SIMPLICAL DECOMPOSITION OF BOUNDARY MEASURES ON CONVEX COMPACT SETS

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Let $K$ be a convex compact subset of a locally convex Hausdorff space $E$; let $M_1^+(K)$ be the convex and vaguely compact set of all positive normalized (Radon-) measures on $K$, and let $M_x^+$ be the subset of $M_1^+(K)$ consisting of all measures with barycenter $x \in K$. The simplicial measures on $K$ are the extreme points of the sets $M_x^+$, $x \in K$.

It was observed by Douglas [10] that a measure $\mu \in M_1^+(K)$ is simplicial iff the space $A(K)$ of continuous affine functions on $K$ is dense in $L^1(\mu)$. (Cf. also Lindenstrauss [15].) It has recently been proved by Vincent-Smith [19], and independently by Mokobodzki and Rogalski [17] that every point in $K$ is the barycenter of a simplicial boundary measure $\mu \in M_1^+(K)$ ($\mu$ vanishes off every boundary set $B_f$ where $f \in C(K)$ [6]). This result is non-trivial since the set $Z_x$ of all boundary measures in $M_x^+$ is non-compact in the vague topology (as well as in any other known topology). In fact, this result is seen to reduce to a well-known theorem of Carathéodory if $K \subset \mathbb{R}^n$. (Cf. e.g. [8].)

In the present paper we shall give complete proofs of the following results stated in the note [2]:

**Theorem 1.** The set $\partial_x Z_x$ of all simplicial boundary measures with barycenter $x \in K$ is a (non-empty) Baire space in the vague topology. If $K$ is metrizable, then $\partial_x Z_x$ is a $G_\delta$-subset of $M_1^+(K)$; hence it is a Polish space.

**Theorem 2.** There exists a locally convex topology $\sigma$ on $M(K) = C(K)^*$ which is stronger than the vague topology and weaker than the norm topology and for which

$$Z_x = (\text{conv} \partial_x Z_x)_\sigma \quad \text{for all } x \in K.$$ 

Specifically, $\sigma$ is the weak topology defined on $M(K)$ by the linear space $F$ obtained by adjoining to $C(K)$ all envelopes $\hat{f}$ of functions $f \in C(K)$.

**Corollary.** $K$ is a Choquet simplex iff every point in $K$ is the barycenter of a unique simplicial boundary measure.

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The next theorem establishes a decomposition into simplicial components within each $Z_x, x \in K$. In this connection we recall that a measure $\theta$ is said to be pseudo-carried by a set $Y$ if $\theta_*([Y]) = 0$, where $\theta_*$ is the interior Baire measure associated with $\theta$ [6].

**Theorem 3.** For every boundary measure $\mu \in M_1^+(K)$ with barycenter $x \in K$ there exists a positive and normalized measure $\theta$ on the vaguely compact set $M_1^+(K)$ such that

$$\mu(f) = \int v(f) \, d\theta(v) \quad \text{for all } f \in C(K),$$

and such that $\theta$ is pseudo-carried by $\partial_e Z_x$. If $K$ is metrizable, then $\theta(\partial_e Z_x) = 1$.

1. **Preliminaries from Choquet boundary theory.**

For the sake of convenience we shall list some basic facts which will be needed in the sequel. Complete proofs can be found in the papers of Bauer [4], Edwards [11] and [12], Davies [7], and Boboc and Cornea [5].

A cone $S$ of continuous real valued functions on a compact Hausdorff space $X$ is said to be *admissible* if it contains the constant functions and separates points. It is said to be *max-stable*, if $fvg \in S$ whenever $f, g \in S$. The max-stable hull $\mathcal{S}$ of an admissible cone $S$ consists of all $f_1 v \ldots v f_n$ where $f_1, \ldots, f_n \in S$. By Stone’s Theorem, $\mathcal{S} - \mathcal{S}$ is dense in $C(K)$.

An admissible cone $S$ over $X$ determines the following (partial) ordering on $M_1^+(X)$:

$$\mu <_S v \iff \mu(f) \leq v(f) \quad \text{for all } f \in S.$$

The $S$-Choquet boundary $\partial_s X$ consists of all points $x \in X$ such that

$$\mu \in M_1^+(X), \varepsilon_x <_S \mu \Rightarrow \varepsilon_x = \mu.$$

By Bauer’s maximum principle, $\partial_s X$ is *non-empty*. In fact for every $f \in S$ there is an $x \in \partial_s X$ such that $f(x) = \sup \{f(y) \mid y \in X\}$ (cf. [3]). Clearly also $\partial_s X = \partial_{\mathcal{S}} X$.

