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SIMPLICIAL DECOMPOSITION OF BOUNDARY MEASURES ON CONVEX COMPACT SETS

ERIK M. ALFSEN and CHRISTIAN F. SKAU

Let K be a convex compact subset of a locally convex Hausdorff space E; let $M_1{}^+(K)$ be the convex and vaguely compact set of all positive normalized (Radon-) measures on K, and let $M_x{}^+$ be the subset of $M_1{}^+(K)$ consisting of all measures with barycenter $x \in K$. The *simplicial measures* on K are the extreme points of the sets $M_x{}^+$, $x \in K$.

It was observed by Douglas [10] that a measure $\mu \in M_1{}^+(K)$ is simplicial iff the space $A(K)$ of continuous affine functions on K is *dense* in $L^1(\mu)$. (Cf. also Lindenstrauss [15].) It has recently been proved by Vincent-Smith [19], and independently by Mokobodzki and Rogalski [17] that every point in K is the barycenter of a simplicial *boundary measure* $\mu \in M_1{}^+(K)$ (μ vanishes off every boundary set B_f where $f \in C(K)$ [6]). This result is non-trivial since the set Z_x of all boundary measures in $M_x{}^+$ is non-compact in the vague topology (as well as in any other known topology). In fact, this result is seen to reduce to a well-known theorem of Carathéodory if $K \subset \mathsf{R}^n$. (Cf. e.g. [8].)

In the present paper we shall give complete proofs of the following results stated in the note [2]:

THEOREM 1. *The set $\partial_e Z_x$ of all simplicial boundary measures with barycenter $x \in K$ is a (non-empty) Baire space in the vague topology. If K is metrizable, then $\partial_e Z_x$ is a G_δ-subset of $M_1{}^+(K)$; hence it is a Polish space.*

THEOREM 2. *There exists a locally convex topology σ on $M(K) = C(K)^*$ which is stronger than the vague topology and weaker than the norm topology and for which*

$$Z_x = \overline{(\operatorname{conv} \partial_e Z_x)_\sigma} \quad \textit{for all } x \in K .$$

Specifically, σ is the weak topology defined on $M(K)$ by the linear space F obtained by adjoining to $C(K)$ all envelopes $\hat{f}$ of functions $f \in C(K)$.

COROLLARY. *K is a Choquet simplex iff every point in K is the barycenter of a unique simplicial boundary measure.*

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The next theorem establishes a decomposition into simplicial components within each Z_x, $x \in K$. In this connection we recall that a measure ϑ is said to be *pseudo-carried* by a set Y if $\vartheta_*(\complement Y)=0$, where ϑ_* is the interior Baire measure associated with ϑ [6].

THEOREM 3. *For every boundary measure $\mu \in M_1{}^+(K)$ with barycenter $x \in K$ there exists a positive and normalized measure ϑ on the vaguely compact set $M_1{}^+(K)$ such that*

$$\mu(f) = \int \nu(f)\, d\vartheta(\nu) \qquad \text{for all } f \in C(K)\,,$$

and such that ϑ is pseudo-carried by $\partial_e Z_x$. If K is metrizable, then $\vartheta(\partial_e z_x)=1$.

1. Preliminaries from Choquet boundary theory.

For the sake of convenience we shall list some basic facts which will be needed in the sequel. Complete proofs can be found in the papers of Bauer [4], Edwards [11] and [12], Davies [7], and Boboc and Cornea [5].

A cone S of continuous real valued functions on a compact Hausdorff space X is said to be *admissible* if it contains the constant functions and separates points. It is said to be *max-stable*, if $f \vee g \in S$ whenever $f,g \in S$. The max-stable hull $\tilde{S}$ of an admissible cone S consists of all $f_1 \vee \ldots \vee f_n$ where $f_1,\ldots,f_n \in S$. By Stone's Theorem, $\tilde{S}-\tilde{S}$ is dense in $C(K)$.

