VECTORVALUED DISTRIBUTIONS COVARIANT UNDER ALGEBRAIC REPRESENTATIONS OF AN ORTHOGONAL GROUP OF ARBITRARY SIGNATURE

A. TENGSTRAND

0. Introduction.

Let $G$ be a Lie group acting on a $C^\infty$-manifold $X$ and let $A$ be a $C^\infty$-representation of $G$ as linear mappings on a finite-dimensional linear space $L$. A distribution $T$ on $X$ with values in $L$ is called covariant under $A$ if

$$ A(g)T(x) = T(gx) $$

for every $g \in G$. Suppose that $X$ locally is a product $V \times Y$ where

$$ Y = [0, 1] \times \ldots \times [0, 1] \subseteq R^s $$

and that $V \times \{y\}$ for every $y \in Y$ is an open part of an orbit in $X$. In this case we are going to prove that, at least locally, a covariant distribution can be written

$$ T(x) = \sum_j f_j(x) \varphi_j(x), $$

where $f_j$ are distributions on $X$ with values in $\mathbb{C}$ which are invariant under $G$, that is $f_j(gx) = f_j(x)$ for every $g \in G$ and $\varphi_j$ are $C^\infty$-functions from $X$ to $L$ which are covariant. Then we use this result to get an explicit description of the covariant distributions when $X = R^n$, $G = SO(p, q)$ and $A$ is an algebraic representation of $SO(p, q)$. Here $SO(p, q)$ is the connected component of the unit $e$ of the orthogonal group which leaves the quadratic form

$$ \langle x, x \rangle = \sum_{r=1}^{n-p+q} \epsilon(r) x_r^2, \quad \epsilon(1) = \ldots = \epsilon(p) = 1, \quad \epsilon(p+1) = \ldots = \epsilon(p+q) = -1 $$

invariant. It turns out that the covariant distributions can be written as (1). The invariant distributions $f$ are described in [2] for $p=1$ and in [5] for the remaining $p$ and $q.$

Received November 12, 1968.
1. Covariant distributions on manifolds which locally are product manifolds.

Let the Lie group $G$ act on the $C^\infty$-manifold $X$ and write $x_1 \sim x_2$ if and only if there is $g \in G$ such that $gx_1 = x_2$. In the following we suppose that the following condition is satisfied:

(*) To every $x_0 \in X$ there exist neighbourhoods $U \subseteq X$ and $V \subseteq Gx_0$ and a diffeomorphism $\varphi = \varphi_U : U \rightarrow V \times Y$ such that $p_Y \varphi(x_1) = p_Y \varphi(x_2)$ if and only if $x_1 \sim x_2$. (Here $p_Y$ is the projection $V \times Y \rightarrow Y$.)

It is easily seen that $\varphi$ can be chosen so that $\varphi(x) = (x, y_0)$ if $x \in V$ and $\varphi(x_0) = (x_0, y_0)$.

**Lemma 1.** To every $x_0 \in X$ there is a neighbourhood $W \subseteq X$ and a $C^\infty$-function $h : W \times W \rightarrow G$ such that $h(x_1, x_2) x_1 = x_2$ if $x_1, x_2 \in W$ and $x_1 \sim x_2$.

**Proof.** Choose neighbourhoods $U \subseteq X$, $V \subseteq Gx_0$ and a diffeomorphism $\varphi = \varphi_U$ as above. It is easily seen that there is a neighbourhood $O \subseteq G$ of $e$ such that $gx \in U$ if $\varphi(x) = (x_0, y)$. Define $\alpha : O \times Y \rightarrow U$ by setting $\alpha(g, y) = g \varphi^{-1}(x_0, y)$. The restriction of $\alpha$ to $O \times \{y_0\}$ is then equal to $\varphi^{-1} \circ (\pi \times 1_Y)$, where $\pi : O \rightarrow V$ and $\pi g = gx_0$. It is well-known that $\operatorname{rank} d\alpha(e) = \dim V$ (see, for example, Helgason [3, pp. 110–113]) and this implies that

$$\operatorname{rank} d\alpha(e, y_0) = \dim V + \dim Y = \dim V \times Y.$$ 

Now it follows that there is a $C^\infty$-function $\beta : W \rightarrow O \times Y$ such that $\alpha \circ \beta = \operatorname{id}_W$ and we can put $h(x_1, x_2) = \gamma(x_2) \gamma(x_1)^{-1}$, where $\gamma = p_O \circ \beta$ and $p_O$ is the projection $O \times Y \rightarrow O$.

