq C\beta_j\, e^{c(|\operatorname{Im}\zeta|+|\zeta|_a)/\beta_j}, \quad \beta_j>0 .$$

If we choose β_j as the integral part of $1+|\operatorname{Im}\zeta|+|\zeta|_a$, it follows that for another constant C

$$|\zeta_j|^{1/b_j} \leqq C(1+|\operatorname{Im}\zeta|+|\zeta|_a) ,$$

and adding this for $j=1,2,\ldots,n$ gives the desired inequality.

Step 2. Assume that all b_j are finite.

If all $a_j<\infty$, then we are ready according to step 1. So assume $a_1=\ldots=a_j=\infty$ and $a_{j+1},\ldots,a_n$ finite. Since $\Gamma^a(\Omega)$ shrinks with a, we have by the assumption in (1) that for $\varepsilon>0$

$$P(D)u=0, \quad u \in \Gamma^{(b_1+\varepsilon,\ldots,b_j+\varepsilon,a_{j+1},\ldots,a_n)}(\Omega) \Rightarrow u \in \Gamma^b(\Omega) .$$

It follows from step 1 that

$$|\zeta|_b \leqq C(1+|\operatorname{Im}\zeta|+|\zeta_1|^{1/(b_1+\varepsilon)}+\ldots+|\zeta_j|^{1/(b_j+\varepsilon)}+|\zeta_{j+1}|^{1/a_{j+1}}+\ldots+|\zeta_n|^{1/a_n})$$

when $P(\zeta)=0$, which gives the desired inequality with a larger constant C.

Step 3. Assume $b_1,\ldots,b_j$ finite and $b_{j+1},\ldots,b_n$ infinite.

It is then enough to take $a_{j+1},\ldots,a_n$ infinite and, by a reduction as in step 2, $a_1,\ldots,a_j$ finite. Now we define the space N and seminorms $\|\cdot\|_K$ as in step 1. In order to get N complete we also use the seminorms

$$\|u\|_{K,m} = \max_{|\alpha|\leqq m}|D^\alpha u, K|_\infty, \quad m=0,1,2,\ldots .$$

Then we get in the same way as in step 1

$$|D^\beta u, K_0|_\infty \leqq C^{|\beta|+1}\, \beta^{b\beta}\, (\|u\|_K+\|u\|_{K,m}), \quad \beta \geqq 0 ,$$

which gives

$$(11) \qquad |\zeta|_b \leqq C\big(1+|\operatorname{Im}\zeta|+|\zeta|_a+\log(1+|\zeta|)\big) \quad \text{if } P(\zeta)=0 .$$

Since a_i and b_i are finite positive rational numbers for $1\leqq i\leqq j$, there exists natural numbers A_i, B_i and N such that

$$1/a_i = A_i/N \quad\text{and}\quad 1/b_i = B_i/N, \qquad 1\leqq i\leqq j ,$$

and (11) is then equivalent to

$$\Sigma_1^j |\zeta_i|^{2B_i} \leqq C(1 + |\mathrm{Im}\,\zeta|^{2N} + \Sigma_1^j |\zeta_i|^{2A_i} + \log(1+|\zeta|)) \quad \text{if } P(\zeta)=0\,.$$

Hence by putting

$$\mu(\tau) = \sup\{\Sigma_1^j |\zeta_i|^{2B_i}(1+|\mathrm{Im}\,\zeta|^{2N}+\Sigma_1^j|\zeta_i|^{2A_i})^{-1}\,;\ P(\zeta)=0,\ |\zeta|\leqq\tau\}$$

we get

$$\mu(\tau) \leqq C + C\log(1+\tau)\,. \tag{12}$$

But according to a lemma by Seidenberg (see Lemma 2.1 in the Appendix of [6])

$$\mu(\tau) = A\tau^r(1+o(1)), \qquad \tau \to +\infty\,.$$

Here $r \leqq 0$ by (12), so that $\mu(\tau)$ is bounded, that is,

$$\Sigma_1^j |\zeta_i|^{2B_i} \leqq C(1+|\mathrm{Im}\,\zeta|^{2N}+\Sigma_1^j|\zeta_i|^{2A_i}) \quad \text{or} \quad |\zeta|_k \leqq C(1+|\mathrm{Im}\,\zeta|+|\zeta|_a)$$

if $P(\zeta)=0$.

