A NOTE ON DESINTEGRATION, TYPE AND
GLOBAL TYPE OF VON NEUMANN ALGEBRAS

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Introduction.

A von Neumann algebra (v.N. algebra) which satisfies
(i) the center $Z$ is $\sigma$-finite,
(ii) the commutant $Z'$ and the v.N. algebra are both generated by $Z$
and a countable set of operators,
has a central decomposition associated with a finite measure space. In
theorem 1.4 we show, that this decomposition reduces the type-classifi-
cation problem to that of factors on a separable Hilbert space.

Proposition 2.1 and 2.2 could be called a sort of Fubini theorem for
direct integrals of v.N. algebras. These results might be a help in under-
standing the globally central decomposition, which is introduced in sec-
tion 3.

In analogy with the usual central decomposition, theorems 3.2 and 3.5
show that we can reduce global classification problems to those of global
factors.

In the special case of a centrally smooth v.N. algebra we end up by
refining the canonical decomposition from [4].

1.

Let $H_n$ be a fixed Hilbert space of dimension $n$ for $n = 1, 2, \ldots, \aleph_0$,
and let $A_n$ be the set of all v.N. algebras on $H_n$. Following [3] and [4]
we have a standard Borel structure on

$$A = \bigcup \{A_n | n = 1, 2, \ldots, \aleph_0 \}.$$ 

The relative structure on the set $F$ of factors is also standard.

**Proposition 1.1.** The set $F_{II}$ of type II factors is an analytic set in $A$.

**Proof.** A v.N. algebra is semi-finite if and only if it is algebraically
isomorphic to the commutant of a finite v.N. algebra. The set $F_{sf}$ of semi-

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finite factors is thus the saturation with respect to algebraic isomorphism of the set $F'_r$ of the commutants of finite factors. $F_r$ is a Borel set [4, theorem 2.8] and thus $F'_r$ is Borel. $F_{sf}$ is then analytic by the proof of [4, theorem 3.4], and we get that $F_{II} = F'_{sf} \setminus F_I$ is an analytic set.

**Corollary 1.2.** Let $x \to B(x)$ be a Borel field of factors on a finite measure space $(X, \mu)$. The set $\{x \in X \mid B(x) \in F_{II}\}$ is measurable. (X need not be countably separated.)

The next theorem due to R. J. Aumann [1] is a generalization of “The principle of measurable choice” [2, appendix V].

**Theorem 1.3.** Let $(T, \mu)$ be a finite measure space and $X$ a standard Borel space. Let $G$ be a Borel set in $T \times X$ such that the projection into $T$ is the whole of $T$. Then there exists a Borel null set $N \subset T$ and a Borel map $g : T \setminus N \to X$ such that $(t, g(t)) \in G$ for all $t \in T \setminus N$.

This theorem enables us to improve some well-known results. We have for example:

**Theorem 1.4.** Let $(X, \mu)$ be equivalent to a finite measure space, $x \to B(x)$ a Borel field of factors, and $B = \int_X B(x) d\mu(x)$. Then $B$ is of type I (resp. $I_n$, $\Pi$, $\Pi_1$, $\Pi_\infty$, III) if and only if $B(x)$ is of type I (resp. $I_n$, $\Pi$, $\Pi_1$, $\Pi_\infty$, III) for almost all $x \in X$.

The theorem is well known in the case where $X$ is a standard Borel space ([2, chap. II and III] and [5]). The proof of the theorem follows the same line as the proof of this special case. The improvement is mainly relying on theorem 1.3. However, in order to use this theorem we must know that the sets $X_1 = \{x \in X \mid B(x) \in F_1\}$, $X_{I_1}$, $X_{\Pi_1}$, $X_{\Pi_\infty}$, $X_{III}$ are measurable, but this follows from [4] and corollary 1.2.

2.

**Proposition 2.1.** Let $(X, \mu)$ and $(Y, \nu)$ be standard Borel spaces with measures. Let $\mu \otimes \nu$ be the product measure, and $(x, y) \to B(x, y)$ a Borel field of v.N. algebras on $X \times Y$. The v.N. algebras

$$
\int_{X \times Y} B(x, y) d\mu(x) \otimes d\nu(y) \text{ and } \int_X \int_Y B(x, y) d\nu(y) d\mu(x),
$$

where $\beta$ is a suitable coherence, are spatially isomorphic.
PROOF. This follows immediately from the Fubini theorem and [4, lemma 4.5].