An admissible cone S over X determines the following (partial) ordering on $M_1{}^+(X)$:

$$\mu \prec_S \nu \iff \mu(f) \leqq \nu(f) \qquad \text{for all } f \in S\,. \tag{1.1}$$

The *S-Choquet boundary* $\partial_S X$ consists of all points $x \in X$ such that

$$\mu \in M_1{}^+(X),\ \varepsilon_x \prec_S \mu \ \Rightarrow\ \varepsilon_x = \mu\,. \tag{1.2}$$

By Bauer's maximum principle, $\partial_S X$ is *non-empty*. In fact for every $f \in S$ there is an $x \in \partial_S X$ such that $f(x)=\sup\{f(y) \mid y \in X\}$ (cf. [3]). Clearly also $\partial_S X = \partial_{\tilde{S}} X$.

If S is an admissible cone over X and $f\colon X \to [\alpha,\infty]$, then $\check{f}_S$ is the pointwise supremum of all $g \in S$, $g \leqq f$. Similarly, if $f\colon X \to [-\infty,\alpha]$, then $\hat{f}_S$ is the pointwise infimum of all $g \in -S$, $g \geqq f$. Clearly $\check{f}_S$ is pointwise limit of an ascending net from $\tilde{S}$, and $\hat{f}_S$ is pointwise limit of a descending net from $-\tilde{S}$. Note also that S-envelopes and $\tilde{S}$-envelopes coincide.

It is easily verified that if S is a max-stable cone over X and $\mu,\nu \in M_1{}^+(X)$, then $\mu \prec_S \nu$ iff

$$(1.3) \qquad \nu(f) \leqq \mu(\hat{f}_S) \quad \text{for all } f \in C(K) .$$

By a standard argument based on the Hahn–Banach Theorem one obtains the following result, which we state as a proposition for later references:

PROPOSITION 1. *If S is an admissible cone over X, if $f \in C(X)$ and if $\mu \in M_1{}^+(X)$, then there is a $\nu \in M_1{}^+(X)$ such that $\mu \prec_S \nu$ and $\nu(f) = \mu(\hat{f}_S)$.*

COROLLARY 1. *If S is a max-stable admissible cone over X and $\mu \in M_1{}^+(X)$, then μ is maximal in the ordering $\prec_S$ iff $\mu(f) = \mu(\hat{f}_S)$ for all $f \in C(K)$.*

Specializing to one-point measures $\mu = \varepsilon_x$, and remembering that $\partial_S X = \partial_{\tilde{S}} X$, one obtains the following result, which is independent of max-stability:

COROLLARY 2. *If S is an admissible cone over X and $x \in X$, then $x \in \partial_S X$ iff $f(x) = \hat{f}_S(x)$ for all $f \in C(X)$.*

If S is an admissible cone over a *metrizable* compact Hausdorff space X, then one may use the density of $\tilde{S} - \tilde{S}$ in $C(X)$ to construct a sequence $\{f_n\}$ from $\tilde{S}$ such that $\|f_n\| \leqq 1$, and

$$\{\alpha f_n - \beta f_m \mid m, n = 1, 2, \ldots ; \alpha, \beta > 0\}$$

is dense in $C(X)$, and now it follows from (1.2), (1.3), and from Corollary 2 above that the function $f = \sum_{n=1}^{\infty} 2^{-n} f_n$ satisfies

$$(1.4) \qquad x \in \partial_S X \iff f(x) = \hat{f}_S(x) .$$

Hence we conclude by the upper semi-continuity of $\hat{f}_S$ that *$\partial_S X$ is a G_δ-subset of X if X is metrizable.*

Corollary 2 is also the starting point in the proof of the Choquet–Edwards Theorem stating that *$\partial_S X$ is a Baire space (in the relativized topology) for every admissible cone S over a compact Hausdorff space X* [9; Appendix B.14], [12].

By a straightforward application of Zorn's Lemma and Corollary 1 above, one can prove the following:

PROPOSITION 2. *If S is a max-stable admissible cone over a compact Hausdorff space X, then there exists for every $x \in X$ a measure $\mu \in M_1{}^+(X)$ such that $\varepsilon_x \prec_S \mu$ and such that μ is maximal in the ordering $\prec_S$. If X is metrizable, then $\mu(\partial_S X) = 1$.*

In the general (non-metrizable) case one can prove by means of a reduction to the metrizable case, based on an idea of Meyer [16], that a $\prec_S$-maximal measure $\mu \in M_1^+(X)$ is pseudo-carried by $\partial_S X$.