If $S$ is an admissible cone over $X$ and $f : X \to [\alpha, \infty]$, then $\check{f}_S$ is the pointwise supremum of all $q \in S, q \leq f$. Similarly, if $f : X \to [-\infty, x]$, then $\hat{f}_S$ is the pointwise infimum of all $g \in -S, g \leq f$. Clearly $\check{f}_S$ is pointwise limit of an ascending net from $S$, and $\hat{f}_S$ is pointwise limit of a descending net from $-S$. Note also that $S$-envelopes and $\mathcal{S}$-envelopes coincide.

It is easily verified that if $S$ is a max-stable cone over $X$ and $\mu, v \in M_1^+(X)$, then $\mu <_S v$ iff
(1.3) \[ v(f) \leq \mu(\hat{f}_S) \quad \text{for all } f \in C(K). \]

By a standard argument based on the Hahn–Banach Theorem one obtains the following result, which we state as a proposition for later references:

**Proposition 1.** If \( S \) is an admissible cone over \( X \), if \( f \in C(X) \) and if \( \mu \in M_1^+(X) \), then there is a \( v \in M_1^+(X) \) such that \( \mu \prec_S v \) and \( v(f) = \mu(\hat{f}_S) \).

**Corollary 1.** If \( S \) is a max-stable admissible cone over \( X \) and \( \mu \in M_1^+(X) \), then \( \mu \) is maximal in the ordering \( \prec_S \) iff \( \mu(f) = \mu(\hat{f}_S) \) for all \( f \in C(K) \).

Specializing to one-point measures \( \mu = \delta_x \), and remembering that \( \partial_S X = \partial_S^X \), one obtains the following result, which is independent of max-stability:

**Corollary 2.** If \( S \) is an admissible cone over \( X \) and \( x \in X \), then \( x \in \partial_S X \) iff \( f(x) = \hat{f}_S(x) \) for all \( f \in C(X) \).

If \( S \) is an admissible cone over a metrizable compact Hausdorff space \( X \), then one may use the density of \( \bar{S} - \bar{S} \) in \( C(X) \) to construct a sequence \( \{f_n\} \) from \( \bar{S} \) such that \( ||f_n|| \leq 1 \), and

\[ \{\alpha f_n - \beta f_m \mid m, n = 1, 2, \ldots; \alpha, \beta > 0\} \]

is dense in \( C(X) \), and now it follows from (1.2), (1.3), and from Corollary 2 above that the function \( f = \sum_{n=1}^{\infty} 2^{-n} f_n \) satisfies

(1.4) \[ x \in \partial_S X \iff f(x) = \hat{f}_S(x). \]

Hence we conclude by the upper semi-continuity of \( \hat{f}_S \) that \( \partial_S X \) is a \( G_\delta \)-subset of \( X \) if \( X \) is metrizable.

Corollary 2 is also the starting point in the proof of the Choquet–Edwards Theorem stating that \( \partial_S X \) is a Baire space (in the relativized topology) for every admissible cone \( S \) over a compact Hausdorff space \( X \) [9; Appendix B.14], [12].

By a straightforward application of Zorn’s Lemma and Corollary 1 above, one can prove the following:

**Proposition 2.** If \( S \) is a max-stable admissible cone over a compact Hausdorff space \( X \), then there exists for every \( x \in X \) a measure \( \mu \in M_1^+(X) \) such that \( \delta_x \prec_S \mu \) and such that \( \mu \) is maximal in the ordering \( \prec_S \). If \( X \) is metrizable, then \( \mu(\partial_S X) = 1 \).

In the general (non-metrizable) case one can prove by means of a reduc-
tion to the metrizable case, based on an idea of Meyer [16], that a \( \prec_S \)-maximal measure \( \mu \in M_1^+(X) \) is pseudo-carried by \( \partial_S X \).

**Lemma 1.** If \( S,T \) are admissible cones over compact Hausdorff spaces \( X,Y \) respectively, and if \( \varphi \) is a continuous map of \( X \) onto \( Y \) such that \( \varphi^*(T) \subseteq S \), then \( \partial_Y Y \subseteq \varphi(\partial_S X) \).