Remark. If we only consider the partial hypoelliptic operators of Gårding and Malgrange then step 1 and 2 is enough so we do not need the lemma of Seidenberg in that case. Step 3 is in order to take care of the somewhat more general operators of Friberg.

Proof that 2) implies 3) in Theorem 3. If $\xi \in \mathsf{R}^n$ choose $\zeta^0 = \xi^0 + i\eta^0$ so that $P(\zeta^0)=0$ and $d(\xi) = |\xi - \zeta^0|$. Then we have

$$|\xi-\xi^0| \leqq d(\xi) \quad \text{and} \quad |\eta^0| \leqq d(\xi)\,.$$

Since all $b_j \geqq 1$ we get

$$|\xi|_b \leqq C(|\xi-\zeta^0|_b+|\zeta^0|_b) \leqq C(1+|\xi-\zeta^0|+|\zeta^0|_b) = C(1+d(\xi)+|\zeta^0|_b)\,. \tag{13}$$

But by 2)

$$|\zeta^0|_b \leqq C(1+|\mathrm{Im}\,\zeta^0|+|\zeta^0|_a)\,.$$

Here $|\mathrm{Im}\,\zeta^0| = |\eta^0| \leqq d(\xi)$ and

$$\begin{aligned} |\zeta^0|_a &\leqq (|\xi^0-\xi|_a+|\eta^0|_a+|\xi|_a) \\ &\leqq C(1+|\xi^0-\xi|+1+|\eta^0|+|\xi|_a) \leqq C(1+d(\xi)+|\xi|_a)\,, \end{aligned}$$

so that

$$|\zeta^0|_b \leqq C(1+d(\xi)+|\xi|_a)\,.$$

Now the desired inequality follows from (13).

In the proof of the next implication in theorem 3 we need the following modification of theorem 4.4.2 in [6].

LEMMA 3. *Suppose*

$$\sum_1^n |\xi_j|^{B_j} \leqq C\big((1+d(\xi))^N + \sum_1^n |\xi_j|^{A_j}\big)\,, \tag{14}$$

where A_j, B_j and N are natural numbers with $A_j \leqq N$ for all j. Then there is a constant C, independent of u, δ and δ_1 such that for every $u \in C^\infty(\Omega)$ with $P(D)u=0$ and all $\delta,\delta_1>0$ we have

$$\sum_1^n \|D_j^{B_j} u, \Omega_{\delta+\delta_1}\|_{P,\delta} \leqq C\sum_1^n \|D_j^{A_j} u, \Omega_{\delta_1}\|_{P,\delta} + C\delta^{-N}\, \|u,\Omega_{\delta_1}\|_{P,\delta}\,, \tag{15}$$

where

$$\|u,\omega\|_{P,\delta}^2 = \sum_{\alpha\neq 0} \delta^{-2|\alpha|} \int_\omega |P^{(\alpha)}(D)\,u(x)|^2\,dx\,.$$

PROOF. We use the same notation as in chapter 4.4 of [6]. It follows from (14) that

$$\sum_1^n |\xi_j|^{B_j} \leqq C\big(\varepsilon^{-N} d_{N,\varepsilon}(\xi) + \sum_1^n |\xi_j|^{A_j}\big) \quad \text{if } 0<\varepsilon<1 \text{ and } \xi \in \mathsf{R}^n\,.$$

Hence Parseval's formula gives

$$\sum_1^n \int |D_j^{B_j} v|^2\,dx \leqq C\left(\varepsilon^{2N} |||v|||_{N,\varepsilon}^2 + \sum_1^n \int |D_j^{A_j} v|^2 dx\right), \quad v \in C_0^\infty(\mathsf{R}^n)\,.$$

Let $\varphi \in C_0^\infty(B_1)$ be equal to 1 in $B_{\frac12}$. If we apply the above estimate to $v = P^{(\alpha)}(D)(\varphi^\varepsilon u)$, we obtain

$$\begin{aligned}\sum_1^n \|D_j^{B_j} u, B_{\frac12\varepsilon}\|_{P,\varepsilon}^2 &\leqq \sum_1^n \sum_{\alpha\neq 0} \varepsilon^{-2|\alpha|} \int |D_j^{B_j} P^{(\alpha)}(D)(\varphi^\varepsilon u)|^2\,dx \\ &\leqq C\varepsilon^{-2N} \sum_{\alpha\neq 0} \varepsilon^{-2|\alpha|}\, |||P^{(\alpha)}(D)(\varphi^\varepsilon u)|||_{N,\varepsilon}^2 + \\ &\quad + \sum_1^n \sum_{\alpha\neq 0} \varepsilon^{-2|\alpha|} \int |D_j^{A_j} P^{(\alpha)}(D)(\varphi^\varepsilon u)|^2\,dx\,.\end{aligned} \tag{16}$$