If $A$ is a representation of $G$ as linear mappings of the finite-dimensional linear space $L$ we put

$$L^x = \{v \in L ; \quad A(g)v = v \text{ for every } g \in G_x\},$$

where $G_x = \{g \in G ; \quad gx = x\}$.

Let $U$ be an open part of $X$ with compact closure and $C^\infty(U, L)$ the set of all $C^\infty$-functions from $U$ to $L$. We put

$$N(U) = \{f \in C^\infty(U, L) ; \quad f(x) \in L^x \text{ for every } x \in U\}.$$ 

$N(U)$ is a module with coefficients in $C^\infty(U, C)$ which is spanned by a finite number of functions $\varphi_j$. Evidently it is sufficient to prove this statement in a neighbourhood of every point $x_0 \in X$ since $U$ has compact closure. Choose a neighbourhood $W$ of $x_0$ and a function $h$ as in lemma 1 and let $\{e_j\}$ be a basis of $L^x$. Then
\[ \psi_j(x) = A(h(x, x_0)) e_j \in C^\infty(W, L) \]

is a basis of \( L^x \) for every \( x \in W \).

If \( g \) is sufficiently close to \( e \in G \) and if \( x \) belongs to a small neighbourhood of \( x_0 \) we have

\[ \psi_j(gx) = A(h(gx, x_0)) e_j = A(h(gx, x)) \psi_j(x) = A(g) \psi_j(x) \]

since \( g^{-1} h(gx, x) \in G_x \). Suppose that \( G \) has the Lie-algebra \( \Gamma \). If \( T \) is covariant, we have

\[ e^{-\gamma^t T(e^\alpha x)} = T(x) \quad \text{for every } \gamma \in \Gamma, \]

where \( \gamma' = dA(\gamma) \). By differentiation with respect to \( t \) we get

\[ \gamma' T(x) = \gamma^+ T(x), \tag{2} \]

where \( \gamma^+ \) is the vectorfield on \( X \) generated by \( \gamma \). On the other hand (2) implies that \( A(g) T = T \circ g \) for every \( g \) in a neighbourhood of \( e \) and consequently for all \( g \) if \( G \) is connected. It is also clear that \( \gamma' \psi_j(x) = \gamma^+ \psi_j(x) \)

when \( x \in W \).

**Theorem 1.** Let \( T \) be a distribution on \( X \) with values in \( L \) such that

\[ \gamma' T = \gamma^+ T \quad \text{for all } \gamma \in \Gamma \tag{3} \]

in the relatively compact open set \( U \subseteq X \). Then

\[ T(x) = \Sigma_j f_j(x) \psi_j(x) \quad \text{in } U, \]

where \( f_j \) are distributions on \( X \) with values in \( C \) such that \( \gamma^+ f_j = 0 \) in \( U \) and \( \psi_j \) belongs to \( C^\infty(U, L) \) and has the property \( \gamma' \psi_j = \gamma^+ \psi_j \) in \( U \).