**Proof.** For a given \( y \in \partial_Y Y \) we define \( S_y = S \mid \varphi^{-1}(y) \). Clearly \( S_y \) is an admissible cone over \( \varphi^{-1}(y) \), and we claim that every \( S_y \)-Choquet point \( x \) of \( \varphi^{-1}(y) \) is an \( S \)-Choquet point of \( X \). In fact if \( \varepsilon_x \prec_S \mu \), then the direct image of \( \mu \) by \( \varphi \) satisfies \( \varepsilon_y \prec \varphi(\mu) \), and so \( \varepsilon_y = \varphi(\mu) \). It follows that \( \text{Supp}(\mu) \subseteq \varphi^{-1}(y) \), and hence \( \varepsilon_x = \mu \), since \( x \) is an \( S_y \)-Choquet point of \( \varphi^{-1}(y) \).

**Lemma 2.** If \( S \) is an admissible cone over a compact Hausdorff space \( X \) and \( \{f_n\} \) is an upper bounded sequence from \( C(X) \) such that \( \limsup_n f_n(x) \leq \alpha \in \mathbb{R} \) for all \( x \in \partial_S X \), then

\[
\limsup_n \tilde{f}_n(x) \leq \alpha \quad \text{for all } x \in X.
\]

**Proof.** For an arbitrary point \( x_0 \in X \) there exist functions \( g_n \in S \) such that

\[
(1.5) \quad g_n \leq f_n, \quad \tilde{f}_n(x_0) < g_n(x_0) + n^{-1}.
\]

Define \( \Phi : X \to \mathbb{R}^\Omega \) by \( \Phi(x) = \{g_n(x)\} \), and write \( Y = \Phi(X) \), \( y_0 = \Phi(x_0) \). Let \( T \) be the subcone of \( C(Y) \) generated by the projections \( p_n \) of \( \mathbb{R}^\Omega \) and the constants. By Lemma 1, \( \Phi(\partial_S X) \subseteq \partial_T Y \), and hence it follows from (1.5) that \( \limsup_n p_n(y) \leq \alpha \) for all \( y \in \partial_T Y \). Applying Proposition 2 to the cone \( T \), we obtain a measure \( \mu \in M_1^+(Y) \) such that \( \mu(\partial_T Y) = 1 \) and such that \( \varepsilon_{y_0} \prec_T \mu \). By Fatou's Lemma:

\[
\limsup_n \tilde{f}_n(x_0) = \limsup_n g_n(x_0) = \limsup_n p_n(y_0) \leq \limsup_n \mu(p_n) \leq \alpha,
\]

and the proof is complete.

**Proposition 3.** If \( S \) is a max-stable admissible cone over a compact Hausdorff space \( X \), then there exists for every \( x \in X \) a measure \( \mu \in M_1^+(X) \) which is pseudo-carried by \( \partial_S X \) such that \( \varepsilon_x \prec_S \mu \).

**Proof.** To show that the \( \prec_S \)-maximal measure \( \mu \) of Proposition 2 is pseudo-carried by \( \partial_S X \), it suffices to show that \( \mu(C) = 0 \) for every compact \( G_\delta \)-set \( C \) such that \( C \cap \partial_S X = \emptyset \). This follows by application of Lemma 2 and (the dual of) Corollary 2 of Proposition 1 to a bounded sequence \( \{f_n\} \) from \( C(X) \) converging pointwise to the indicator function \( \chi_C \).

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2. Unilateral representation theorems in ordered convex compacts.

We shall use the term ordered convex compact to denote a convex compact subset $K$ of a locally convex Hausdorff space $E$ provided with a (partial) ordering defined by a cone $E^+$ which is closed and proper (that is, $E^+ \cap (-E^+) = \{0\}$). The isotone (order preserving) functions in $A(K)$ form a cone, which we denote by $L(K)$, or briefly by $L$. It follows by a standard separation argument (based on the Hahn–Banach Theorem) that $L$ determines the ordering of $K$, in that

\[ x \leq y \iff l(x) \leq l(y) \quad \text{for all } l \in L. \tag{2.1} \]

In particular, $L$ separates the points of $K$.

The set of all maximal points of an ordered convex compact $K$ will be denoted by $Z(K)$ or briefly by $Z$. It follows by a standard application of Zorn's Lemma that there exists for every point $x \in K$ a maximal point $z \in K$ such that $x \leq z$. In particular $Z \neq \emptyset$. It is also easily verified that $Z$ is a union of faces. (It is a "σ-face" in the terminology of Goullet de Rugy [14].) Hence, if $Z$ is convex, then it is a face of $K$.