But, according to Lemma 4.4.3 in [6]

$$\sum_{\alpha\neq 0} \varepsilon^{-2|\alpha|}\, |||P^{(\alpha)}(D)(\varphi^\varepsilon u)|||_{N,\varepsilon}^2 \leqq C\|u, B_\varepsilon\|_{P,\varepsilon}^2\,, \tag{17}$$

and from the identity

$$\begin{aligned}&\varepsilon^{-|\alpha|}\, D_j^{A_j} P^{(\alpha)}(D)(\varphi^\varepsilon u) \\ &\quad = \sum_\beta \sum_{0\leqq k_j\leqq A_j} \varepsilon^{-k_j} \binom{A_j}{k_j} \left(\varepsilon^{k_j+|\beta|} D_j^{k_j} D^\beta \varphi^\varepsilon/\beta!\right) \varepsilon^{-|\alpha+\beta|} D_j^{A_j-k_j} P^{(\alpha-\beta)}(D)u\end{aligned}$$

we get from Lemma 4.4.2 in [6] that

$$\begin{aligned} I &= \sum_{\alpha\neq 0} \varepsilon^{-2|\alpha|} \int |D_j^{A_j} P^{(\alpha)}(D)(\varphi^\varepsilon u)|^2\,dx \\ &\leqq C \sum_{\alpha\neq 0} \varepsilon^{-2|\alpha|} \sum_{0\leqq k_j\leqq A_j} \varepsilon^{-2k_j} \int_{B_\varepsilon} |D_j^{A_j-k_j} P^{(\alpha)}(D)u|^2\,dx\,.\end{aligned}$$

Now (see e.g. Ehrling [2, p. 270])

$$\int_{-1}^{1} |f^{(k)}|^2\,dx \leqq C\int_{-1}^{1}(|f|^2+|f^{(n)}|^2)\,dx, \quad 0<k<n,\ f\in C^\infty(R)\ ,$$

and this gives

$$\varepsilon^{-2k_j}\int_{B_\varepsilon} |D_j^{A_j-k_j}P^{(\alpha)}(D)u|^2\,dx \leqq C\int_{B_\varepsilon}\left(|D_j^{A_j}P^{(\alpha)}(D)u|^2+\varepsilon^{-2A_j}|P^{(\alpha)}(D)u|^2\right)dx$$

so that

$$I \leqq C\|D_j^{A_j}u,B_\varepsilon\|_{P,\varepsilon}^2 + C\varepsilon^{-2A_j}\|u,B_\varepsilon\|_{P,\varepsilon}^2\ . \tag{18}$$

Since $A_j \leqq N$, we get from (16), (17) and (18) that

$$\sum_1^n \|D_j^{B_j}u,B_{\frac{1}{2}\varepsilon}\|_{P,\varepsilon}^2 \leqq C\sum_1^n \|D_j^{A_j}u,B_\varepsilon\|_{P,\varepsilon}^2 + C\varepsilon^{-2N}\|u,B_\varepsilon\|_{P,\varepsilon}^2\ ,$$

and (15) follows from this as in the proof of theorem 4.4.2 in [6].

Proof that 3) implies 4) in Theorem 3. Since $u\in\Gamma^a(\Omega)$ is a local property of u, we can assume that Ω is bounded. If we choose the natural members A_j, B_j and N such that

$$1/a_j = A_j/N \quad\text{and}\quad 1/b_j = B_j/N, \quad j=1,2,\ldots,n,$$

then the assumption in (3) gives that (14) holds and from $a_j \geqq 1$ we obtain $A_j \leqq N$ for all j. Hence lemma 3 gives

$$\|D_1^{B_1}u,\Omega_{\delta+\delta_1}\|_{P,\delta} \leqq C\sum_{|\alpha|=1}\delta^{-N\alpha}\|D^{\alpha A}u,\Omega_{\delta_1}\|_{P,\delta}\ ,$$

where

$$\alpha = (\alpha_0,\alpha_1,\ldots,\alpha_n), \quad \delta^{-N\alpha} = \delta^{-N\alpha_0}, \quad D^{\alpha A} = D_1^{\alpha_1 A_1}\ldots D_n^{\alpha_n A_n}\ .$$