**Proof.** It is evident that it suffices to prove the theorem in a neighbourhood of every point in \( U \). Let \( x_0 \in U \) and choose a neighbourhood \( U' \subseteq X \) of \( x_0 \) according to (*) and a neighbourhood \( O \subseteq G \) of \( e \) such that \( O U' \subseteq U \). From (3) it is easily seen that \( A(g) T(x) = T(gx) \) in \( U' \) for every \( g \in O \). Let \( V \) be a neighbourhood of \( x_0 \) in \( G x_0 \) and \( \varphi = \varphi_{U'}: U' \to V \times Y \) a diffeomorphism (see *). Clearly we can choose \( U' \) so that there is a function \( h: U' \times U' \to G \) as in lemma 1. Now we put

\[ \overline{T} = T \circ \varphi^{-1}, \quad \overline{\varphi} = \varphi \circ g \circ \varphi^{-1} \quad \text{for } g \in O, \quad \overline{h} = h \circ (\varphi^{-1} \times \varphi^{-1}). \]

\( \overline{T} \) is a distribution on \( V \times Y \) such that

\[ A(\overline{\varphi}) \circ T = \overline{T} \circ \overline{\varphi} \quad (A(\overline{\varphi}) = A(g)). \]

If \( R^Y_x \) is a regulisator on \( Y \) (see de Rham [4]), then for every \( y \in Y \)
\( \overline{T}_{e,y} = R_{e} V \overline{T}(y) \) is a distribution on \( V \times \{y\} \) which satisfies

\[
A(\overline{h}(z_1, z_2)) \overline{T}_{e,y} = \overline{T}_{e,y} \circ \overline{h}(z_1, z_2)
\]

for every \( z_1 \) and \( z_2 \) in \( V \).

Let \( \omega^\alpha \) be an infinitely differentiable \( m \)-form on \( V \) \((m = \dim V)\) such that

\[
\omega^\alpha \to \delta_{z_0} \quad \text{as} \quad \alpha \to 0
\]

in distribution-sense. If \( \omega^\alpha_z = \omega^\alpha \circ \overline{h}(z_0, z) \) it is easily seen that \( \omega_z \to \delta_z \) in distribution-sense and if \( S = \overline{T}_{e,y} \) we have

\[
\langle S, \omega^\alpha_z \rangle \to S(z) \quad \text{as} \quad \alpha \to 0
\]

in distribution-sense. For every \( z_1 \) and \( z_2 \) in \( V \) we have

\[
A(\overline{h}(z_1, z_2)) \langle S, \omega^\alpha_z \rangle = \langle S \circ \overline{h}(z_1, z_2), \omega^\alpha_z \rangle.
\]

Here the function on the left is a \( C^\infty \)-function and we can put \( z_2 = z \) and get

\[
A(\overline{h}(z_1, z)) \langle S, \omega^\alpha_z \rangle = \langle S, \omega^\alpha_z \circ \overline{h}(z_1, z) \rangle.
\]

But as \( \omega^\alpha_z \circ \overline{h}(z_1, z)^{-1} = \omega^\alpha \circ \overline{h}(z_0, z) \overline{h}(z_1, z)^{-1} = \omega^\alpha_{z_1} \) we have

\[
A(\overline{h}(z_1, z)) \langle S, \omega^\alpha_z \rangle = \langle S, \omega^\alpha_{z_1} \rangle.
\]

Here the function on the left converges to \( A(\overline{h}(z_1, z) \circ S(z)) \) which is independent of \( z \) because the function on the right is independent of \( z \). Furthermore \( A(\overline{h}(z_1, z)) \circ S(z) \) is a \( C^\infty \)-function in \( z_1 \) and as \( \langle S, \omega^\alpha_{z_1} \rangle \to S(z_1) \) we conclude that \( S \in C^\infty(V, L) \). Now we have proved that the distribution \( S = \overline{T}_{e,y} \) is infinitely differentiable in \( z \) and \( y \). The function \( T_{e,y} = \overline{T}_{e,y} \circ \varphi \) is an element of the module \( N(U) \) and consequently we can write

\[
T_{e}(x) = \sum f_{j,e}(x) \psi_j(x),
\]

where \( \psi_j \) are the functions above. As \( \psi_j \) are linear independent it follows that \( f_{j,e} \) converges in \( D'(U) \) to distributions \( f_j \). We have proved that

\[
T(x) = \sum f_j(x) \psi_j(x)
\]

in a neighbourhood of every point in \( U \) and consequently in all of \( U \).