The $L$-envelopes of functions on an ordered convex compact $K$ will be called monotone convex and concave envelopes. They are related to the customary convex and concave envelopes by the inequalities

\[ \tilde{J}_L \leq \tilde{f} \leq f \leq \check{f} \leq \check{J}_L. \tag{2.2} \]

The ordering $<_L$ defined on $M_1^+(K)$ by the max-stable cone $\check{L}$ will be denoted by $<<$. It is related to the customary ordering of Choquet by the formula

\[ \mu << v \Rightarrow \mu \lll v. \tag{2.3} \]

Observe that if $\mu << v$ and if $x_\mu, x_v$ are the barycenters of $\mu$ and $v$ respectively, then for every $l \in L$ we shall have $l(x_\mu) = \mu(l)$, $l(x_v) = v(l)$, and hence also $l(x_\mu) \leq l(x_v)$. By virtue of (2.1) this gives the implication

\[ \mu << v \Rightarrow x_\mu \leq x_v. \tag{2.4} \]

**Lemma 3.** A point $x$ of an ordered convex compact $K$ belongs to $Z$ iff $\hat{a}_L(x) = a(x)$ for all $a \in A(K)$.

**Proof.** Assume first that $x \in Z$, and consider an arbitrary $a \in A(K)$. By Proposition 1 there is a $\mu \in M_1^+(K)$ such that $\epsilon_x << \mu$ and $\mu(a) = \hat{a}_L(x)$. By (2.4) and by the maximality of $x$, this gives $x = x_\mu$. Since $a \in A(K)$, we obtain $a(x) = \mu(a) = \hat{a}_L(x)$.

Assume next that $x \notin Z$, say $x < y \in K$. By (2.1) there is an $l \in L$ such that $l(x) < l(y)$. Now
\[ l(x) < l(y) \leq \hat{l}_L(y) = \hat{l}_L(x), \]

and the proof is complete.

**Proposition 4.** The set \( Z \cap \partial_{e}K \) of maximal extreme points of an ordered convex compact \( K \) is equal to the L-Choquet boundary \( \partial_{L}K \).

**Proof.** Assume first that \( x \in Z \cap \partial_{e}K \), and consider an arbitrary \( f \in C(K) \). By Corollary 2 to Proposition 1 it suffices to prove that \( \hat{f}_L(x) = f(x) \).

By a known characterization of extreme points due to Hervé [13], there is, for given \( \varepsilon > 0 \), an \( a \in A(K) \) such that

\[
(2.5) \quad f \leq a, \quad a(x) < f(x) + \frac{1}{2} \varepsilon.
\]

By the Lemma \( \hat{a}_L(x) = a(x) \); and by the definition of monotone envelopes there is an \( l \in -L \) such that

\[
(2.6) \quad a \leq l, \quad l(x) < a(x) + \frac{1}{2} \varepsilon.
\]

Combining (2.5) and (2.6) we obtain \( f \leq l \) and \( l(x) < f(x) + \varepsilon \). Since \( \varepsilon > 0 \) was arbitrary, this gives \( \hat{f}_L(x) = f(x) \).

Assume next that \( x \in \partial_{L}K \). Consider first an arbitrary \( y \in K \) such that \( x \leq y \). Observing that every function in \( \hat{L} \) is isotone, we obtain \( \varepsilon_x \ll \varepsilon_y \).

By assumption \( x \in \partial_{L}K = \partial_{L}^L K \), and so \( x = y \). This proves \( x \) to be a maximal point of \( K \).

Consider next an arbitrary \( \mu \in M_1^+(K) \) such that \( x = x_\mu \). Now \( \varepsilon_x \ll \mu \), and it follows by (2.3) that \( \varepsilon_x \ll \mu \). Since \( x \in \partial_{L}K \), we obtain \( \varepsilon_x = \mu \).

This shows that \( x \) is an extreme point of \( K \), and the proof is complete.

