NOTE ON A PAPER BY STETKÆR-HANSEN
CONCERNING ESSENTIAL SELFADJOINTNESS
OF SCHROEDINGER OPERATORS

JOHANN WALTER

Introduction.

By $G$ we denote an open set in $\mathbb{R}^n$, by $(u, v) = \int_G u \overline{v} \, dx$ the scalar product defined in the Hilbert space $L^2(G)$. By $H_{2, \text{loc}}(G)$ we denote the space of all functions which are defined in $G$ and possess locally square integrable derivatives up to the second order, and $Q_{x, \text{loc}}(G)$ is the set of all functions satisfying in $G$ a local Stummel condition. (A description of this condition can be found, for example, in [5] and [3]. Atomic Coulomb potentials are included in $Q_{x, \text{loc}}$.)

Let

$$a_{jk}(x) \in C^2(G), \quad b_j(x) \in C^1(G), \quad q(x) \in Q_{x, \text{loc}}(G)$$

be realvalued functions and $(a_{ik})$ a positive definite symmetric matrix. If we denote by $A$ the symmetric operator defined in $C^0_0(G)$ by the differential expression

$$Du = \sum_{i, k=1}^n D_j a_{jk} D_k u + qu, \quad D_j = i \frac{\partial}{\partial x_j} + b_j,$$

it is known (cf. [2], [3]) that the adjoint operator $A^*$ has the domain of definition

$$(1) \quad \mathcal{D}(A^*) = \{ u \mid u \in L^2(G) \cap H_{2, \text{loc}}(G), Du \in L^2(G) \}.$$

Now choose nonnegative lipschitzean functions $\varrho(x)$ and $\sigma(x)$ with the following properties in $G$:

$$(2) \quad \sum a_{jk} \varrho_{x_j} \varrho_{x_k} \leq 1 \quad \text{a.e.,}$$

$$(3) \quad \sum a_{jk} \sigma_{x_j} \sigma_{x_k} \leq e^{2\sigma} \quad \text{a.e.,}$$

$$(4) \quad \lim_{x \to \partial G} \{ \varrho(x) + \sigma(x) \} = \infty. \quad (*)$$

Received May 10, 1968.

*) The condition (2) was first used by Jörgens [3], the conditions (3) and (4) are due to the author [7] resp. [8].

If (2) and (3) hold with $\varphi^2(\varrho)$ and $\varphi^2(\sigma)$ respectively instead of 1 and $e^{2\sigma}$ respectively at the right side and if
Theorem. If \( \delta \) is a positive number and

\[
(Au, u) \geq (1 + \delta)(e^{2\sigma}u, u) \quad \text{for all } u \in C_0^2(G),
\]

then \( A \) in \( C_0^2(G) \) is essentially selfadjoint.

Clearly \( \sigma(x) \) has to be chosen as small as possible to make condition (5) less restrictive.

The proof of our theorem (which in the case \( \sigma \equiv 0 \) is due to Stetkær-Hansen [4]) is a suitable generalization of a proof of Wienholtz [9]. In the case \( \sigma \equiv 0 \) Triebel [6] deduces a special result in a similar way.

Proof of the theorem.

Since \( A \) is bounded from below by 1, it is sufficient to show that \( h \in L^2(G) \) and \( h(Au) = 0 \) for all \( u \in C_0^2(G) \) imply \( h = \Theta \) (\( \Theta \) denoting the zero element of \( L^2(G) \)); cf. [1, p. 159]. Making essential use of (1), we deduce from (5) in the same way as in [9, p. 60] and [4] that

\[
\int_G |h|^2(\sum a_{jk}
\gamma_j \gamma_k) \, dx \geq (1 + \delta) \int_G |h|^2 e^{2\sigma} \, dx
\]

holds for all lipschitzian functions \( \gamma(x) \) with a compact support in \( G \).

Let \( f(t), g(t) \) be functions defined in \([0, \infty)\) with piecewise continuous first derivatives and compact support. Following an idea of Jörgens [3], we put \( \gamma(x) = f(\sigma(x))g(\sigma(x)) \). Because of (4), \( \gamma(x) \) has compact support in \( G \). We insert this \( \gamma \) into (6) and arrive at the inequality

\[
(1 + \epsilon) \int_G |h|^2 e^{2\sigma} f'(t)^2 \, dx + (1 + 1/\epsilon) \int_G |h|^2 (f')^2 g^2 \, dx \geq \frac{1}{2} \epsilon \int_G |h|^2 f^2 g^2 \, dx + (1 + \frac{1}{2} \epsilon) \int_G |h|^2 e^{2\sigma} f^2 g^2 \, dx
\]

(for any \( \epsilon > 0 \)) by using (2), (3) and some easy estimates.

We now choose

\[
f(t) = \begin{cases} 
1 & \text{for } 0 \leq t \leq R, \\
\text{linear for } R \leq t \leq R + 1, \\
0 & \text{for } R + 1 \leq t,
\end{cases} \quad g(t) = \begin{cases} 
e^{-t} - e^{-1} e^{-t} & \text{for } 0 \leq t \leq \alpha, \\
0 & \text{for } \alpha \leq t.
\end{cases}
\]

It follows that

\[
\int_0^\infty dt/\varphi(t) = \infty \quad \text{and} \quad \int_0^\infty dt/\psi(t) < \infty,
\]

the new functions

\[
r(x) = \int_0^x dt/\varphi(t) \quad \text{and} \quad s(x) = \text{Max} \left\{ 0; -\log \int_0^\infty dt/\varphi(t) \right\}
\]

satisfy (2) and (3) respectively; cf. [4], [7].
\( f' \equiv 0 \) for \( t \notin [R, R+1] \), \( |f'| \equiv 1 \) for \( t \in (R, R+1) \),

\[ f \leq 1, \quad g \leq e^{-t}, \quad |g'| \leq (1 + 1/\alpha)e^{-t}, \]

and \( g(t) \) converges uniformly to \( e^{-t} \) as \( \alpha \to \infty \). Inserting this into (7), taking into consideration that \( h \in L^2(G) \) and letting \( \alpha \to \infty \), we finally get the inequality

\[
(1 + \varepsilon) \int_G |h|^2 f^2 \, dx + (1 + 1/\varepsilon) \int_{R \leq \varepsilon \leq R+1} |h|^2 e^{-2\varepsilon} \, dx \\
\leq \frac{1}{2} \delta \int_{\varepsilon \leq R} |h|^2 e^{-2\varepsilon} \, dx + (1 + \frac{1}{2}\delta) \int_G |h|^2 f^2 \, dx.
\]

For \( \varepsilon < \frac{1}{2}\delta \), \( h \neq \Theta \) and \( R \) sufficiently large this is a contradiction.

REFERENCES


INSTITUTE OF MATHEMATICS

TECHNICAL UNIVERSITY, 51 AACHEN

GERMANY