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TOPOLOGICAL GROUPS IN WHICH MULTIPLICATION ON ONE SIDE IS DIFFERENTIABLE OR LINEAR

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Our aim in this paper is to study Hilbert's fifth problem for infinite-dimensional groups. We will imitate the historical development for finite-dimensional groups and so we study groups with a left-condition on the group multiplication. In 1938 it was proved by Birkhoff [1] that locally Banach local groups where $(x,y) \to x \cdot y$ is continuously differentiable in a sense defined below, are analytic local groups. In 1946 it was proved by Segal [3] that locally Euclidean groups where $x \to x \cdot y$ is differentiable for fixed y, are Lie groups. Contrary to Birkhoff's proof Segal's proof is valid only for finite-dimensional groups, since it makes use of Haar measure and also for other reasons. The main result of this paper, Theorem 2.1, shows how Segal's theorem generalizes to the infinite-dimensional case. We also give two types of counter-examples to show, that only continuity of the group multiplication and left differentiability do not imply analyticity. I wish to thank H. Rådström, who suggested that I should study infinite-dimensional groups, for helpful discussions and stimulating interest.

Definitions.

Let f be a map from a Banach space B to a Banach space C. We say that f has a strong Gateaux derivative at x_0, if

$$\|f(x)-f(x_0)\| = O(\|x-x_0\|) \quad \text{as} \quad x \to x_0 ,$$

and if there is a continuous linear operator $u\colon B \to C$ such that

$$\|f(x)-f(x_0)-u(x-x_0)\| = o(\|x-x_0\|)$$

in every finite-dimensional subspace of B. We say that u is the Frechet derivative of f at x_0, if

$$\|f(x)-f(x_0)-u(x-x_0)\| = o(\|x-x_0)\| \quad \text{as} \quad x \to x_0 .$$

We say that f is continuously Frechet differentiable, if u depends continuously on x_0 in the norm topology for u.

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A local group is called a left differentiable local group if it satisfies the conditions 1) and 2a) below. It is called a local L-group (left linear group) if it satisfies the conditions 1) and 2b) below. It is called an analytic local group if it satisfies the conditions 1) and 2c) below.

1) A neighbourhood of the unit element is a neighbourhood of zero in a Banach space and zero is unit element.

2a) $x \to x \cdot y$ is continuously Frechet differentiable, if x and y are sufficiently near zero.

2b) The group multiplication satisfies $x \cdot y = y + T_y x$ where T_y is a linear transformation depending on y, if x and y are sufficiently near zero.

2c) $(x,y) \to x \cdot y$ is continuously Frechet differentiable, if x and y are sufficiently near zero.

The above mentioned theorem by Birkhoff explains our choice of terminology in the last definition.

An L-group is a group in which a neighbourhood of the unit element is a local L-group.

1. Some elementary properties and examples of L-groups.

In a local L-group we have:

1) The operator T_y is bounded since $x \to x \cdot y = y + T_y x$ is continuous.

2) $\|T_y\|$ is bounded for y in some neighbourhood of zero since $(x,y) \to x \cdot y$ is continuous at $(0,0)$.

3) $T_{y \cdot z} = T_z T_y$. This follows from the associative law.

4) $y \to T_y$ is continuous in strong operator topology for T_y. This follows from the formula $T_y x - T_y x_0 = x \cdot y - x \cdot y_0 + y_0 - y$, where the right hand side tends to zero as $y \to y_0$.

5) If $y \to T_y$ is continuous at some point y_0 in the norm topology for T_y, it is also continuous at the point $z \cdot y_0$, and so $y \to T_y$ is continuous in a neighbourhood of zero in the norm topology for T_y. This follows from 1) and 3), since

$$\|T_{z \cdot y} - T_{z \cdot y_0}\| = T_y T_z - T_{y_0} T_z\| \leqq \|T_y - T_{y_0}\| \, \|T_z\|$$

which tends to zero as $y \to y_0$.

We now give two examples of L-groups which are not analytic local groups.

