A NOTE ON A PROBLEM OF BUSEMANN

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Let $K$ and $K'$ be two bodies in $n$-dimensional euclidean space $E^n$, $n \geq 3$, which are convex and centrally symmetric about the origin. For any unit vector $u$ in $E^n$, the $(n-1)$-dimensional measures (or areas) of the intersection of $K$ or $K'$ and the hyperplane $(x,u) = 0$ are denoted by $A(u)$ and $A'(u)$, respectively. The volumes of $K$ and $K'$ are denoted by $V$ and $V'$. The problem in question [1] is:

Does $A(u) \geq A'(u)$ for all unit vectors $u$ imply $V \geq V'$?

Although easily stated, this important question seems remarkably difficult to answer. Much effort has been spent in trying to establish an affirmative answer, or to find a counterexample. The answer is, indeed, affirmative if $K'$ is an ellipsoid, or if $A'(u) = cA(u)$ (see [1] for references), but no other special case seems to have been settled. The purpose of this note is to show that the answer is yes also if $K$ and $K'$ are 3-dimensional bodies of rotation with a common axis.

In a plane equipped with polar coordinates $r, \psi$ we consider continuous closed curves represented by $\pi$-periodic expressions $r(\psi)$ of the following type: For $0 \leq \psi \leq \frac{1}{2} \pi$ we have $r(\pi - \psi) = r(\psi) > 0$, and as $\psi$ increases from 0 to $\frac{1}{2} \pi$, $r(\psi) \sin \psi$ is nondecreasing, i.e. any line parallel to the polar axis meets the curve in at most two points or segments. When such a curve is rotated around the polar axis we shall call the result a central symmetric monotonic body of rotation generated by $r(\psi)$. The set of all such bodies is denoted by $M$. Clearly, all central symmetric convex bodies of rotation belong to $M$.

Given a body $B$ in $M$, let $A_B(\varphi)$ denote the area which $B$ cuts out of a plane through the center, whose normal makes an angle $\varphi$, $0 \leq \varphi \leq \frac{1}{2} \pi$, with the axis of rotation. We then have the following

**Theorem.** If for two bodies 1 and 2 in $M$ it is true that $A_1(\varphi) \geq A_2(\varphi)$ for all $\varphi$ in $[0, \frac{1}{2} \pi]$, and $A_1(\varphi) > A_2(\varphi)$ for some $\varphi$ in this interval, then the volume of 1 is greater than that of 2.

Received June 15, 1969.

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PROOF. With the notation used in the accompanying figure, we have
\[ \sin \gamma \sin \varphi = \cos \varphi. \]
For a body \( B \), generated by \( r_B(\varphi) \), this relation gives us
\[
A_B(\varphi) = 2 \int_0^{\frac{1}{2} \pi} r_B^2(\varphi) d\gamma = \int_0^\varphi r_B^2(\varphi) f(\varphi, \theta) d\theta,
\]
where \( \varphi = \frac{1}{2} \pi - \theta \), and
\[
f(\varphi, \theta) = (\sin^2 \varphi - \sin^2 \theta)^{-\frac{1}{2}} \cos \theta, \quad 0 \leq \theta < \varphi \leq \frac{1}{2} \pi.
\]
Let the two bodies be generated by \( r_1(\varphi) = r(\theta) + \varrho(\theta) \), and \( r_2(\varphi) = r(\theta) \). Then by assumption
\[
A_1(\varphi) - A_2(\varphi) = 2 \int_0^\varphi (2r\varrho + \varrho^2) f(\varphi, \theta) d\theta \geq 0
\]
for all \( \varphi \) in \( (0, \frac{1}{2} \pi) \), and positive for some \( \varphi \).

Assuming first that the number of intervals in which \( \varrho \) is positive (or negative) is finite, we let \( (\theta_k) \), \( 0 = \theta_0 < \theta_1 < \ldots < \theta_N = \frac{1}{2} \pi \), be a sequence dividing \( [0, \frac{1}{2} \pi] \) into intervals where \( \varrho \) has constant sign. Clearly \( \varrho \) can not be negative in \( [\theta_k, \theta_{k+1}] \), so \( \varrho \) is nonnegative in \( [\theta_{2k}, \theta_{2k+1}] \) and nonpositive in \( [\theta_{2k-1}, \theta_{2k}] \). Thus
\[
A_1(\theta_n) - A_2(\theta_n) = 2 \sum_{k=0}^{n-1} (-1)^k \int_{\theta_k}^{\theta_{k+1}} |2r\varrho + \varrho^2| f(\theta_n, \theta) d\theta, \quad n = 1, 2, \ldots, N.
\]
Putting
\[
\sigma_k = \int_{\theta_k}^{\theta_{k+1}} |2r\varrho + \varrho^2| d\theta
\]
we have
\[ \int_{\theta_k}^{\theta_{k+1}} |2r' \varphi + \varphi' | f(\theta, \varphi) \, d\theta = \alpha_{k,n} \sigma_k , \]

where

\[ \alpha_{k,n} = f(\theta_n, \xi_{k,n}) \quad \text{for some } \xi_{k,n} \text{ in } [\theta_k, \theta_{k+1}] . \]

For any \( \varphi, \ 0 < \varphi < \frac{1}{2} \pi \), the function \( f \) is strictly increasing in \( \theta \), so

\[ 0 < \alpha_{0,n} < \alpha_{1,n} < \alpha_{2,n} < \ldots . \]

But our assumption

\[ \sum_{k=0}^{n-1} (-1)^k \alpha_{k,n} \sigma_k \geq 0, \quad n = 1, 2, \ldots , N , \]

then implies (by Abel's formula for partial summation)

\[ \sum_{k=0}^{n-1} (-1)^k \beta_k \sigma_k > 0 , \]

for any decreasing sequence \( (\beta_k) \). This proves that \( V_1 > V_2 \) since \( V_1 - V_2 \) can be written as a sum of this type, with \( \beta_k = \pi y_k \), where \( y_k \) is the distance from the axis of rotation to the center of gravity of the region with area \( \sigma_k \).

If \( r + \varphi \) and \( r \) intersect at infinitely many points, we still must have \( \varphi(\theta) > 0 \) in some interval \( \theta' < \theta < \theta'' < \frac{1}{2} \pi \). Denote the area enclosed by \( r + \varphi \) and \( r \) in this interval by \( \sigma \). Let \( p \) be any outer polygonal approximation to \( r + \varphi \), and \( q \) any inner polygonal approximation to \( r \). Let \( \theta_k, \sigma_k \) and \( \beta_k \) be defined as above for \( p \) and \( q \). For some even number \( n \) the interval \( [\theta_n, \theta_{n+1}] \) contains \( [\theta', \theta''] \), and \( \sigma_n \geq \sigma \). There must also exist a positive \( \delta \), independent of \( p \) and \( q \), such that \( \beta_n - \beta_{n+1} > \delta \). Since (again using partial summation)

\[ V_p - V_q = \sum_{k=0}^{N-1} (-1)^k \beta_k \sigma_k > (\beta_n - \beta_{n+1}) \sigma_n > \delta \sigma , \]

we conclude that

\[ V_1 - V_2 \geq \delta \sigma > 0 . \]

This completes the proof.

It was recently brought to the author's attention by J. Schaefer (Calgary) that H. Hadwiger (Bern) has obtained by different methods a very similar result, as a special case of a theorem appearing in volume 23 of this journal [2]. Neither result contains the other. The author also wishes to express his gratitude to Professor Hadwiger for pointing out that a certain condition assumed in the first draft of this note was redundant.
REFERENCES


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