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AUTOMORPHISMS OF ABSTRACT AFFINE NEAR-RINGS

JOSEPH J. MALONE, JR.

0. Introduction.

Let γ be a homomorphism of the near-ring N into the near-ring M. Then γ carries the maximal sub-C-ring of N into the maximal sub-C-ring of M and carries the maximal sub-Z-ring of N into the maximal sub-Z-ring of M. Moreover, the homomorphism is completely determined by its restrictions to these sub-near-rings. Conversely, one may ask which homomorphisms on the sub-C-ring and which homomorphisms on the sub-Z-ring may be mated to produce a homomorphism on N to M, that is, which such homomorphisms may occur as restrictions. In general, a satisfactory answer has not been given.

This paper investigates the homomorphism construction problem for abstract affine near-rings. In particular, automorphisms of abstract affine near-rings are studied. Information about automorphisms of near-rings seems important as a preliminary to obtaining Galois-like results for near-rings.

1. Preliminaries.

Let M be a left R-module. On $R\times M$ define a coordinatewise addition and define multiplication such that

$$(r_1,m_1)\cdot(r_2,m_2) = (r_1r_2,\, r_1m_2+m_1), \qquad r_1,r_2\in R \text{ and } m_1,m_2\in M\ .$$

The system $(R\times M,\ +,\ \cdot)$ is an example of the type of near-ring known as an abstract affine near-ring. In [3] it is shown that every abstract affine near-ring arises from such a construction on a module.

Gonshor introduced abstract affine near-rings in [3] and completely described their ideal structure. He generalizes the results in [2] and [4]. In the terminology of [1], an abstract affine near-ring is a near-ring in which the maximal sub-C-ring, that is $(R,0)$, is left-distributive and in which the maximal sub-Z-ring is $(0,M)$. Note that in [1] near-rings are left near-rings whereas in [3] near-rings are right near-rings.

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2. Compatible endomorphisms.

Definition. Let N be a left R-module. If α is a ring endomorphism of R such that

$$(r-\alpha r)n = 0, \qquad r \in R \text{ and } n \in N ,$$

α will be called an N-compatible endomorphism of R.

We note in passing that the kernel of an N-compatible endomorphism is contained in the annihilator of N.

Let M and N be left R-modules giving rise to the abstract affine near-rings $R \times M$ and $R \times N$. The following theorems show the manner in which R-homomorphisms on M to N may be extended to near-ring homomorphisms on $R \times M$ to $R \times N$. Essentially, any R-homomorphism may be mated with any N-compatible endomorphism of R.

Theorem 1. *Let M and N be left R-modules, let α be an N-compatible endomorphism of R, and let β be an R-homomorphism on M to N. The map*

$$\varphi\colon\ R \times M \to R \times N \quad \text{defined by} \quad \varphi(r,m) = (\alpha r, \beta m)$$

is a near-ring homomorphism on $R \times M$ to $R \times N$.

Proof. We verify that multiplication is preserved by φ. The rest is immediate. Consider

$$\begin{aligned}\varphi\big((r_1,m_1)(r_2,m_2)\big) &= \varphi(r_1r_2,\, r_1m_2+m_1)\\ &= \big(\alpha(r_1r_2),\, \beta(r_1m_2+m_1)\big) = \big(\alpha(r_1r_2),\, r_1\beta m_2+\beta m_1\big)\end{aligned}$$

and

$$\begin{aligned}\varphi(r_1,m_1)\,\varphi(r_2,m_2) &= (\alpha r_1,\, \beta m_1)(\alpha r_2,\, \beta m_2)\\ &= (\alpha r_1 \alpha r_2,\, \alpha r_1 \beta m_2+\beta m_1) = \big(\alpha(r_1r_2),\, \alpha r_1\beta m_2+\beta m_1\big) .\end{aligned}$$

The result follows from the compatibility condition.