Example 1. Let G be the set of continuously differentiable real-valued functions on $[0,1]$ with $f(0) = 0$, $f(1) = 1$, $f'(x) > 0$. As group operation we take $(f,g) \to f \circ g$ and as metric we take $d(f,g) = \sup |f'(x) - g'(x)|$. In this way G becomes a topological group. For

$$\begin{aligned} d(f\circ g, f_0\circ g_0) \leqq\ & \sup|f'(g(x))\cdot g'(x)-f_0'(g(x))\cdot g'(x)| + \\ & +\sup|f_0'(g(x))\cdot g'(x)-f_0'(g_0(x))\cdot g'(x)| + \\ & +\sup|f_0'(g_0(x))\cdot g'(x)-f_0'(g_0(x))\cdot g_0'(x)| . \end{aligned}$$

As $(f,g) \to (f_0,g_0)$ all three terms become small, the middle term since $x \to f_0'(x)$ is uniformly continuous. The continuity of $f \to f^{-1}$ follows easily from the inequality

$$\begin{aligned} \sup|(f^{-1})'(f(x))-(f_0^{-1})'(f(x))| \leqq\ & \sup|(f^{-1})'(f(x))-(f_0^{-1})'(f_0(x))| + \\ & +\sup|(f_0^{-1})'(f_0(x))-(f_0^{-1})'(f(x))| . \end{aligned}$$

By the mapping $f \to f-x$, G is mapped isometrically onto a neighbourhood of the zero element in the Banach space, which consists of the continuously differentiable functions h on $[0,1]$ with $h(0)=h(1)=0$ and with the norm $\|h\|=\sup|h'(x)|$. The image of G under this mapping is an L-group since

$$h_1\cdot h_2 = (h_1+x)\circ(h_2+x) - x = h_1\circ(h_2+x) + h_2 = h_2 + T_{h_2}h_1 ,$$

where T_{h_2} is linear. But we observe that the group multiplication is not uniformly continuous in any neighbourhood of the unit element as a function $G\times G \to G$. For

$$\begin{aligned} d(f\circ g, f\circ e) &= \sup|f'(g(x))\cdot g'(x) - f'(x)| \\ &\geqq \sup|\big(f'(g(x))-f'(x)\big)\cdot g'(x)| - \sup|(g'(x)-1)\cdot f'(x)| \end{aligned}$$

and even if $d(g,e)$ is small, this does not imply that the first term on the right hand side is small since the f':s do not form an equicontinuous family. In the same way we see that $\lim_{g\to h}\|T_g-T_h\|=0$ does not hold.

EXAMPLE 2. We let (x,y) stand for an element in R_3, $x\in R_1$, $y\in R_2$. We define a group multiplication in R_3 in the following way: $(x_1,y_1)\cdot(x_2,y_2)=(x_1+x_2,y_2+V_{x_2}y_1)$ where $x \to V_x$ is a periodic one-parameter group of unitary linear transformations of R_2. Let p be the smallest positive number such that $V_p=I$. In this way R_3 becomes an L-group. Since $V_{\frac12 p}=-I$, we see that $(\frac12 p,y)\cdot(\frac12 p,y)=(p,0)$ and so (p,y) has no square root for $y\neq 0$, and many square roots for $y=0$. We see that if $z\in R_3$ and $z\neq 0$, then $\|z^n\| \to \infty$ as $n\to\infty$ for if $z=(x,y)$ and $x\neq 0$, then $\|z^n\|=\|(nx,y_1)\|\geqq n|x|$ and if $x=0$, then $z^n=(0,ny)$. We also see that if $z_1\in R_3$, $z_2\in R_3$, then $\|z_1\cdot z_2\|\leqq\|z_1\|+\|z_2\|$. We now take the Hilbert sum H of a sequence of such groups with periods $p_j\to 0$. The sum is a Hilbert space with the norm $\|z\|=(\sum\|z_j\|^2)^{\frac12}$, where $\|z_j\|$ is the Euclidean norm in R_3. H is also an L-group. For if ε is an arbitrary positive number and $z^0\in H$, $w^0\in H$, then

$$z^0 \cdot w^0 = (z_1{}^0 \cdot w_1{}^0, \ldots, z_N{}^0 \cdot w_N{}^0, z_{N+1}^0 \cdot w_{N+1}^0, \ldots),$$

where

$$\begin{aligned}&\|(0,\ldots,0,z_N{}^0 \cdot w_N{}^0, z_{N+1}^0 \cdot w_{N+1}^0,\ldots)\| \\ &\qquad \leqq \|(0,\ldots,0,z_N{}^0,z_{N+1}^0,\ldots)\| + \|(0,\ldots,0,w_N{}^0,w_{N+1}^0,\ldots)\| < \tfrac{1}{4}\varepsilon\end{aligned}$$

if N is sufficiently large. Now if (z,w) is sufficiently near (z^0,w^0), we have

$$\begin{aligned}&\|(0,\ldots,0,z_N \cdot w_N, z_{N+1} \cdot w_{N+1},\ldots)\| \\ &\qquad \leqq \|(0,\ldots,0,z_N,z_{N+1},\ldots)\| + \|(0,\ldots,0,w_N,w_{N+1},\ldots)\| < \tfrac{1}{2}\varepsilon .\end{aligned}$$