PROPOSITION 2.2. Let \((X, \mu)\) and \((Y, \nu)\) be as in proposition 2.1 and \(x \to B(x)\) and \(y \to C(y)\) Borel fields of v.N. algebras. Then

(i) \((x, y) \to B(x) \otimes C(y)\) is a Borel field,

(ii) the v.N. algebras

\[
\int_X B(x) \, d\mu(x) \otimes \int_Y C(y) \, d\nu(y) \quad \text{and} \quad \int_{X \times Y} B(x) \otimes C(y) \, d\mu \otimes d\nu(x, y)
\]

are spatially isomorphic.

PROOF. When we have shown (i), we get (ii) from proposition 2.1 and [2, chap. II, \$3, proposition 3].

Identify \(H_n \otimes H_m\) and \(H_{nm}\) by suitable fixed isometries. Then \(\otimes: A_n \times A_m \to A_{nm}\) is well defined for all \(n\) and \(m\) in \(\{1, 2, \ldots, k_0\}\). Let \(I_n\) denote the set of scalar multiples of the unit operator on \(H_n\). By [4, remark after lemma 2.1],

\[B \to B \otimes I_m \quad \text{and} \quad C \to I_n \otimes C\]

are Borel maps.

Further, \((B, C) \to (B \cup C)''\) is a Borel map: \(A_{nm} \times A_{nm} \to A_{nm}\) [3, theorem 3, corollary 2]. Combining the above we see that

\[(B, C) \to B \otimes C = (B \otimes I_n \cup I_m \otimes C)''\]

is a Borel map. Now (i) follows easily.

3.

In the following let \(B\) denote a v.N. algebra on a separable Hilbert space \(H\), let \(Z\) be the center of \(B\) and \(Z^o\) the lattice of projections in \(Z\). Let \(\sim\) denote spatial equivalence in \(Z^o\), that is, \(E \sim F\) if and only if \(B_E\) and \(B_F\) are spatially isomorphic, and if and only if there exists a partial isometry \(U \in L(H)\) such that \(UBU^* \subseteq B\) and \(U^*BU \subseteq B\), \(U^*U = E\) and \(UU^* = F\). \(\text{See [4, \$6].}\) Let \(Z_G^o\) be the set of globally central projections, and \(Z_G = (Z_G^o)''\). We define

\[G(B) = \{U \in L(H) \mid U \text{ is unitary, } UBU^* = B\}.\]

Then we have \(G(B) = G(B')\).

LEMMA 3.1. \(Z_G = G(B)\)' and \(Z_G^o\) is the lattice of projections in \(Z_G\).
Proof. $G(B)$ contains all unitaries in $B \cup B'$, thus $G(B)' \subset Z$. From [4, lemma 6.1], we have $Z_G^o = Z \cap G(B)'$ and we get indeed $Z_G = (Z \cap G(B)')'' = G(B)'$.

For $E \in Z^o$ let $\hat{E}$ denote the minimal projection in $Z_G^o$ greater than $E$; $\hat{E}$ is called the globally central support of $E$.

Define global type as in [4, § 6], and define $B$ to be of global type $I_n^p$, denoted $I_n^G$, if there exist $E_i \in Z^o$, $i = 1, 2, \ldots, n$, pairwise orthogonal, spatially equivalent and globally multiplicity free such that $I = \sum E_i$.

It is a well-known procedure to show that a v.N. algebra $B$ has a unique decomposition into a direct sum of v.N. algebras of the types $I_{n}^G$, $n = 1, 2, \ldots, \mathfrak{n}_0$, $\Pi_1^G$, $\Pi_\infty^G$, $\Pi_{\infty}^G$, such that the projections $E_{I_n}$, \ldots, $E_{I_{\infty}}$ determining it are globally central.