**Remark.** It follows from Proposition 4 that \( Z \cap \partial_{e}K \neq \emptyset \). It is also possible to give a more direct proof of this statement. In fact, Mokobodzki and Rogalski [17] have shown by a direct argument that \( Z \cap \partial_{e}F \neq \emptyset \) for every closed face \( F \) of \( K \) which is “hereditary” in the sense that

\[ x \in F, \ y \in K, \ x \leq y \Rightarrow y \in F. \]

**Proposition 5.** If \( K \) is an ordered convex compact, then \( Z \cap \partial_{e}K \) is a non-empty Baire space in the relativized topology. If \( K \) is metrizable, then \( Z \cap \partial_{e}K \) is a \( G_\delta \)-subset of \( K \). For every maximal point \( z \in Z \) there exists a measure \( \mu \in M_1^+(K) \) pseudo-carried by \( Z \cap \partial_{e}K \) such that

\[
(2.7) \quad a(z) = \int a \, d\mu \quad \text{for all} \quad a \in A(K).
\]

If \( K \) is metrizable, then \( \mu(\partial_{e}K) = 1 \).
The proof is a direct application of the results of Section 1, in particular of Proposition 3. Note that $\varepsilon_2 \ll \mu$ implies $z \leq x_\mu$ by virtue of (2.4). By maximality $z = x_\mu$, which gives formula (2.7).

3. Simplicial boundary measures.

The following proposition was proved by Douglas in a slightly different setting [10]. For the sake of completeness we give the proof.

**Proposition 6.** If $K$ is a convex compact set in a locally convex Hausdorff space and $\mu \in M_1^+(K)$, then $\mu$ is simplicial iff $A(K)$ is dense in $L^1(\mu)$.

**Proof.** Assume first that $A(K)$ is non-dense in $L^1(\mu)$. We shall show that $\mu$ is non-extreme in $M_{x_\mu}^+$ where $x = x_\mu$.

By assumption there is a non-zero element $h$ of $L^\infty(\mu)$ such that $\|h\|_{\infty} \leq 1$ and $\mu(ah) = 0$ for all $a \in A(K)$. The measure $\nu$ defined by $d\nu = h \, d\mu$, satisfies $-\mu \leq \nu \leq \mu$. Hence the two measures $\mu_1 = \mu + \nu$, $\mu_2 = \mu - \nu$ are positive and non-equal. Moreover, $\mu_i(a) = \mu(a(x))$ for $i = 1, 2$ and $a \in A(K)$. Hence $\mu_1, \mu_2 \in M_{x_\mu}^+$, and $\mu = \frac{1}{2} \mu_1 + \frac{1}{2} \mu_2$ is non-extreme.

Assume next that $\mu$ is a non-extreme point of $M_{x_\mu}^+$, say $\mu = \frac{1}{2} \mu_1 + \frac{1}{2} \mu_2$ where $\mu_1, \mu_2 \in M_{x_\mu}^+$ and $\mu_1 \neq \mu_2$. Now $0 \leq \mu_1 \leq 2 \mu$, and so $d\mu_1 = h \, d\mu$, where $h \in L^\infty(\mu)$ and $0 \leq h \leq 2$ a.e. (\mu). Also $1 - h$ is a non-zero element of $L^\infty(\mu)$ since $\mu_1 + \mu_2$, and $1 - h$ annihilates $A(K)$ since $\mu((a(1 - h)) = \mu(a) - \mu_1(a) = 0$ for all $a \in A(K)$. This proves that $A(K)$ is non-dense in $L^1(\mu)$.

**Proposition 7.** Let $K$ be a convex compact set in a locally convex Hausdorff space and let $\mu \in M_1^+(K)$ be a measure of finite support, say $\mu = \sum_{j=1}^n \lambda_j \varepsilon_{x_j}$. Then $\mu$ is simplicial iff $\{x_1, \ldots, x_n\}$ is an affinely independent set of points.

**Proof.** Assume first that $\{x_1, \ldots, x_n\}$ is affinely dependent, say

\begin{equation}
\sum_{i=1}^n \beta_i x_j = 0, \quad \sum_{j=1}^n \beta_j = 0,
\end{equation}

where $\{\beta_1, \ldots, \beta_n\} \neq \{0, \ldots, 0\}$. Define $h \in L^\infty(\mu)$ as follows:

$$h(y) = \begin{cases} 
\beta_j \lambda_j & \text{if } y = x_j, \ j \in \{1, \ldots, n\}, \\
0 & \text{if } y \notin \{x_1, \ldots, x_n\}.
\end{cases}$$