From the construction of \( h \) in lemma 1 and from the proof above it follows that \( f_{j,e} \) only depend on \( y \) and consequently \( \gamma^+ f_{j,e} = 0 \) and then \( \gamma^+ f_j = 0 \).

Remark. The theorem holds for an arbitrary open set \( U \subseteq X \) if there is a finite number of linear independent covariant functions \( \psi_j \) which
span \(N(U)\). If \(U = X\) this implies that the distributions \(f_j\) are invariant under \(G\).

2. Description of \(L^2\) when \(G = SO(p, q)\).

Let \(A\) be an irreducible representation of \(G = SO(p, q)\) as linear mappings of a finite-dimensional linear space \(L\). For every \(x \in \mathbb{R}^n\), \(L^x\) is the direct sum of the one-dimensional subspaces of \(L\) which is invariant under \(A(G_x)\). From Weyl [6] it follows that the representations of \(G_x\), which is isomorphic to \(SO(p', q')\) where \(p' + q' = n - 1\), have degree 1 if and only if they have the weight \((0, \ldots, 0)\). If \(A\) is irreducible it follows from Boerner ([1], p. 251–254) that there are one-dimensional subspaces of \(L\) invariant under \(A(G_x)\) if and only if \(A\) has the weight \((k, 0, \ldots, 0)\) where \(k\) is a positive integer and that there is only one such subspace. If the weight of \(A\) is \((k, 0, \ldots, 0)\) \(L\) is isomorphic to \(\pi_k\), where \(\pi_k\) is the space of all homogeneous polynomials

\[
p(\xi) = \sum_\alpha a_\alpha \xi_\alpha = \sum_\alpha a_{\alpha_1 \ldots \alpha_k} \xi_{\alpha_1} \cdots \xi_{\alpha_k}
\]

in \(\xi = (\xi_1, \ldots, \xi_n)\) of degree \(k\) and where \(a_\alpha\) are symmetric in \(\alpha\) and

\[
\text{Tr}(p) = (\sum_j \epsilon(j) a_{jj^*}) \xi_{j^*} = 0,
\]

where \(\alpha^* = (x_2, \ldots, x_k)\). (See Weyl [6].)

Evidently the polynomials \(\langle \xi, \eta \rangle^j = (\sum_{i=1}^n \epsilon_i \eta_i)^j\) are invariant under \(G\) and the polynomials \(\langle x, \xi \rangle^s = (\sum_{i=1}^n \epsilon_i x_i \xi_i)^s\) are invariant under \(G_x\).

We put

\[
P_{j,k}(x, \xi) = c_{j,k} \langle \xi, \xi \rangle^j \langle x, \xi \rangle^{k - 2j}, \quad 0 \leq 2j \leq k,
\]

where

\[
c_{j,k} = \binom{k}{2j} (2j)! / 2^n.
\]

Then \(P_{j,k}\) is homogeneous in \(\xi\) of degree \(k\) and after a calculation we get

\[
\text{Tr}(P_{j,k}) = \begin{cases} (n + 1) P_{j-1, k-2} + \langle x, x \rangle P_{j-1, k-2} & \text{if } 2j \leq k - 2, \\ (n + 1) P_{j-1, k-2} & \text{if } 2j = k - 1, \\ n P_{j-1, k-2} & \text{if } 2j = k. \end{cases}
\]

Here \(P_{-1, k-2} = 0\).

Now it is easily seen that, if \(u_j = (-1/n + 1)^j\) when \(j < 2k\) and \(u_j = (-1/n + 1)^j (n + 1)/n\) when \(j = 2k\), we have

\[
\text{Tr} \sum_{0 \leq 2j \leq k} u_j \langle x, x \rangle^j P_{j,k}(x, \xi) = 0.
\]

We have proved the following.
Lemma 2. If \( A \) is an irreducible representation of \( SO(p,q) \) with the weight \( (k, 0, \ldots, 0) \), then \( L^\xi \) is spanned by the polynomial

\[
P_k(x, \xi) = \sum_{0 \leq 2j \leq k} u_j c_{jk} \langle x, x \rangle^j \langle \xi, \xi \rangle^j \langle x, \xi \rangle^{k-2j}.
\]

For other irreducible representations \( L^x = \{0\} \).