LEMMA 1. *If S,T are admissible cones over compact Hausdorff spaces X, Y respectively, and if φ is a continuous map of X onto Y such that $\varphi^*(T) \subset S$, then $\partial_T Y \subset \varphi(\partial_S X)$.*

PROOF. For a given $y \in \partial_T Y$ we define $S_y = S \mid \varphi^{-1}(y)$. Clearly S_y is an admissible cone over $\varphi^{-1}(y)$, and we claim that every S_y-Choquet point x of $\varphi^{-1}(y)$ is an S-Choquet point of X. In fact if $\varepsilon_x \prec_S \mu$, then the direct image of μ by φ satisfies $\varepsilon_y \prec_T \varphi(\mu)$, and so $\varepsilon_y = \varphi(\mu)$. It follows that $\operatorname{Supp}(\mu) \subset \varphi^{-1}(y)$, and hence $\varepsilon_x = \mu$, since x is an S_y-Choquet point of $\varphi^{-1}(y)$.

LEMMA 2. *If S is an admissible cone over a compact Hausdorff space X and $\{f_n\}$ is an upper bounded sequence from $C(X)$ such that* $\limsup_n f_n(x) \leqq \alpha \in \mathsf{R}$ *for all $x \in \partial_S X$, then*

$$\limsup_n \check{f}_{n,S}(x) \leqq \alpha \quad \textit{for all } x \in X .$$

PROOF. For an arbitrary point $x_0 \in X$ there exist functions $g_n \in S$ such that

$$g_n \leqq f_n, \quad \check{f}_{n,S}(x_0) < g_n(x_0) + n^{-1} . \tag{1.5}$$

Define $\Phi : X \to \mathsf{R}^{\aleph_0}$ by $\Phi(x) = \{g_n(x)\}$, and write $Y = \Phi(X)$, $y_0 \in \Phi(x_0)$. Let T be the subcone of $C(Y)$ generated by the projections p_n of $\mathsf{R}^{\aleph_0}$ and the constants. By Lemma 1, $\Phi(\partial_S X) \subset \partial_T Y$, and hence it follows from (1.5) that $\limsup_n p_n(y) \leqq \alpha$ for all $y \in \partial_T Y$. Applying Proposition 2 to the cone T, we obtain a measure $\mu \in M_1^+(Y)$ such that $\mu(\partial_T Y) = 1$ and such that $\varepsilon_{y_0} \prec_T \mu$. By Fatou's Lemma:

$$\begin{aligned}\limsup_n \check{f}_{n,S}(x_0) &= \limsup_n g_n(x_0) = \limsup_n p_n(y_0) \\ &\leqq \limsup_n \mu(p_n) \leqq \mu(\limsup_n p_n) \leqq \alpha ,\end{aligned}$$

and the proof is complete.

PROPOSITION 3. *If S is a max-stable admissable cone over a compact Hausdorff space X, then there exists for every $x \in X$ a measure $\mu \in M_1^+(X)$ which is pseudo-carried by $\partial_S X$ such that $\varepsilon_x \prec_S \mu$.*

PROOF. To show that the $\prec_S$-maximal measure μ of Proposition 2 is pseudo-carried by $\partial_S X$, it suffices to show that $\mu(C) = 0$ for every compact G_δ-set C such that $C \cap \partial_S X = \emptyset$. This follows by application of Lemma 2 and (the dual of) Corollary 2 of Proposition 1 to a bounded sequence $\{f_n\}$ from $C(X)$ converging pointwise to the indicator function χ_C.

2. Unilateral representation theorems in ordered convex compacts.

We shall use the term *ordered convex compact* to denote a convex compact subset K of a locally convex Hausdorff space E provided with a (partial) ordering defined by a cone E^+ which is closed and proper (that is, $E^+ \cap (-E^+) = \{0\}$). The *isotone* (order preserving) functions in $A(K)$ form a cone, which we denote by $L(K)$, or briefly by L. It follows by a standard separation argument (based on the Hahn–Banach Theorem) that L *determines* the ordering of K, in that

$$(2.1) \qquad x \leqq y \iff l(x) \leqq l(y) \quad \text{for all } l \in L .$$

In particular, L *separates the points of* K.