And then we also have

$$\|(z_1 \cdot w_1 - z_1{}^0 \cdot w_1{}^0, \ldots, z_{N-1} \cdot w_{N-1} - z_{N-1}^0 \cdot w_{N-1}^0, 0, 0, \ldots)\| < \tfrac{1}{4}\varepsilon .$$

Thus if (z,w) is sufficiently near (z^0,w^0), $\|z\cdot w - z^0 \cdot w^0\| < \frac{1}{4}\varepsilon + \frac{1}{2}\varepsilon + \frac{1}{4}\varepsilon = \varepsilon$ and hence $(z,w) \to z\cdot w$ is continuous. The continuity of $z \to z^{-1}$ is equally easy to show. It is obvious that the group multiplication satisfies $z\cdot w = w + T_w z$. Also in H, $\|z^n\| \to \infty$ if $z \neq 0$ and so H does not contain small subgroups. However in every neighbourhood of the unit element there are elements z and w, with $z \neq w$ and $z^2 = w^2$. This shows that the condition of uniform continuity in Theorem 2.2 below cannot be removed. We also observe that $T_z \to I$ in the norm topology if z tends to zero along a line of the form $(tz_1, tz_2, \ldots, tz_n, 0, 0, \ldots)$. But also in this case $\lim_{z\to 0}\|T_z - I\| = 0$ does not hold.

In Examples 1 and 2 we have $\lim_{z\to 0}\|T_z\| = 1$, but if we do not choose V unitary in Ex. 2 we can get an L-group where this is not the case. We can choose the one-parameter group of transformations of R_2 to be

$$x \to \begin{pmatrix} \cos 2\pi x p_j{}^{-1} & a \sin 2\pi x p_j{}^{-1} \\ -a^{-1}\sin 2\pi x p_j{}^{-1} & \cos 2\pi x p_j{}^{-1} \end{pmatrix}$$

where $a > 1$. However, Theorem 3.2 in this paper shows that the spectral radius of T_z behaves well.

2. Groups with locally uniformly continuous group multiplication.

In this section we will often use the following well-known fact: If $f\colon B \to C$ is continuously Frechet differentiable in a sphere containing x and y, then

$$\|f(x) - f(y)\| \leqq \sup \|f'(tx + (1-t)y)\| \, \|x - y\| . \tag{1}$$

THEOREM 2.1. *A left differentiable local group is an analytic local group* *if and only if* $(x,y) \to x\cdot y$ *is uniformly continuous in some neighbourhood of the unit element.*

(1) gives that the group multiplication in an analytic local group is uniformly continuous in a neighbourhood of the unit element and so we have only to show the converse. The theorem will follow from the lemmas below. In the sequel we do not always explicitly point out in how large a neighbourhood of zero we work.

Put $x\cdot y = y + f_y(x)$. The associative law gives

$$f_z(y+f_y(x)) = f_z(y) + f_{y\cdot z}(x)\,.$$

Derivation with respect to x gives

$$f_z'(y+f_y(x))\circ f_y'(x) = f'_{y\cdot z}(x)\,.$$

And if we put $x=0$, we get

$$f_z'(y)\circ f_y'(0) = f'_{y\cdot z}(0) \tag{2}$$

Lemma 1. *In a left differentiable local group* $(z,y)\to f_z'(y)$ *is continuous in the norm topology for* $f_z'(y)$ *if* $y \to f_y'(0)$ *has a point of continuity.*

Proof. Equation (2) gives: If $y\to f_y'(0)$ is continuous at a point y, then it is also continuous at $y\cdot z$. We see this by keeping z fixed and varying y in (2), for $y \to f_z'(y)$ is continuous. Thus $y\to f_y'(0)$ is continuous in a neighbourhood of zero. Since $f_0'(0)=I$, this implies that $f_y'(0)^{-1}$ exists and that $y\to f_y'(0)^{-1}$ is continuous in a neighbourhood of zero. By multiplying both sides of (2) to the right with $f_y'(0)^{-1}$ we then get the continuity of $(z,y)\to f_z'(y)$.

Let U be a sphere around zero in a left differentiable local group such that $(x,y)\to x\cdot y$ is uniformly continuous in $U\times U$. Put $f_y'(0)=T_y$.