Corollary 1a. *Let β be an R-homomorphism on M to N. Then $\varphi'\colon R \times M \to R \times N$ defined such that $\varphi(r,m)=(r,\beta m)$ is a near-ring homomorphism.*

Proof. The identity map on R is N-compatible.

Corollary 1b. *Let α be an N-compatible endomorphism of R. Then $\varphi''\colon R \times N \to R \times N$ defined such that $\varphi(r,n)=(\alpha r,n)$ is a near-ring endomorphism.*

Proof. The identity map on N is an R-homomorphism.

Theorem 2. *Let φ be a near-ring homomorphism on $R \times M$ to $R \times N$.*

Let α be the restriction of φ to R, let β be the restriction of φ to M, and let β be onto N. Then α is an N-compatible endomorphism of R iff β is an R-homomorphism on M to N.

Proof. As remarked before, these restrictions determine the homomorphism. Consider

$$\begin{aligned}\varphi(0,rm) = \varphi((r,0)\,(0,m)) &= \varphi(r,0)\,\varphi(0,m)\\ &= (\alpha r,0)\,(0,\beta m) = (0,\alpha r\beta m)\end{aligned}$$

and

$$\varphi(0,rm) = (0,\beta(rm)), \qquad r\in R \text{ and } m \in M\,.$$

Hence

$$\alpha r\beta m = \beta(rm)\,.$$

If β is an R-homomorphism, $\alpha r\beta m = r\beta m$ and $(r-\alpha r)\beta m = 0$. Since an arbitrary element of N can be put in the form βm, α is an N-compatible endomorphism of R. Conversely, if α is compatible we have $(r-\alpha r)\beta m = 0$. Thus $r\beta m = \alpha r\beta m = \beta(rm)$ and β is an R-homomorphism.

Turning to automorphisms we find that the mating process described above yields all of the automorphisms for certain abstract affine near-rings $R\times M$, i.e. every near-ring automorphism of $R\times M$ is an extension of an R-automorphism of M. This is the case, for instance, for a trivial module or if R is the ring of integers and operator multiplication is the taking of natural multiples. On the other hand, consider the abstract affine near-ring arising from the module for which R is the field of complex numbers and M is the additive group of complex numbers. Let each of α and β be the map which takes an element into its conjugate. Define

$$\varphi:\ R\times M \to R\times M \qquad \text{such that} \qquad \varphi(r,m) = (\alpha r,\beta m)\,.$$

Then φ is a near-ring automorphism of $R\times M$ but α is not compatible. This may be seen by taking $r = i$ and $m = 1$. So not every automorphism of an abstract affine near-ring has restrictions which are, respectively, M-compatible and an R-automorphism.

3. Non-compatible automorphisms.

In this section we investigate the manner in which non-compatible automorphisms of R and non-R-automorphisms of M are mated to yield the remaining near-ring automorphism of $R\times M$.

Theorem 3. *With M as a left R-module, let α be a ring automorphism of R and let β be a group automorphism of M. Then the map*

$$\varphi\colon\ R\times M \to R\times M \quad \textit{defined by} \quad \varphi(r,m) = (\alpha r, \beta m)$$

is a near-ring automorphism of $R\times M$ *iff* $\alpha r\beta m=\beta(rm)$, $r\in R$ *and* $m\in M$.

PROOF. This theorem is immediate from the proofs of Theorems 1 and 2.

We have seen that any compatible automorphism of R may be mated with any R-automorphism of M. We now discuss the uniqueness of extensions as it concerns non-compatible automorphisms of R and non-R-automorphisms of M.

THEOREM 4. *With M as a left R-module, let A be the group of all ring automorphisms of R and let X be the subset of compatible ring automorphisms of R. Then congruence modulo X is an equivalence relation on A.*

PROOF. Since a subgroup of a group induces an equivalence relation on the elements of the group, we only need show that X determines a subgroup of A.