The set of all *maximal* points of an ordered convex compact K will be denoted by $Z(K)$ or briefly by Z. It follows by a standard application of Zorn's Lemma that there exists for every point $x \in K$ a maximal point $z \in K$ such that $x \leqq z$. In particular $Z \neq \emptyset$. It is also easily verified that Z is *a union of faces.* (It is a "σ-face" in the terminology of Goullet de Rugy [14].) Hence, if Z is convex, then it is a face of K.

The L-envelopes of functions on an ordered convex compact K will be called *monotone* convex and concave envelopes. They are related to the customary convex and concave envelopes by the inequalities

$$(2.2) \qquad \check{f}_L \leqq \check{f} \leqq f \leqq \hat{f} \leqq \hat{f}_L .$$

The ordering $\prec_L$ defined on $M_1{}^+(K)$ by the max-stable cone $\tilde{L}$ will be denoted by $\prec\!\prec$. It is related to the customary ordering of Choquet by the formula

$$(2.3) \qquad \mu \prec \nu \Rightarrow \mu \prec\!\prec \nu .$$

Observe that if $\mu \prec\!\prec \nu$ and if x_μ, x_ν are the barycenters of μ and ν respectively, then for every $l \in L$ we shall have $l(x_\mu) = \mu(l)$, $l(x_\nu) = \nu(l)$, and hence also $l(x_\mu) \leqq l(x_\nu)$. By virtue of (2.1) this gives the implication

$$(2.4) \qquad \mu \prec\!\prec \nu \Rightarrow x_\mu \leqq x_\nu .$$

LEMMA 3. *A point x of an ordered convex compact K belongs to Z iff $\hat{a}_L(x) = a(x)$ for all $a \in A(K)$.*

PROOF. Assume first that $x \in Z$, and consider an arbitrary $a \in A(K)$. By Proposition 1 there is a $\mu \in M_1{}^+(K)$ such that $\varepsilon_x \prec\!\prec \mu$ and $\mu(a) = \hat{a}_L(x)$. By (2.4) and by the maximality of x, this gives $x = x_\mu$. Since $a \in A(K)$, we obtain $a(x) = \mu(a) = \hat{a}_L(x)$.

Assume next that $x \notin Z$, say $x < y \in K$. By (2.1) there is an $l \in L$ such that $l(x) < l(y)$. Now

$$l(x) < l(y) \leqq \hat{l}_L(y) = \hat{l}_L(x) ,$$

and the proof is complete.

PROPOSITION 4. *The set $Z \cap \partial_e K$ of maximal extreme points of an ordered convex compact K is equal to the L-Choquet boundary $\partial_L K$.*

PROOF. Assume first that $x \in Z \cap \partial_e K$, and consider an arbitrary $f \in C(K)$. By Corollary 2 to Proposition 1 it suffices to prove that $\hat{f}_L(x) = f(x)$.

By a known characterization of extreme points due to Hervé [13], there is, for given $\varepsilon > 0$, an $a \in A(K)$ such that

$$f \leqq a, \quad a(x) < f(x) + \tfrac{1}{2}\varepsilon . \tag{2.5}$$

By the Lemma $\hat{a}_L(x) = a(x)$; and by the definition of monotone envelopes there is an $l \in -L$ such that

$$a \leqq l, \quad l(x) < a(x) + \tfrac{1}{2}\varepsilon . \tag{2.6}$$

Combining (2.5) and (2.6) we obtain $f \leqq l$ and $l(x) < f(x) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this gives $\hat{f}_L(x) = f(x)$.

Assume next that $x \in \partial_L K$. Consider first an arbitrary $y \in K$ such that $x \leqq y$. Observing that every function in $\tilde{L}$ is isotone, we obtain $\varepsilon_x \prec\prec \varepsilon_y$. By assumption $x \in \partial_L K = \partial_{\tilde{L}} K$, and so $x = y$. This proves x to be a maximal point of K.

Consider next an arbitrary $\mu \in M_1^+(K)$ such that $x = x_\mu$. Now $\varepsilon_x \prec \mu$, and it follows by (2.3) that $\varepsilon_x \prec\prec \mu$. Since $x \in \partial_{\tilde{L}} K$, we obtain $\varepsilon_x = \mu$. This shows that x is an extreme point of K, and the proof is complete.