Lemma 2. *Let* ε *be a positive number and let* S *be a closed sphere,* $S\subset U$. *Then there is a closed sphere* $S'\subset \mathring{S}$ *(with arbitrarily small radius) such that for every* $y\in S'$

$$\overline{\lim_{w\to y}}\, \|T_w - T_y\| \leqq 4\varepsilon\,. \tag{3}$$

Proof. For $y\in S$ we have $x\cdot y = y + T_y x + r_y(x)$, where $\|r_y(x)\|\leqq \varepsilon\|x\|$ for all x with $\|x\|\leqq n^{-1}$ where n depends on y. If M_n is the set of y:s for which n will work, we have $S=\bigcup_{n=1}^{\infty} M_n$. Thus $\bar{M}_n$ has interior points for some n, since a complete metric space is of the second category. Let S' be a closed sphere in $\mathring{\bar{M}}_n$. For $y\in S'$ we have (3). To see this let $y\in S'$, $z\in M_n$, $\|r_y(x)\|\leqq\varepsilon\|x\|$ if $\|x\|\leqq m^{-1}$. Now

$$\begin{aligned}\|x\cdot y-x\cdot z\| &= \|y-z+(T_y-T_z)x+r_y(x)-r_z(x)\| \\ &\geqq \|(T_y-T_z)x\| - 2\varepsilon\|x\| - \|y-z\|,\end{aligned}$$

if $\|x\|=\min(n^{-1},m^{-1})$. Since the left hand side of this inequality tends to zero uniformly in x as $z\to y$, because of the uniformly continuous group multiplication, we get $\overline{\lim}_{z\to y}\|T_z-T_y\|\leqq 2\varepsilon$. Since M_n is dense in S', the triangle inequality gives (3)

Lemma 3. *If in a left differentiable local group $(x,y)\to x\cdot y$ is uniformly continuous in a neighbourhood of the unit element, then $y\to f_y'(0)$ has a point of continuity.*

Proof. Let S_0 be a closed sphere in U, and if S_{n-1} is defined let $S_n\subset\mathring{S}_{n-1}$ be a closed sphere with radius less than n^{-1}, such that for all $y\in S_n$,

$$\overline{\lim_{w\to y}}\,\|T_w-T_y\| \leqq n^{-1}.\bigcap S_n$$

is a point at which $y\to f_y'(0)$ is continuous.

Lemma 1 and Lemma 3 give the important

Lemma 4. *If in a left differentiable local group $(x,y)\to x\cdot y$ is uniformly continuous in a neighbourhood of zero, then $(z,y)\to f_z'(y)$ is continuous in the norm topology for $f_z'(y)$.*

In the sequel we show that the continuity of $(z,y)\to f_z'(y)$ implies that the left differentiable local group is an analytic local group.

Lemma 5. *Let B be a left differentiable local group. If*

$$U = \{(x,w)\mid \|x\|\leqq\delta, \|w-y\|\leqq\delta\}$$

is a neighbourhood of $(0,y)$ in $B\times B$ such that $\|f_x'(w)-I\|<\varepsilon$ for (x,w) in U, and if $(x^k,y\cdot x)\in U$ for $k=1,2,\ldots,n-1$, then

$$\|y\cdot x^n - y - n(y\cdot x-y)\| \leqq n\varepsilon\|y\cdot x - y\|.$$

Proof. From (1) we have

$$\begin{aligned}\|y\cdot x^{k+1}-y\cdot x^k-(y\cdot x-y)\| &= \|x^k+f_{x^k}(y\cdot x)-x^k-f_{x^k}(y)-(y\cdot x-y)\| \\ &= \|f_{x^k}(y\cdot x)-f_{x^k}(y)-(y\cdot x-y)\| \leqq \varepsilon\|y\cdot x-y\|.\end{aligned}$$

Thus

$$\begin{aligned}\|y\cdot x^n-y-n(y\cdot x-y)\| &= \Big\|\sum_{k=0}^{n-1}(y\cdot x^{k+1}-y\cdot x^k-(y\cdot x-y))\Big\| \\ &\leqq \sum_{k=0}^{n-1}\|y\cdot x^{k+1}-y\cdot x^k-(y\cdot x-y)\| \leqq n\varepsilon\|(y\cdot x-y)\|.\end{aligned}$$

The lemma is proved.