It is quite easy to prove that any v.N. algebra $B$ has a greatest central projection $E_1^* C$ such that $B_{E_1}$ is centrally smooth. It follows that $B_F$ is not centrally smooth for any $F \in Z^o$ with $F \leq I - E_1^* C$, and that $E_1 E_{I_{\infty}} = I$ in general is not known.

We know [4, § 6], that $E_{I_0} E_{I_{\infty}}$, but which relations there are between $E_{I_0}$ and $E_1$ (resp. $E_{I_{\infty}}$) or whether $E_1 = I$ in general is not known.

We have $Z_G \subset B \subset Z_G^o$. We can take $Z_G$ as the set of diagonal operators $Z_G(\mu)$ on a direct integral of Hilbert spaces $H = \int_X H(x) d\mu(x)$. Then $B$ is decomposable: $B = \int_X B(x) d\mu(x)$. This decomposition of $B$ over $Z_G$ is essentially unique. We call it the globally central decomposition of $B$.

Theorem 3.2. (i) Let $B = \int_X B(x) d\mu(x)$ be the globally central decomposition of $B$. Then $B(x)$ is a global factor for almost all $x \in X$.

(ii) If $B = \int_X B(x) d\mu(x)$, and $B(x)$ is a global factor for almost all $x \in X$, and the set of diagonal operators $Z(\mu)$ is contained in $Z_G$, then $Z(\mu) = Z_G$, and we have the globally central decomposition.

Proof. (i). The space $G(B)$ is a Polish space in the strong operator topology. Let $T_k, k = 1, 2, \ldots$, be a strongly dense sequence in $G(B)$. Let $T_k = \int_X T_k(x) d\mu(x)$. Since the set of $T_k$'s generates $G(B)' = Z_G'$, the set of $T_k(x)$'s generates $L(H(X))$ for almost all $x \in X$.

For all $k$ we have $T_k BT_k^* = B$, and thus $T_k(x) B(x) T_k(x)^* = B(x)$ for almost all $x \in X$. From this we deduce that, for almost all $x \in X$,

$$\{ T_k(x) \mid k = 1, 2, \ldots \} \subset G(B(x)),$$

and therefore $L(H(x)) = G(B(x))'$, which shows that $B(x)$ is a global factor for almost all $x \in X$.

(ii). Take $E$ in $Z_G^o$. Since $Z(\mu) \subset Z_G$, we can write $E = \int_X E(x) d\mu(x)$. We can assume that, for all $x \in X$, $B(x)$ is a global factor and $E(x)$ is a central projection in $B(x)$. The set $X' = \{ x \in X \mid E(x) \neq 0, I \}$ is Borel.
Suppose that $\mu(X') \neq 0$. We can assume that $H(x) = H_n$ for a fixed $n$ and all $x \in X'$.

Since $B(x)$ is a global factor, and $E(x) \neq 0, I$ for all $x \in X'$, there exists for all $x \in X'$ a unitary operator in $L(H_n)$ such that $UB(x)U^* = B(x)$ and $UE(x)U^* = E(x)$.

The set $G_n$ of unitaries on $H_n$ is a Polish space in the strong topology. Take the set

$$M = \{(x, U) \in X' \times G_n | UB(x)U^* = B(x), UE(x)U^* = E(x)\}$$

which is a Borel set [4, lemma 2.1].

From theorem 1.3 we get a Borel null set $N \subset X'$ and a Borel map $x \to U'(x)$ from $X' \setminus N$ into $L(H_n)$ such that, for all $x \in X' \setminus N$,

$$U'(x) \in G(B(x)) \quad \text{and} \quad U'(x)E(x)U'(x)^* = E(x) .$$

Now extend $U'$ to a Borel field $U$ on the whole of $X$ by defining $U(x)$ to be the unit operator on $H(x)$ for $x$ not in $X' \setminus N$. For $U = \int_X U(x)d\mu(x)$ we have $U \in G(B)$ and $UEU^* = E$, which is a contradiction since $E \in Z_G$.

Thus we have $\mu(X') = 0$, and therefore $E \in Z(\mu)$.

This completes the proof.

**Proposition 3.3.** Let $E$ and $F$ be central projections and $E \sim F$, let $U$ be a partial isometry determining the equivalence. Then $U \in Z_G'$.