REMARK. It follows from Proposition 4 that $Z \cap \partial_e K \neq \emptyset$. It is also possible to give a more direct proof of this statement. In fact, Mokobodzki and Rogalski [17] have shown by a direct argument that $Z \cap \partial_e F \neq \emptyset$ for every closed face F of K which is "hereditary" in the sense that

$$x \in F, \ y \in K, \ x \leqq y \ \Rightarrow \ y \in F .$$

PROPOSITION 5. *If K is an ordered convex compact, then $Z \cap \partial_e K$ is a non-empty Baire space in the relativized topology. If K is metrizable, then $Z \cap \partial_e K$ is a G_δ-subset of K. For every maximal point $z \in Z$ there exists a measure $\mu \in M_1^+(K)$ pseudo-carried by $Z \cap \partial_e K$ such that*

$$a(z) = \int a \, d\mu \quad \text{for all } a \in A(K) . \tag{2.7}$$

If K is metrizable, then $\mu(\partial_e K) = 1$.

THE PROOF is a direct application of the results of Section 1, in particular of Proposition 3. Note that $\varepsilon_z \prec\prec \mu$ implies $z \leqq x_\mu$ by virtue of (2.4). By maximality $z = x_\mu$, which gives formula (2.7).

3. Simplicial boundary measures.

The following proposition was proved by Douglas in a slightly different setting [10]. For the sake of completeness we give the proof.

PROPOSITION 6. *If K is a convex compact set in a locally convex Hausdorff space and $\mu \in M_1{}^+(K)$, then μ is simplicial iff $A(K)$ is dense in $L^1(\mu)$.*

PROOF. Assume first that $A(K)$ is non-dense in $L^1(\mu)$. We shall show that μ is non-extreme in $M_x{}^+$ where $x = x_\mu$.

By assumption there is a non-zero element h of $L^\infty(\mu)$ such that $\|h\|_\infty \leqq 1$ and $\mu(ah) = 0$ for all $a \in A(K)$. The measure ν defined by $d\nu = h d\mu$, satisfies $-\mu \leqq \nu \leqq \mu$. Hence the two measures $\mu_1 = \mu + \nu$, $\mu_2 = \mu - \nu$ are positive and non-equal. Moreover, $\mu_i(a) = \mu(a) = a(x)$ for $i = 1, 2$ and $a \in A(K)$. Hence $\mu_1, \mu_2 \in M_x{}^+$, and $\mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$ is non-extreme.

Assume next that μ is a non-extreme point of $M_x{}^+$, say $\mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$ where $\mu_1, \mu_2 \in M_x{}^+$ and $\mu_1 \neq \mu_2$. Now $0 \leqq \mu_1 \leqq 2\mu$, and so $d\mu_1 = h d\mu$, where $h \in L^\infty(\mu)$ and $0 \leqq h \leqq 2$ a.e. (μ). Also $1 - h$ is a non-zero element of $L^\infty(\mu)$ since $\mu_1 \neq \mu_2$, and $1 - h$ annihilates $A(K)$ since $\mu(a(1-h)) = \mu(a) - \mu_1(a) = 0$ for all $a \in A(K)$. This proves that $A(K)$ is non-dense in $L^1(\mu)$.

PROPOSITION 7. *Let K be a convex compact set in a locally convex Hausdorff space and let $\mu \in M_1{}^+(K)$ be a measure of finite support, say $\mu = \sum_{j=1}^n \lambda_j \varepsilon_{x_j}$. Then μ is simplicial iff $\{x_1, \ldots, x_n\}$ is an affinely independent set of points.*

PROOF. Assume first that $\{x_1, \ldots, x_n\}$ is affinely dependent, say

$$\sum_{i=1}^n \beta_j x_j = 0, \quad \sum_{j=1}^n \beta_j = 0 , \tag{3.1}$$

where $\{\beta_1, \ldots, \beta_n\} \neq \{0, \ldots, 0\}$. Define $h \in L^\infty(\mu)$ as follows:

$$h(y) = \begin{cases} \beta_j/\lambda_j & \text{if } y = x_j,\ j \in \{1, \ldots, n\} , \\ 0 & \text{if } y \notin \{x_1, \ldots, x_n\} . \end{cases}$$