LEMMA 6. *If in a left differentiable local group $(z,y) \to f_z'(y)$ is continuous, then $x \to y \cdot x$ satisfies a first order Lipschitz condition uniformly in y, if x and y are sufficiently small.*

PROOF. We first observe that if in every neighbourhood of zero there are x, y and z, with $x \neq z$ and $\|y \cdot x - y \cdot z\| > N\|x-z\|$, then in every neighbourhood of zero there are y and w, with $w \neq 0$ and $\|y \cdot w - y\| > \frac{1}{4}N\|w\|$. To see this define w by $w \cdot x = z$. If both x and z are small enough, then $\|w\| < 2\|x-z\|$ from (1). Then

$$\begin{aligned}\tfrac{1}{2}N\|w\| < \|y\cdot x - y\cdot z\| &= \|y\cdot x - y\cdot w\cdot x\| \\ &= \|x + f_x(y\cdot w) - x - f_x(y)\| < 2\|y\cdot w - y\|,\end{aligned}$$

where the last inequality also follows from (1).

Now choose an ε, $0<\varepsilon<1$ and a δ, such that $\|f_z'(y)-I\|<\varepsilon$ if $\|z\| \leqq 3\delta$ and $\|y\| \leqq 3\delta$ and such that $\|y\cdot x\| < 3\delta$ if $\|y\| \leqq \delta$, $\|x\| \leqq \delta$. We see by putting $y=0$ in Lemma 5 that if $\|x\| \leqq \delta$, then $\|x^k\| \leqq \delta$ for all k such that $(1+\varepsilon)\|x\|k \leqq \delta$. For such k Lemma 5 also gives

$$3\delta+\delta \geqq \|y\cdot x^k - y\| \geqq k(1-\varepsilon)\|y\cdot x - y\|.$$

If $(1+\varepsilon)\|x\|k = \delta$, we see that $\|y\cdot x - y\| = O\,(\|x\|)$ uniformly in y.

REMARK. Lemma 4 and Lemma 6 show that continuity of $(z,y) \to f_z'(y)$ is equivalent with local uniform continuity of $(x,y) \to x\cdot y$.

LEMMA 7. *If in a left differentiable local group $(z,y) \to f_z'(y)$ is continuous, then $x \to y\cdot x$ has a Frechet derivative at $x=0$.*

PROOF. We first show that $t \to y\cdot tx$, $t \geqq 0$, has a derivative at $t=0$. Let δ be a positive number. If we choose t sufficiently small and after that s sufficiently small, we get $\|(sx)^{[t/s]} - tx\| \leqq \delta\|tx\|$. We see this by putting $y=0$ and sx instead of x in Lemma 5. Then Lemma 6 gives

$$\|y\cdot(sx)^{[t/s]} - y\cdot tx\| \leqq K\delta\|tx\|,$$

where K is the Lipschitz constant from Lemma 6. When we divide both sides of it by t, this inequality becomes

$$\left\|\frac{y\cdot(sx)^{[t/s]} - y}{t} - \frac{y\cdot tx - y}{t}\right\| \leqq K\delta\|x\|.$$

If t is sufficiently small, then

$$\|y\cdot(sx)^{[t/s]} - y - [t/s](y\cdot sx - y)\| \leqq [t/s]\delta\|y\cdot sx - y\|,$$

from Lemma 5. Thus we get

$$\|y\cdot(sx)^{[t/s]} - y - (t/s)(y\cdot sx - y)\| \leqq ([t/s]+\delta^{-1})\delta\|y\cdot sx - y\|,$$

and if s is sufficiently small, the right hand side of this inequality is less than $2\delta t s^{-1}\|y\cdot sx - y\| \leqq 2\delta t K\|x\|$. Division by t in this inequality gives

$$\left\|\frac{y\cdot(sx)^{[t/s]} - y}{t} - \frac{y\cdot sx - y}{s}\right\| \leqq 2K\delta\|x\|.$$

These inequalities give

$$\left\|\frac{y\cdot tx - y}{t} - \frac{y\cdot sx - y}{s}\right\| \leqq 3K\delta\|x\|,$$

but since δ is arbitrary this implies that $t \to y\cdot tx$, $t \geqq 0$ has a derivative at $t=0$.