We know that the identity map on R is compatible. Let $\alpha\in X$. Then

$$(r-\alpha r)m = 0, \qquad r\in R \text{ and } m\in M\,.$$

For $r\in R$, there exists $r_1\in R$ such that $\alpha^{-1}r_1=r$. Hence

$$(\alpha^{-1}r_1-\alpha(\alpha^{-1}r_1))m = (\alpha^{-1}r_1-r_1)m = 0$$

or

$$(r_1-\alpha^{-1}r_1)m = 0, \qquad r_1\in R \text{ and } m\in M\,.$$

Hence $\alpha^{-1}\in X$. Let $\alpha,\gamma\in X$. Then each of these may be mated with the identity map on M. (We will use the notation $\varphi=[\alpha,\beta]$ to indicate a near-ring automorphism of $R\times M$ whose restriction to R is α and whose restriction to M is β.) Consider $\varphi=[\alpha,\iota]$ and $\varphi'=[\gamma,\iota]$, ι the identity map on M. Then $\varphi\varphi'=[\alpha\gamma,\iota]$ is a near-ring automorphism of $R\times M$. Since ι is an R-automorphism, $\alpha\gamma$ is a compatible automorphism of M. Hence the theorem follows.

THEOREM 5. *Let $\varphi=[\alpha,\beta]$ and let $\gamma\equiv\alpha\ (\mathrm{mod}\,X)$. Then $[\gamma,\beta]$ is a near-ring automorphism of $R\times M$. If $[\alpha,\beta]$ and $[\gamma,\beta]$ are near-ring automorphisms of $R\times M$, then $\gamma\equiv\alpha\ (\mathrm{mod}\,X)$.*

PROOF. By hypothesis, $\gamma\alpha^{-1}\in X$. Then $[\gamma\alpha^{-1},\iota]$ is a near-ring automorphism and $[\gamma\alpha^{-1},\iota][\alpha,\beta]=[\gamma,\beta]$ is a near-ring automorphism of $R\times M$ as desired.

The product of the near-ring automorphisms $[\gamma,\beta]$ and $[\alpha^{-1},\beta^{-1}]$ is the near-ring automorphism $[\gamma\alpha^{-1},\iota]$. Hence $\gamma\equiv\alpha\ (\mathrm{mod}\,X)$.

Thus we see that, if one element of a coset of A (mod X) can be mated with a β, then so can all the members of the same coset. Moreover, only the members of this coset may be mated with β. Since each coset (mod X) has the same cardinality as X, we have that, if a non-R-automorphism of M can be extended to an automorphism of $R \times M$, it has as many extensions as an R-automorphism of M.

We have viewed the near-ring automorphism of $R \times M$ as extensions of the group automorphisms of M. We may also view the near-ring automorphisms as extensions of the ring automorphisms of R. The following theorems are analagous to those just stated. Proofs will not be given.

THEOREM 6. *With M as a left R-module, let B be the group of all group automorphisms of M and let Y be the subset of all R-automorphisms of M. Then congruence modulo Y is an equivalence relation on B.*

THEOREM 7. *Let $\varphi = [\alpha, \beta]$ and let $\delta \equiv \beta \pmod{Y}$. Then $[\alpha, \delta]$ is a near-ring automorphism of $R \times M$. If $[\alpha, \beta]$ and $[\gamma, \delta]$ are near-ring automorphisms of $R \times M$, then $\beta \equiv \delta \pmod{Y}$.*

REFERENCES

1. G. Berman and R. J. Silverman, *Near-rings*, Amer. Math. Monthly 66 (1959), 23–34.
2. D. W. Blackett, *The near-ring of affine transformations*, Proc. Amer. Math. Soc. 7 (1956), 517–519.
3. H. Gonshor, *On abstract affine near-rings*, Pacific J. Math. 14 (1964), 1237–1240.
4. K. G. Wolfson, *Two-sided ideals of the affine near-ring*, Amer. Math. Monthly 65 (1958), 29–30.

DEPARTMENT OF MATHEMATICS, TEXAS A & M UNIVERSITY,
COLLEGE STATION, TEXAS 77843, U.S.A.