Clearly h is a non-zero element of $L^\infty(\mu)$ and for every continuous linear functional a on the given locally convex space,

$$\mu(ah) = \sum_{j=1}^n \beta_j a(x_j) = a\left(\sum_{j=1}^n \beta_j x_j\right) = 0, \qquad \mu(h) = \sum_{j=1}^n \beta_j = 0 ,$$

and so $\mu((a+\alpha)h) = 0$ for all $\alpha \in \mathbf{R}$. By density $\mu(a'h) = 0$ for all $a' \in A(K)$. It follows that $A(K)$ is non-dense in $L^1(\mu)$, and by Proposition 6, μ is non-simplicial.

Assume next that μ is a non-extreme point of $M_x{}^+$ where $x=x_\mu=\sum_{j=1}^n \lambda_j x_j$, say $\mu=\frac{1}{2}\mu_1+\frac{1}{2}\mu_2$ where $\mu_1,\mu_2 \in M_x{}^+$ and $\mu_1 \neq \mu_2$. Necessarily $\mu_i=\sum_{j=1}^n \alpha_{ij}\varepsilon_{x_j}$, where $0 \leqq \alpha_{ij} \leqq 2\lambda_j$ for $j=1,\ldots,n$ and $i=1,2$. Also $x=\sum_{j=1}^n \alpha_{ij}x_j$ for $i=1,2$, and so

$$\sum_{j=1}^n (\alpha_{1j}-\alpha_{2j})x_j \;=\; 0\;. \tag{3.2}$$

The coefficients of (3.2) have zero sum, and they do not all vanish since $\mu_1 \neq \mu_2$. Hence $\{x_1,\ldots,x_n\}$ is affinely dependent.

Proposition 8. *If K is a convex compact subset of R^n and $\mu \in M_1{}^+(K)$, then μ is simplicial iff μ is supported by an affinely independent set of (at most $n+1$) points.*

Proof. By Proposition 7 it suffices to assume that μ is simplicial and to show $\operatorname{Supp}(\mu)$ affinely independent. To this end we define F to be the affine span of $\operatorname{Supp}(\mu)$, and we consider an affinely independent subset $J=\{x_0,\ldots,x_k\}$ of $\operatorname{Supp}(\mu)$ such that F is the affine span of J. Now we assume that there is a point $y \in \operatorname{Supp}(\mu) \setminus J$, and we shall show that this contradicts the simpliciality of μ.

Let U be a bounded neighbourhood of 0 in R^n such that the sets $x_0+U,\ldots,x_k+U;\ y+U$ are pairwise disjoint. By Proposition 6 there exists a sequence $\{a_n\}$ from $A(F)$ which converges in $L^1(\mu)$ to the indicator function of the set $y+U$. Writing $W=\bigcup_{j=0}^k (x_j+U)$, we obtain

$$\int_W |a_n|\, d\mu \to 0\;, \tag{3.3}$$

and

$$\int_{y+U} a_n d\mu \to \mu(y+U)\;. \tag{3.4}$$

(For brevity we write $\mu(A)$ in the place of $\mu(A \cap K)$ for $A \subset \mathsf{R}^n$.) We claim that the functional

$$p\colon\, a \mapsto \int_W |a|\, d\mu$$

is a norm on $A(F)$. Clearly, it is a semi-norm, and if $p(a)=0$, then $a(x_0)=\ldots=a(x_k)=0$ since a is continuous and $x_j \in \operatorname{Supp}(\mu)$ for $j=0,\ldots,k$, and this in turn implies $a=0$ on F since a is an affine function and F is spanned by $x_0,\ldots,x_k$.

The norm p on the finite dimensional space $A(F)$ must be topologically equivalent to the supremum norm over $K \cap F$. Hence $\{a_n\}$ converges uniformly to 0 on K by virtue of (3.3). However, $\mu(y+U) \neq 0$ since $y \in \operatorname{Supp}(\mu)$, and hence (3.4) gives a contradiction.