Now put $\lim_{t\to 0} t^{-1}(y\cdot tx - y) = H(x)$. We have

$$y\cdot(tx+tz) = y\cdot tx_t\cdot tz,$$

where $x_t \to x$ as $t \to 0$. Thus

$$\begin{aligned}\lim_{t\to 0}\frac{y\cdot(tx+tz)-y}{t} &= \lim_{t\to 0}\frac{y\cdot tx_t\cdot tz - y}{t} \\ &= \lim_{t\to 0}\frac{(y+tH(x)+o(t))\cdot tz - y}{t} = \lim_{t\to 0}\frac{y\cdot tz + tH(x) + o(t) - y}{t},\end{aligned}$$

since $\|f'_{tz}(y) - I\| \to 0$ as $t \to 0$. And

$$\lim_{t\to 0}\frac{y\cdot tz + tH(x) + o(t) - y}{t} = H(x) + H(z).$$

Thus there is a linear operator H such that $\|y\cdot x - y - Hx\| = o(\|x\|)$ as x tends to zero along a line. We also immediately get $\|H\| \leqq K$ and since the estimates when we prove that $t \to y\cdot tx$ has a derivative at $t=0$ hold uniformly for $\|x\|=1$, we have

$$\|y\cdot x - y - Hx\| = o(\|x\|) \quad \text{as} \quad x \to 0.$$

The lemma is proved.

Proof of Theorem 2.1. We now determine the Frechet derivative of $x \to y\cdot x$ at x_0. We recall that $f'_{x_0}(0) = T_{x_0}$. When $z \to 0$ we have

$$y\cdot(x_0+z) = y\cdot\big((T_{x_0}^{-1}z+o(\|z\|))\cdot x_0\big) = \big(y + HT_{x_0}^{-1}z + o(\|z\|)\big)\cdot x_0$$
$$= y\cdot x_0 + f'_{x_0}(y)HT_{x_0}^{-1}z + o(\|z\|)\,.$$

From this we see that $x \to y\cdot x$ is continuously Frechet differentiable for fixed y. The simultaneous continuity of the partial derivatives of $(x,y) \to x\cdot y$ follows i.e. from Lemma 4. Thus Theorem 2.1 is proved.

We also give the following theorem which is a generalization of a lemma by Gleason (see [2 pp. 120–121]). Example 2 in Section 1 shows the necessity of assuming a condition like local uniform continuity of the group multiplication. We recall that if U is a set of ordered pairs (x,y), $U[x]$ denotes the set of points y such that $(x,y) \in U$.

Theorem 2.2. *Let G be a topological group without small subgroups, where the group multiplication is uniformly continuous in some neighbourhood of the unit element in some uniform structure for G. Then there is a neighbourhood W of e such that $x \in W$, $y \in W$ and $x^2=y^2$ implies $x=y$.*

Proof. Let U and V be members of the uniformity for G such that

1) $U[e]$ is a symmetric neighbourhood of e such that $(U[e])^2$ does not contain a non-trivial subgroup;
2) $(x,y) \to xy$ is uniformly continuous for $(x,y) \in (U[e])^4 \times (U[e])^4$;
3) $V \subset U$ and $(x,x_0) \in V$ and $(y,y_0) \in V$ imply $(xy,x_0y_0) \in U$ if $x,y,x_0,y_0 \in (U[e])^4$.

If $x^2=y^2$ and $a=x^{-1}y$, then $x^{-1}ax=a^{-1}$ and $x^{-1}a^mx=a^{-m}$.

Suppose that the theorem is false. Then we can find x and y, $x\neq y$ and $x^2=y^2$ in such a neighbourhood of e that $a=x^{-1}y \in V[e]$ and such that $(x^{-1}bx,b) \in V$ for all $b \in (U[e])^2$ since we have uniformly continuous group multiplication in $(U[e])^4 \times (U[e])^4$.

Now we have $a^m \in (U[e])^2$ for all positive odd m and $a^m \in (U[e])$ for all positive even m. We prove this by induction. $m=1$ gives $a \in V[e] \subset (U[e])^2$. Suppose that the proposition is valid for all integers $\leqq p$. If p is even, then $a^p \in U[e]$ from the induction hypothesis and since $a \in U[e]$, we have $a^{p+1} \in (U[e])^2$. If p is odd, $p+1$ is even and $a^{\frac{1}{2}(p+1)} \in (U[e])^2$ from the induction hypothesis. Then $(x^{-1}a^{\frac{1}{2}(p+1)}x, a^{\frac{1}{2}(p+1)}) \in V$ from (1), and since $(a^{\frac{1}{2}(p+1)}, a^{\frac{1}{2}(p+1)}) \in V$, assumption 3) gives

$$(x^{-1}a^{\frac{1}{2}(p+1)}xa^{\frac{1}{2}(p+1)},\ a^{\frac{1}{2}(p+1)}a^{\frac{1}{2}(p+1)}) \in U\,.$$

But $x^{-1}a^{\frac{1}{2}(p+1)}xa^{\frac{1}{2}(p+1)}=e$ and so $a^{p+1} \in U[e]$ which was to be proved. But since $U[e]$ is symmetric, $a^m \in (U[e])^2$ for all integers m which contradicts our assumption that $(U[e])^2$ does not contain a non-trivial subgroup. The theorem is proved.