Clearly $h$ is a non-zero element of $L^\infty(\mu)$ and for every continuous linear functional $a$ on the given locally convex space,

$$\mu(ah) = \sum_{j=1}^n \beta_j a(x_j) = a(\sum_{j=1}^n \beta_j x_j) = 0, \quad \mu(h) = \sum_{j=1}^n \beta_j = 0,$$

and so $\mu((a + \alpha)h) = 0$ for all $\alpha \in \mathbb{R}$. By density $\mu(a'h) = 0$ for all $a' \in A(K)$. It follows that $A(K)$ is non-dense in $L^1(\mu)$, and by Proposition 6, $\mu$ is non-simplicial.
Assume next that \( \mu \) is a non-extreme point of \( M_x^+ \) where \( x = x_\mu = \sum_{j=1}^n \lambda_j x_j \), say \( \mu = \frac{1}{2} \mu_1 + \frac{1}{2} \mu_2 \) where \( \mu_1, \mu_2 \in M_x^+ \) and \( \mu_1 \perp \mu_2 \). Necessarily \( \mu_2 = \sum_{j=1}^n \alpha_{ij} e_{x_j} \), where \( 0 \leq \alpha_{ij} \leq 2 \lambda_j \) for \( j = 1, \ldots, n \) and \( i = 1, 2 \). Also \( x = \sum_{j=1}^n \alpha_{ij} x_j \) for \( i = 1, 2 \), and so

\[
\sum_{j=1}^n (x_{1j} - x_{2j}) x_j = 0 .
\]

The coefficients of (3.2) have zero sum, and they do not all vanish since \( \mu_1 \perp \mu_2 \). Hence \( \{x_1, \ldots, x_n\} \) is affinely dependent.

**Proposition 8.** If \( K \) is a convex compact subset of \( \mathbb{R}^n \) and \( \mu \in M_1^+(K) \), then \( \mu \) is simplicial iff \( \mu \) is supported by an affinely independent set of (at most \( n+1 \)) points.

**Proof.** By Proposition 7 it suffices to assume that \( \mu \) is simplicial and to show \( \text{Supp}(\mu) \) affinely independent. To this end we define \( F \) to be the affine span of \( \text{Supp}(\mu) \), and we consider an affinely independent subset \( J = \{x_0, \ldots, x_k\} \) of \( \text{Supp}(\mu) \) such that \( F \) is the affine span of \( J \). Now we assume that there is a point \( y \in \text{Supp}(\mu) \setminus J \), and we shall show that this contradicts the simpliciality of \( \mu \).

Let \( U \) be a bounded neighbourhood of 0 in \( \mathbb{R}^n \) such that the sets \( x_0 + U, \ldots, x_k + U; y + U \) are pairwise disjoint. By Proposition 6 there exists a sequence \( \{a_n\} \) from \( A(F) \) which converges in \( L^1(\mu) \) to the indicator function of the set \( y + U \). Writing \( W = \bigcup_{j=0}^k (x_j + U) \), we obtain

\[
\int_W |a_n| \, d\mu \to 0 ,
\]

and

\[
\int_{y+U} a_n \, d\mu \to \mu(y + U) .
\]

(For brevity we write \( \mu(A) \) in the place of \( \mu(A \cap K) \) for \( A \subset \mathbb{R}^n \).) We claim that the functional

\[ p : a \mapsto \int_W |a| \, d\mu \]

is a norm on \( A(F) \). Clearly, it is a semi-norm, and if \( p(a) = 0 \), then \( a(x_0) = \ldots = a(x_k) = 0 \) since \( a \) is continuous and \( x_j \in \text{Supp}(\mu) \) for \( j = 0, \ldots, k \), and this in turn implies \( a = 0 \) on \( F \) since \( a \) is an affine function and \( F \) is spanned by \( x_0, \ldots, x_k \).

The norm \( p \) on the finite dimensional space \( A(F) \) must be topologically equivalent to the supremum norm over \( K \cap F \). Hence \( \{a_n\} \) converges uniformly to 0 on \( K \) by virtue of (3.3). However, \( \mu(y + U) \neq 0 \) since \( y \in \text{Supp}(\mu) \), and hence (3.4) gives a contradiction.
Corollary. If $K$ is a convex compact subset of $\mathbb{R}^n$, then a simplicial boundary measure $\mu \in M_1^+(K)$ is supported by a set of at most $n + 1$ extreme points.