COROLLARY. *If K is a convex compact subset of R^n, then a simplicial boundary measure $\mu \in M_1{}^+(K)$ is supported by a set of at most $n+1$ extreme points.*

4. Completion of the proofs of the main theorems.

Theorem 1 and Theorem 3 follow easily from the results of Section 2, while the proof of Theorem 2 requires a separate argument based on Bauer's maximum principle. The specialization to the classical Carathéodory Theorem is evident by the results of Section 3.

PROOF OF THEOREM 1 AND THEOREM 3. The set $M_x{}^+$ is an ordered convex compact in the vague topology and in the ordering of Choquet. Specifically, the positive cone of $M(K)=C(K)^*$ for this ordering consists of all μ such that $\mu(f) \geqq 0$ for every convex function $f \in C(K)$. By a known characterization [6] of boundary measures, Z_x is the set of maximal elements of $M_x{}^+$. (Z_x is a face of $M_x{}^+$ in the present case.) Now an application of Proposition 5 completes the proof.

PROOF OF THEOREM 2. The set Z_x consists of all $\mu \in M_x{}^+$ such that $\mu(\hat{f}-f)=0$ for all $f \in C(K)$ [6]. Hence Z_x is closed in the topology σ. To prove that Z_x is contained in the σ-closed convex hull of $\partial_e Z_x$, we assume the contrary, say

$$\mu \in Z_x \setminus \overline{(\operatorname{conv} \partial_e Z_x)_\sigma} \,. \tag{4.1}$$

Now there exists (by Hahn–Banach separation) a function $k \in F$ such that

$$\sup_{\nu \in \partial_e Z_x} \nu(k) = \alpha < \mu(k) \,. \tag{4.2}$$

By the definition of F,

$$k = \beta_0 + \textstyle\sum_{j=1}^n \beta_j \hat{f}_j \,,$$

where $f_0, \ldots, f_n \in C(K)$ and $\beta_1, \ldots, \beta_n \in \mathsf{R}$. All measures occuring in (4.2) are boundary measures, and therefore one may replace the function k by the continuous function

$$k' = \beta_0 + \textstyle\sum_{j=1}^n \beta_j f_j \,.$$

Next we may replace k' by its (l.s.c.) convex lower envelope $\check{k}'$. By a well-known theorem (based on the Hahn–Banach Theorem [6]) there is a net $\{g_\alpha\}$ of continuous convex functions on K such that $g_\alpha \nearrow \check{k}'$. Hence there is a continuous convex function g on K such that $g \leqq k'$ and such that

$$\sup_{\nu \in \partial_e Z_x} \nu(g) \leqq \alpha < \mu(g) \,. \tag{4.3}$$

Now $\nu \mapsto \nu(g)$ is a continuous, affine and *isotone* function on $M_x{}^+$. By Bauer's maximum principle it attains its maximum value at the Choquet boundary defined by the cone of such functions. By Proposition 4 this Choquet boundary is equal to the set $Z_x \cap \partial_e M_x{}^+ = \partial_e Z_x$. (Recall that Z_x is a face of $M_x{}^+$.) By (4.3) this is a contradiction, and the proof is complete.

REMARK. The conclusion of Theorem 2 does not subsist if σ is replaced by the vague topology or by the norm topology. In fact let $K = \operatorname{conv}(D \cup L)$ where D is a plane disk in R^3 and L is a line segment orthogonal to D which meets D in a point $y \in \partial_e D \cap (L \setminus \partial_e L)$. The set Z_x determined by the center x of D is non-closed in the vague topology; and the set $\partial_e Z_x$ consists of measures supported by at most four points (Proposition 8), hence the norm closed convex hull of $\partial_e Z_x$ consists of discrete measures only. Thus we shall have

$$\overline{(\operatorname{conv} \partial_e Z_x)}_{\mathrm{norm}} \subset Z_x \subset \overline{(\operatorname{conv} \partial_e Z_x)}_{\mathrm{vague}},$$

and both inclusions are *strict*.

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Note added in proof, February 1970.

After the present paper was submitted, we have become aware that the result quoted in the Remark of Section 2 above has already been proved by G. Lumer in 1963 [21]. Also we take the opportunity to mention Andenæs' forthcoming paper [20], where Vincent-Smith's construction of simplicial boundary measures is presented in a more general setting.

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UNIVERSITY OF OSLO, NORWAY
UNIVERSITY OF AARHUS, DENMARK