3. Distributions covariant under algebraic representations of \( SO(p,q) \).

Now we can use the results in section 1 and 2 to characterize the distributions on \( R^n_0 = R^n \setminus \{0\} \) which are covariant under irreducible representations of \( SO(p,q) \).

Lemma 3. Let \( A \) be an irreducible representation of \( SO(p,q) \) as linear mappings on the finite dimensional linear space \( L \). Then there are covariant distributions on \( R^n_0 \) if and only if \( A \) has the weight \( (k, 0, \ldots, 0) \) and in that case \( T \) is covariant if and only if

\[
T(x)(\xi) = f(x) P_k(x, \xi),
\]

where \( f \) is an invariant distribution on \( R^n \) with values in \( C \).

Remark. If \( A \) has the weight \( (k, 0, \ldots, 0) \) every distribution with values in \( L \) can be written as a homogeneous polynomial in \( \xi \) of degree \( k \) with distributions \( T_\alpha \) as coefficients that is

\[
T(x)(\xi) = \sum_\alpha T_\alpha(x) \xi_\alpha,
\]

where \( T_\alpha \) is symmetric.

Proof. From theorem 1 and lemma 2 it follows that it is sufficient to prove that \( R^n_0 \) has the property \( (\ast) \). If \( x_0 = (x_1^0, \ldots, x_n^0) \in R^n \), then, for example, \( x_1^0 \neq 0 \) and the function \( \varphi: R^n \rightarrow R^n \) defined by

\[
y_1 = \varphi_1(x) = \langle x, x \rangle, \quad y_k = \varphi_k(x) = x_k \quad \text{if} \quad k \geq 2
\]

is a diffeomorphism in a neighbourhood of \( x_0 \). Furthermore \( \{y' \mid |y' - x_0'| < \varepsilon\} \) where \( y' = (y_2, \ldots, y_n), \ x_0' = (x_2, \ldots, x_n) \) is diffeomorphic to an open part of \( Gx_0 \) if \( \varepsilon > 0 \) is sufficiently small.

Now it remains to characterize the distributions with support in the origin which are covariant under an irreducible representation \( A \). If the distribution \( T \) has values in \( L \) and support in the origin we can write

\[
T(x)(\xi) = P(D^*, \xi) \delta(x),
\]

where \( D^* = (D_1^*, \ldots, D_n^*) \), \( D_k = \varepsilon(k)(\partial/\partial x_k) \). Furthermore \( T \) is covariant if and only if
that is, when the polynomial \( P(x, \xi) \) is covariant under \( A \). Then evidently \( P(x, \xi) \in L^2 \) for every \( x \) and consequently

\[
P'(x, \xi) = Q(\langle x, x' \rangle) P_k(x, \xi)
\]

with \( P_k \) as in lemma 2. But as \( P \) is a polynomial it follows if we observe the construction of \( P_k \) that \( Q \) also is a polynomial and we have proved

**Lemma 4.** Let \( A \) be an irreducible representation of \( SO(p, q) \) as linear mappings on the finite-dimensional linear space \( L \). There are covariant distributions \( T \) with values in \( L \) and support in the origin if and only if \( A \) has the weight \((k, 0, \ldots, 0)\) and in that case a distribution \( T \) is covariant if and only if

\[
T(x)(\xi) = Q(\Box) P_k(D^2, \xi) \delta(x)
\]

where

\[
\Box = \sum_{\nu=1}^n \epsilon_\nu \partial^2 \partial^2 x_\nu .
\]