By induction over k we get

$$\|D_1^{kB_1}u,\Omega_{k\delta+\delta_1}\|_{P,\delta} \leqq C^k\sum_{|\alpha|=k}\binom{k}{\alpha}\delta^{-N\alpha}\|D^{\alpha A}u,\Omega_{\delta_1}\|_{P,\delta}\ .$$

Let now K be a compact set contained in Ω and take a constant c, $0<c<2$, such that $K\subset\Omega_c$ and set $\delta=c/2k$ and $\delta_1=c/2$. Since $\delta<1$ and all the derivatives of u are bounded on the relatively compact set Ω_{δ_1}, we obtain with m equal to the degree of P

$$\delta^m\|D^{\alpha A}u,\Omega_{\delta_1}\|_{P,\delta} \leqq C^{|\alpha A|}(\alpha A)^{a(\alpha A)} = C^{\alpha_1A_1+\ldots+\alpha_nA_n}(\alpha_1A_1)^{N\alpha_1}\ldots(\alpha_nA_n)^{N\alpha_n}.$$

Now $P^{(\alpha)}$ is a constant $\neq 0$ for some α with $|\alpha|=m$ and it follows that

$$\begin{aligned}
\left(\int_{\Omega_c}|D_1^{kB_1}u|^2\,dx\right)^{\frac{1}{2}} &\leqq C\delta^m\|D_1^{kB_1}u,\Omega_c\|_{P,\delta}\\
&\leqq C^k\sum_{|\alpha|=k}\binom{k}{\alpha}(\tfrac{1}{2}c)^{-N\alpha_0}k^{N(k-\alpha_1-\ldots-\alpha_n)}C^{\alpha_1A_1+\ldots+\alpha_nA_n}(kA_1)^{N\alpha_1}\ldots(kA_n)^{N\alpha_n}\\
&= C^k k^{kN}(c^{-N}+C^{A_1}A_1^N+\ldots+C^{A_n}A_n^N)^k \leqq C^{kB_1}(kB_1)^{b_1(kB_1)}\ .
\end{aligned}$$

If we apply this estimate to $D_1^i u$ for $i=0,1,\ldots,B_1-1$ we get

$$\left(\int_{\Omega_c}|D_1^j u|^2\,dx\right)^{\frac{1}{2}} \leqq C^j j^{b_1 j}, \quad j=1,2,\ldots\,,$$

and this gives with the help of formula (4.4.7) in [6] that

$$\sup_{x\in K}|D_1^j u(x)| \leqq C^j j^{b_1 j}, \quad j=1,2,\ldots\,,$$

that is, $u \in \Gamma^b(\Omega)$ if $b_2=\ldots=b_n=\infty$. If both b_1 and b_2 are finite, then one starts from the inequality

$$\|D_1^{k_1B_1}D_2^{k_2B_2}u, \Omega_{(k_1+k_2)\delta+\delta_1}\|_{P,\delta}$$
$$\leqq C^{k_1+k_2}\sum_{|\alpha'|=k_1}\sum_{|\alpha''|=k_2}\binom{k_1}{\alpha'}\binom{k_2}{\alpha''}\delta^{-N(\alpha'+\alpha'')}\|D^{(\alpha'+\alpha'')A}u, \Omega_{\delta_1}\|_{P,\delta}$$

and calculates as above. The other cases are treated similarily.

Proof that 4) implies 5) in Theorem 3. Take $\varphi \in C_0^\infty(B)$. Then by definition $u*\varphi \in \Gamma^a(\Omega_\sigma)$. Furthermore

$$P(D)(u*\varphi) = \left(P(D)u\right)*\varphi = 0 \quad \text{in } \Omega_\sigma\,.$$

Hence $u*\varphi \in \Gamma^b(\Omega_\sigma)$, which again by definition gives $u \in \tilde{\Gamma}^b(\Omega)$, since σ and φ are arbitrary.

This finishes the proof of theorem 3, since 5) implies 6) is a triviality and 6) implies 1) is a direct consequence of the corollary of theorem 2.

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