3. Some theorems on local L-groups.

The results of this section are stated only for local L-groups. It would however be interesting to extend especially Theorem 3.3 to a more general class of left differentiable local groups. We begin by giving another necessary and sufficient condition for a local L-group to be an analytic local group than that of locally uniformly continuous group multiplication. Since in a local L-group $T_x \to I$ in strong operator topology as $x \to 0$, we have

$$\|x^2 - 2x\| = \|x + T_x x - 2x\| = \|(T_x - I)x\| = o(\|x\|) \quad \text{as} \quad x \to 0$$

in every finite-dimensional subspace of the Banach space. Thus in a local L-group $x \to x^2$ has the strong Gateaux derivative $x \to 2x$ at zero.

THEOREM 3.1. *A local L-group is an analytic local group if and only if $x \to x^2$ has a Frechet derivative at $x = 0$.*

PROOF. It follows from Theorem 2.1 and the remark after Lemma 6 in the same section that a local L-group is an analytic local group if and only if $x \to T_x$ is continuous in the norm topology for T_x. From the observation 5 in section 1 we see that this is the case if and only if $x \to T_x$ is continuous at $x = 0$ in the norm topology for T_x. Thus we show that norm continuity of $x \to T_x$ at $x = 0$ is equivalent with $x \to x^2$ having a Frechet derivative at $x = 0$. If $x \to T_x$ is continuous at $x = 0$ in the norm topology for T_x, then

$$\|x^2 - 2x\| = \|(T_x - I)x\| = o(\|x\|) \quad \text{as } x \to 0$$

and so $x \to x^2$ has the Frechet derivative $x \to 2x$ at zero. Thus we have only to show the converse. We first prove a lemma.

LEMMA. *If in a local L-group $\lim_{x \to 0} \|T_x - I\| = 0$ does not hold, then there is also a line $t \to tx$ through zero where $\lim_{t \to 0} \|T_{tx} - I\| = 0$ does not hold.*

PROOF OF THE LEMMA. Put $E = \{x \mid \|x\| = 1\}$ and let δ_n be a sequence of positive numbers $\delta_n \to 0$. If $\|T_x - I\| \to 0$ as $x \to 0$ along every straight line through zero, then for every $\varepsilon > 0$ and every $x \in E$ there is a δ_n such that $\|T_{tx} - I\| \leqq \varepsilon$ if $|t| \leqq \delta_n$ where δ_n depends on x. Since the space E is of the second category there is a set $M \subset E$ where the same δ_n will work and such that $\bar{M}$ has interior in E. Then also $\|T_{tx}-I\| \leqq \varepsilon$ if $|t| \leqq \delta_n$ for $x \in \bar{M}$ since $x \to T_x$ is continuous in strong operator topology for T_x. Let y be an interior point of $\bigcup_{|t| \leqq \delta_n} t\bar{M}$, then $y^{-1} \cdot \bigcup_{|t| \leqq \delta_n} t\bar{M}$ is a neighbourhood of zero. We see that $\|T_x - I\| < 3\varepsilon$, if x belongs to this neighbourhood of zero (and ε is sufficiently small). This implies that $\lim_{x \to 0} \|T_x - I\| = 0$. The lemma is proved.

Proof of Theorem 3.1 continued. Now suppose that $x \to x^2$ has a Frechet derivative at zero and that we have not $\lim_{x \to 0} \|T_x - I\| = 0$. We consider a neighbourhood U of zero where $\|T_x y\| \geqq k\|y\|$ for all $x \in U$ and all y, and a line where $\overline{\lim}_{t \to 0} \|T_{tx} - I\| > \varepsilon$. We consider

$$z \cdot tx \cdot z \cdot tx - 2(z \cdot tx) = T_{tx} T_z tx - tx + T_{tx} T_z T_{tx} z - T_{tx} z ,$$