**Proof.** It is sufficient to show, that $UG = GU$ for all $G \in Z_G'$.

It is obvious that $GE \sim UGEU^* \leq F$. Since $G \in Z_G'$, we have $UGEU^* \leq G$, and thus $UGEU^* \leq GF$. In the same way we may obtain $U^*GFU \leq GE$. Combining these inequalities we get $UGEU^* = GF$ or $UGU^* = GUU^*$, and thus $UG = UEG = UGE = GFU = GU$.

**Proposition 3.4.** Let $B = \int_X B(x)d\mu(x)$ be the globally central decomposition of $B$. Let $E = \int_X E(x)d\mu(x)$ and $F = \int_X F(x)d\mu(x)$ be central projections in $B$. Then

(i) $E \sim F$ if and only if $E(x) \sim F(x)$ for almost all $x \in X$,

(ii) $\widehat{E} = \int_X \widehat{E(x)}d\mu(x)$.

**Proof.** (i). Let $U$ be a partial isometry giving the equivalence. By proposition 3.3, $U$ is decomposable: $U = \int_X U(x)d\mu(x)$. For almost all $x \in X$ we have that $U(x)$ is a partial isometry such that

$$U(x)^*U(x) = E(x), \quad U(x)U(x)^* = F(x) ,$$
and thus $E(x) \sim F(x)$ for almost all $x \in X$.

On the other hand, suppose that $E(x) \sim F(x)$ for almost all $x \in X$, that is, $B(x)_{E,x}$ is spatially isomorphic to $B(x)_{F,x}$. It follows from [2, chap. II, § 3, Proposition 6] and [4, lemma 4.1] that $B_E$ is spatially isomorphic to $B_F$, and thus $E \sim F$.

(ii). $E = \int_X E(x) d\mu(x)$ is a globally central projection if and only if $\mu(\{x \in X \mid \hat{E}(x) \neq 0, I\} = 0$. Since $B(x)$ is a global factor, $\hat{E}(x) = 0$ if $E(x) = 0$ and $\hat{E}(x) = I$ if $E(x) \neq 0$. From this (ii) follows easily.

**Theorem 3.5.** If $B$ is a v.N. algebra on a separable Hilbert space and $\int_X B(x) d\mu(x)$ is its globally central decomposition, then we have:

(i) $B$ is of type $I^G$ (resp. $I_1^G$, $I_\infty^G$) if and only if $B(x)$ is of type $I^G$ (resp. $I_1^G$, $I_\infty^G$) for almost all $x \in X$.

(ii) If $B$ is of type $I_\infty^G$, then $B(x)$ is of type $I_\infty^G$ for almost all $x \in X$.

(iii) If $B(x)$ is of type $III^G$ for almost all $x \in X$, then $B$ is of type $III^G$.

We omit the proof of this theorem. It is quite extensive and involves a thorough discussion of the spatial equivalence relation on $Z^\circ$. However, it uses the same technique and is rather similar to the proof of theorem 1.4, and the basic difficulties are tackled in propositions 3.3 and 3.4.

It is not known whether (ii) or (iii) has a converse. This problem might be accessible similar to [5], by some kind of trace argument.

**Remark 3.6.** If $B$ is centrally smooth and

$$B = \sum_{n=0}^{\infty} \int B(x) d\mu_n(x) \bigotimes W_n = \sum_{n=0}^{\infty} \int B(x) \bigotimes B_n d\mu_n(x)$$

is the canonical decomposition, as described in [4, § 5], it is easily seen from theorem 3.2 (ii) that this gives the globally central decomposition by taking $(X, \mu)$ to be the disjoint sum of $(W_n, \mu_n)$, $n = 0, 1, 2, \ldots, \mathcal{N}_0$. Also, the type $I_n^G$ part of $B$ is the part associated with $W_n$ for $n = 1, 2, \ldots, \mathcal{N}_0$, and the type $III^G$ part of $B$ is that associated with $W_0$.

If every v.N. algebra on a separable Hilbert space is centrally smooth, then theorem 3.5 almost trivially follows from the above remark.

**References**


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