4. Completion of the proofs of the main theorems.

Theorem 1 and Theorem 3 follow easily from the results of Section 2, while the proof of Theorem 2 requires a separate argument based on Bauer's maximum principle. The specialization to the classical Carathéodory Theorem is evident by the results of Section 3.

Proof of Theorem 1 and Theorem 3. The set $M_x^+$ is an ordered convex compact in the vague topology and in the ordering of Choquet. Specifically, the positive cone of $M(K) = C(K)^*$ for this ordering consists of all $\mu$ such that $\mu(f) \geq 0$ for every convex function $f \in C(K)$. By a known characterization [6] of boundary measures, $Z_x$ is the set of maximal elements of $M_x^+$. ($Z_x$ is a face of $M_x^+$ in the present case.) Now an application of Proposition 5 completes the proof.

Proof of Theorem 2. The set $Z_x$ consists of all $\mu \in M_x^+$ such that $\mu(f^* - f) = 0$ for all $f \in C(K)$ [6]. Hence $Z_x$ is closed in the topology $\sigma$. To prove that $Z_x$ is contained in the $\sigma$-closed convex hull of $\partial_x Z_x$, we assume the contrary, say

\begin{equation}
\mu \in Z_x \setminus (\text{conv } \partial_x Z_x)_{\sigma}.
\end{equation}

Now there exists (by Hahn–Banach separation) a function $k \in F$ such that

\begin{equation}
\sup_{\nu \in \partial_x Z_x} \nu(k) = \alpha < \mu(k).
\end{equation}

By the definition of $F$,

$$k = \beta_0 + \sum_{j=1}^n \beta_j f_j,$$

where $f_0, \ldots, f_n \in C(K)$ and $\beta_1, \ldots, \beta_n \in \mathbb{R}$. All measures occurring in (4.2) are boundary measures, and therefore one may replace the function $k$ by the continuous function

$$k' = \beta_0 + \sum_{j=1}^n \beta_j f_j.$$

Next we may replace $k'$ by its (l.s.c.) convex lower envelope $\tilde{k}'$. By a well-known theorem (based on the Hahn–Banach Theorem [6]), there is a net $\{g_\alpha\}$ of continuous convex functions on $K$ such that $g_\alpha \rightarrow \tilde{k}'$. Hence there is a continuous convex function $g$ on $K$ such that $g \leq k'$ and such that

\begin{equation}
\sup_{\nu \in \partial_x Z_x} \nu(g) \leq \alpha < \mu(g).
\end{equation}
Now \( \nu \mapsto \nu(g) \) is a continuous, affine and isotone function on \( M_x^+ \). By Bauer's maximum principle it attains its maximum value at the Choquet boundary defined by the cone of such functions. By Proposition 4 this Choquet boundary is equal to the set \( Z_x \cap \partial_c M_x^+ = \partial_c Z_x \). (Recall that \( Z_x \) is a face of \( M_x^+ \).) By (4.3) this is a contradiction, and the proof is complete.

**Remark.** The conclusion of Theorem 2 does not subsist if \( \sigma \) is replaced by the vague topology or by the norm topology. In fact let \( K = \text{conv}(D \cup L) \) where \( D \) is a plane disk in \( \mathbb{R}^2 \) and \( L \) is a line segment orthogonal to \( D \) which meets \( D \) in a point \( y \in \partial_c D \cap (L \setminus \partial_c L) \). The set \( Z_x \) determined by the center \( x \) of \( D \) is non-closed in the vague topology; and the set \( \partial_c Z_x \) consists of measures supported at at most four points (Proposition 8), hence the norm closed convex hull of \( \partial_c Z_x \) consists of discrete measures only. Thus we shall have

\[
\overline{\text{conv} \partial_c Z_x}_{\text{norm}} \subseteq Z_x \subseteq (\overline{\text{conv} \partial_c Z_x})_{\text{vague}},
\]

and both inclusions are strict.

**References**

17. G. Mokobodzki and M. Rogalski, Private communication.
19. F. G. Vincent-Smith, Private communication.

**Note added in proof, February 1970.**

After the present paper was submitted, we have become aware that the result quoted in the Remark of Section 2 above has already been proved by G. Lumer in 1963 [21]. Also we take the opportunity to mention Andenæs’ forthcoming paper [20], where Vincent-Smith’s construction of simplicial boundary measures is presented in a more general setting.

**FURTHER REFERENCES**


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