where we choose $\|z\| = \|x\|$. If we choose t sufficiently small, we see that $\|T_{tx} T_z tx - tx\| < \delta \|tx\|$, where we can choose δ arbitrary small. We can choose arbitrary small elements t and z such that $\|T_{tx} z - z\| > \varepsilon \|z\|$. But then $\|T_z T_{tx} z - T_z z\| > k\varepsilon \|z\|$ and if we choose t so small that $\|T_z z - z\| < \frac{1}{2} k\varepsilon \|z\|$, which is possible from our assumption on $x \to x^2$, we get $\|T_z T_{tx} z - z\| > \frac{1}{2} k\varepsilon \|z\|$. And then $\|T_{tx} T_z T_{tx} z - T_{tx} z\| > \frac{1}{2} k^2 \varepsilon \|z\|$. Thus we have

$$\|z \cdot tx \cdot z \cdot tx - 2(z \cdot tx)\| > (\tfrac{1}{2} k^2 \varepsilon - \delta) \|z\| ,$$

but since δ and t can be chosen arbitrary small this contradicts the existence of a Frechet derivative of $x \to x^2$ at zero.

We now give a theorem on the spectral radius $\sigma(T_x)$ of T_x. The remark after Example 2 in section 1 shows that the norm may not behave so well.

Theorem 3.2. *In a local L-group* $\sigma(T_x) = 1 + O(\|x\|)$ *as* $x \to 0$.

Proof. If $\|T_x\| \leqq K$ for $\|x\| \leqq \delta$, then $\|x^n\| \leqq \delta$ if $0 \leqq n \leqq \delta/(K\|x\|)$. To see this we observe that if $\|x^m\| \leqq mK\|x\| \leqq \delta$, then

$$\|x^{m+1}\| = \|x^m + T_{x^m} x\| \leqq mK\|x\| + K\|x\| = (m+1)K\|x\| .$$

Now if $n = \delta/(K\|x\|)]$, we have

$$K \geqq \sigma(T_{x^n}) = \sigma((T_x)^n) = (\sigma(T_x))^n ,$$

where the first inequality follows since the spectral radius does not exceed the norm, the first equality follows from observation 3 in Section 1, and the second equality follows from the spectral mapping theorem. And this formula gives $\sigma(T_x) \leqq 1 + M\|x\|$, for some M. The inequality

$\sigma(T_x) \geqq 1 - N\|x\|$ for some N follows from the spectral mapping theorem by taking inverses.

We conclude this section with a theorem on the non-existence of small subgroups in a class of local L-groups. Let a local H-group be a local L-group whose underlying Banach space has the property that every bounded sequence contains a weakly convergent subsequence. Let U be the sphere $\|z\| \leqq M$.

THEOREM 3.3. *If G is a local H-group such that x^{-1} exists for $x \in U$ and $x \cdot y = y + T_y x$ for $x \in U$, $y \in U$, then U does not contain a non-trivial subgroup of G.*

PROOF. We assume that the theorem is false. Let $\{x^n\}_{-\infty}^{\infty}$ be a non-trivial subgroup in U. We define $z_n = (x + x^2 + \ldots + x^{n-1})/n$. Then $\|z_n\| \leqq M$. We have $z_n = ((n-1)x + (n-2)T_x x + \ldots + T_x{}^{n-2}x)/n$. Thus we have

$$\begin{aligned} z_n \cdot x - z_n &= x + T_x z_n - z_n \\ &= x + \frac{T_x(n-1)x + T_x{}^2(n-2)x + \ldots + T_x{}^{n-1}x}{n} - \\ &\quad - \frac{(n-1)x + T_x(n-2)x + \ldots + T_x{}^{n-2}x}{n} \\ &= \frac{x + T_x x + T_x{}^2 x + \ldots + T_x{}^{n-1}x}{n} = \frac{x^n}{n}. \end{aligned}$$

Thus $\|z_n \cdot x - z_n\| \leqq M/n$. Now z_n contains a weakly convergent subsequence z_{n_ν} which converges to some z, $\|z\| \leqq M$. Then

$$z_{n_\nu} \cdot x - z_{n_\nu} = x + T_x z_{n_\nu} - z_{n_\nu} \rightarrow x + T_x z - z = z \cdot x - z$$

in weak topology. But then

$$\|z \cdot x - z\| \leqq \underline{\lim} \|z_{n_\nu} \cdot x - z_{n_\nu}\| = 0\,,$$

which is a contradiction. We immediately get the

COROLLARY. *A local H-group does not contain small subgroups.*

REMARK. If in the L-group described in Example 1 we choose a function h and form the corresponding z_n as in the proof of Theorem 3.3, then the functions z_n will converge pointwise to a function f with the property $(f + x) \circ (h + x) = f + x$. If we consider the complex plane under multiplication, choose an y with $|y| = 1$, $y \neq 1$ and form the corresponding z_n, then z_n will converge to zero.

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