In order to combine lemmas 3 and 4 we now prove

**Lemma 5.** \( P_k(x, \xi) \Box^{k+j} \delta(x) = k! P(D^2_k, \xi) (1 + \Box + \Box^2 + \ldots + \Box^j) \delta(x) \).

**Proof.** At first we observe that \( P_k(x, \xi) = P_k(\xi, x) \) and consequently we have \( \text{Tr}_x P_k(x, \xi) = 0 \) if we regard \( P_k \) as a polynomial in \( x \) with polynomials in \( \xi \) as coefficients. We put \( P_k(x, \xi) = \sum a_\alpha(\xi) x_\alpha \) where \( a_\alpha(\xi) \) is symmetric in \( x \). If \( \varphi \in D(\mathbb{R}^n) \) we get after a calculation

\[
\Box(P_k(x, \xi) \varphi(x)) = \binom{k}{2} \text{Tr}_x P_k(x, \xi) \varphi(x) + \sum_{j=1}^n \sum a_\alpha(\xi)(x_a(x_\alpha \partial \partial^2 x_{\alpha j}))D_\alpha \varphi(x) + P_k(x, \xi) \Box \varphi(x) .
\]

Now if we use that \( \text{Tr}_x P_k(x, \xi) = 0 \) it follows by induction that if \( r \leq k \), then

\[
\Box^r(P_k(x, \xi) \varphi(x)) = \sum_{\mu=0}^r c_\mu \sum_{j_1, \ldots, j_\mu} a_\alpha(x_a(x_{\alpha i_1} \ldots x_{\alpha i_\mu})D_{\alpha i_1} \ldots D_{\alpha i_\mu} \Box^{r-\mu} \varphi(x) ,
\]

where \( c_\mu \) are combinatorial coefficients with \( c_\mu = 1 \). Now the lemma easily follows.

We have now proved the following theorem.

**Theorem 2.** If \( A \) is an irreducible representation of \( SO(p, q) \) as linear mappings on the finite-dimensional linear space \( L \) then there are distributions on \( \mathbb{R}^n \) with values in \( L \) which are covariant under \( A \) if and only if \( A \) has the weight \((k, 0, \ldots, 0)\). In that case a distribution \( T \) is covariant if and only if it can be written
\[ T(x)(\xi) = f(x) \, P_k(x, \xi), \]
where \( f \) is an invariant distribution with values in \( \mathbb{C} \).

If \( A \) is an algebraic representation of \( SO(p,q) \) as linear mappings on the finite-dimensional linear space \( L \) it is well known that \( A \) is reducible (see, for example, Boerner [1]). Then we can write

\[ L = \bigoplus_m a_m L_m, \]
where \( m=(m_1, \ldots, m_p) \), \( p=[n/2] \), \( L_m \) is the invariant subspace which belongs to the irreducible representation \( A_m \) with weight \( m \), and \( a_m \) are non-negative integers a finite number of which are \( >0 \). It is easily seen that

\[ L^x = \bigoplus_m a_m L_m^x = \bigoplus a_{(k,0,\ldots,0)} L_{(k,0,\ldots,0)}^x = \bigoplus a_k L_k^x \]
because \( L_m^x = \{0\} \) if \( m \neq (k,0,\ldots,0) \). The space \( L^x \) is spanned by \( a_k \, P_k(x, \xi^{jk}) \), where \( \xi^{jk}=(\xi_1^{jk}, \ldots, \xi_n^{jk}) \) and \( 1 \leq j_k \leq a_k \).

**Theorem 3.** If \( A \) is an algebraic representation of \( SO(p,q) \) as linear mappings on the finite-dimensional linear space \( L \) then the distribution \( T \) on \( \mathbb{R}^n \) with values in \( L \) is covariant under \( A \) if and only if

\[ (T(x))(\xi) = \sum_{k} \sum_{j_k=1}^{a_k} a_k f_{k,j_k}(x) \, P_k(x, \xi^{j_k}), \]
where \( f_{k,j_k} \) are distributions in \( \mathbb{R}^n \) with values in \( \mathbb{C} \) which are invariant under \( SO(p,q) \) and \( \xi=(\xi_1^1, \ldots, \xi_n^1, \xi_1^2, \ldots, \xi_n^2, \ldots) \).

**References**


University of Lund, Sweden